The Economic Journal, 98 (*September* 1988), 746–754
Printed in Great Britain

THE THEORY OF RATIONAL BUBBLES IN STOCK PRICES*

Behzad T. Diba and Herschel I. Grossman

In a model in which risk-averse asset holders with infinite planning horizons maximise expected utility, the product of a stock's price and the marginal utility of consumption satisfies a first-order linear expectational difference equation that has an eigenvalue greater than unity and a stochastic forcing term that reflects the expected evolution of the stock's dividends. In this model, the market-fundamentals component of the stock price is defined to be the particular solution to this expectational difference equation that equates the product of the stock price and the marginal utility of consumption to the expected present value of the products of future dividends and future marginal utilities of consumption.

The general solution to the expectational difference equation allows the stock price to have a rational-bubbles component in addition to the market-fundamentals component. The existence of a rational-bubbles component would reflect a self-confirming belief that the stock price depends on a variable (or a combination of variables) that is intrinsically irrelevant – that is, not part of market fundamentals – or on truly relevant variables in a way that involves parameters that are not part of market fundamentals.

The property that the eigenvalue of the expectational difference equation is greater than unity has two important consequences. First, it guarantees the existence of an economically meaningful (i.e. forward-looking) market-fundamentals solution except in extreme cases of the process generating dividends. Secondly, it implies that rational bubbles have explosive conditional expectations. Specifically, the expected value of a rational-bubbles component of a stock price either would increase or would decrease geometrically into the infinite future.

The results of Mussa (1984) underscore the association of economically interesting market fundamentals with nonconvergent rational bubbles. Mussa shows that various examples of attempts to construct alternative models in which potential rational bubbles are convergent all preclude a forward-looking market-fundamentals solution for some relevant price variable.[1]

* We thank Robert Flood, Danny Quah, David Runkle and Kenneth West for helpful discussions relating to the ideas in this paper. We have benefitted also from comments by the editor and by two anonymous referees. The views expressed in this paper are those of the authors and do not necessarily represent the views of the Federal Reserve Bank of Philadelphia or of the Federal Reserve System.

[1] Quah (1986) develops an example in which the stock price coincides with the expected present value of future dividends, and yet rational bubbles are convergent for certain parameter values. In this example, however, the market-fundamentals solution is essentially backward looking, because dividends depend on a set of state variables that have no apparent relation to currently available information about current and future earnings and other potentially relevant variables.

The fact that rational bubbles have explosive conditional expectations implies that a negative rational-bubbles component cannot exist, because, given free disposal, stock holders cannot rationally expect a stock price to decrease without bound and, hence, to become negative at a finite future date. The property of explosive conditional expectations also suggests that if a positive rational bubble exists, stockholders might expect it eventually to dominate the stock price, which would then bear little relation to market fundamentals. Positive rational bubbles are empirically plausible only if, despite explosive conditional expectations, the probability is small that a rational bubble would become arbitrarily large. This observation focuses attention on processes, like one suggested by Blanchard and Watson (1982), that apparently can generate rational bubbles that are likely to start, burst and restart repeatedly.

The present paper explores more deeply the theoretical possibility of rational bubbles in stock prices by focusing on the circumstances of the inception of rational bubbles. The inception of a rational bubble after the first date of trading of a stock would involve an innovation in the stock price. Accordingly, any rational-bubbles component that starts after the first date of trading has an expected initial value of zero. Moreover, because free disposal rules out negative rational bubbles, this expected initial value can equal zero only if any initial realisation of a rational bubble after the first date of trading equals zero with probability one.

This theoretical argument means that the impossibility of negative rational bubbles also rules out the inception of a positive rational bubble except at the first date of trading of a stock. One important implication of this argument is that the process suggested by Blanchard and Watson for generating empirically interesting positive rational bubbles is inconsistent with the impossibility of negative rational bubbles. Once a positive rational bubble that began at the first date of trading has burst, it cannot restart.

In the existing literature, Brock (1982) and Tirole (1982) already have constructed an argument against the existence of positive rational bubbles in stock prices.[2] This argument assumes a constant number of asset holders with infinite planning horizons and shows that the existence of a positive rational bubble would violate a transversality condition that, as Benveniste and Scheinkman (1982) demonstrate, is implied by the optimising behaviour of each asset holder. As Gray (1984) explains, the existence of a positive rational bubble in this setting would imply that stockholders expect to gain utility from selling the stock now (or at any finite future date) and never buying it back.

This argument does not necessarily preclude the existence of a positive rational bubble in an economy with a growing number of potential asset holders. Tirole (1985) shows that a rational bubble can arise in asset prices in a model with an infinite succession of overlapping generations of asset holders

[2] Lucas (1978) presents another argument for uniqueness of rational expectations equilibrium based on contraction mappings. Brock (1982) points out that this argument rules out multiple stationary equilibria but does not preclude the non-stationary price paths associated with rational bubbles.

with finite planning horizons, as long as the growth rate of the economy is greater than or equal to the required rate of return. O'Connell and Zeldes (1988) derive the same result in a model with a growing number of asset holders with infinite planning horizons. They show that each agent can rationally plan to sell the overvalued asset to another agent at some finite future date. Accordingly, although the transversality condition of every agent is satisfied, aggregate demand for the overvalued asset need not vanish.

Quah (1986) raises another objection to invoking the transversality condition in order to rule out rational bubbles that almost surely burst at a date in the finite future. The idea seems to be that, even if such rational bubbles can restart repeatedly, the probability that stockholders will gain utility from a strategy that involves selling shares now and never buying them back is zero. Accordingly, the transversality condition would not rule out such rational bubbles if stockholders ignore zero probability events.

The argument developed in the present paper applies to all forms of rational bubbles, including those that apparently can burst and restart repeatedly. Moreover, unlike the analyses of Brock and Tirole, the present argument does not exploit the properties of either infinite planning horizons or a finite number of potential stockholders. Accordingly, although we formalise the analysis within the model of infinite planning horizons, an analogous argument would apply to the inception of rational bubbles in an overlapping-generations framework.

In this regard, note that the results derived below are directly applicable to the simpler model in which the required rate of return on equity is constant. This model can represent a special case, which arises under risk neutrality, of either the model of infinite planning horizons or the model of overlapping generations.

In what follows, Section I reviews the basic properties of rational bubbles in stock prices. Section II derives the result that, if a rational bubble exists, it must have started on the first date of trading. Section III derives the further result that rational bubbles cannot burst and simultaneously restart. Section IV offers concluding remarks.

I. PROPERTIES OF RATIONAL BUBBLES

Assume that a representative household maximises expected utility over an infinite horizon,

$$\mathrm{E}_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau), \quad 0 < \beta < 1, \tag{1}$$

where $\{c_\tau\}$ is a stochastic process representing consumption of a single perishable good, and β is a discount factor for future consumption. Positive time preference implies that β is less than unity. The utility function, $u(\cdot)$, is strictly concave, increasing and continuously differentiable. The conditional expectations operator E_t is based on an information set that contains, at least, current and past values of all the variables entering the model.

Each period, the household receives an endowment, y_τ, of the consumption good. The household can attempt to smooth consumption by acquiring shares, s_τ, at the price of p_τ (units of the consumption good) per share. Each share pays a dividend of d_τ units of the consumption good per period. The budget constraint faced by the household at date τ is

$$c_\tau + p_\tau(s_{\tau+1} - s_\tau) \leqslant y_\tau + d_\tau s_\tau.$$

The stochastic process $\{d_\tau, y_\tau\}$ is exogenous to the model and assumed stationary.

The first-order condition for the household's utility maximisation problem is

$$p_t u'(c_t) = \beta \mathrm{E}_t[(p_{t+1} + d_{t+1}) u'(c_{t+1})]. \tag{2}$$

The left-hand side of equation (2) is the marginal utility from selling a share this period. The right-hand side is the present value of this period's expectation of the marginal utility from selling a share next period. Thus, equation (2) implies that in equilibrium the household cannot increase its expected utility by buying or selling shares at date t and planning to sell or to buy them back at date $t+1$ (or at any date in the finite future).

If the market for the consumption good (and, by implication, the market for shares) is clearing, then the price of shares is such that the representative household's consumption, as given by equation (2), equals the *per capita* supply of the consumption good. If we normalise the number of existing shares *per capita*, s_τ, to unity, this market-clearing condition becomes

$$c_\tau = y_\tau + d_\tau \quad \text{for all} \quad \tau \geqslant t. \tag{3}$$

Following Lucas (1978), substituting equation (3) into equation (2) yields the asset-pricing equation

$$\mathrm{E}_t q_{t+1} - \beta^{-1} q_t = -\mathrm{E}_t[u'(y_{t+1} + d_{t+1}) d_{t+1}], \tag{4}$$

where

$$q_t \equiv u'(y_t + d_t) p_t.$$

Equation (4) is a first-order expectational difference equation. Because the eigenvalue, β^{-1}, is greater than unity, the forward-looking solution for q_t involves a convergent sum, as long as $\mathrm{E}_t[u'(y_{t+j} + d_{t+j}) d_{t+j}]$ does not grow with j at a geometric rate equal to or greater than β^{-1}. The forward-looking solution, denoted by F_t and referred to as the market-fundamentals component of q_t, is

$$F_t = \sum_{j=1}^{\infty} \beta^j \mathrm{E}_t[u'(y_{t+j} + d_{t+j}) d_{t+j}]. \tag{5}$$

This market-fundamentals solution to equation (4) sets the current product of the stock price and the marginal utility of consumption equal to the present value of expected future products of dividends and the marginal utility of consumption. If the representative household is risk neutral, equation (5) reduces to the simpler specification of market fundamentals, which equates the stock price to the present value of expected future dividends.

The general solution to equation (4) is the sum of the market-fundamentals component, F_t and the rational-bubbles component, B_t – that is,

$$q_t = B_t + F_t, \tag{6}$$

where B_t is the solution to the homogeneous expectational difference equation,

$$E_t B_{t+1} - \beta^{-1} B_t = 0. \tag{7}$$

A non-zero value of B_t would reflect the existence of a rational bubble at date t – that is, a self-confirming belief that q_t does not conform to the market-fundamentals component, F_t.

The assumption of rational expectations implies that in forming $E_t B_{t+j}$, for all $j > 0$, potential asset holders behave as if they know that any rational-bubbles component would conform to equation (7) in all future periods. Accordingly, any solution to equation (7) would have the property

$$E_t B_{t+j} = \beta^{-j} B_t \quad \text{for all} \quad j > 0. \tag{8}$$

Equation (8) says that the existence of a non-zero rational-bubbles component at date t would imply that the expected value of the rational-bubbles component at date $t+j$ either increases or decreases with j at the geometric rate β^{-1}. Therefore, because the eigenvalue β^{-1} exceeds unity, the existence of a rational bubble would imply that $\{E_t q_{t+j}\}_{j=1}^{\infty}$ either increases or decreases without bound.

In particular, the existence of a negative rational-bubbles component at date t would imply that $E_t q_{t+j}$ becomes negative for some finite j. But, given free disposal of shares, stockholders cannot rationally expect a stock price to become negative at a finite future date. Therefore, a negative rational-bubbles component would be a contradiction and, hence, cannot exist.

Solutions to equation (7) satisfy the stochastic difference equation

$$B_{t+1} - \beta^{-1} B_t = z_{t+1}, \tag{9}$$

where z_{t+1} is a random variable (or combination of random variables) generated by a stochastic process that satisfies

$$E_{t-j} z_{t+1} = 0 \quad \text{for all} \quad j \geqslant 0. \tag{10}$$

The random variable z_{t+1} is an innovation, comprising new information available at date $t+1$. This information can be intrinsically irrelevant – that is, unrelated to F_{t+1} – or it can be related to truly relevant variables, like d_{t+1}, through parameters that are not present in F_{t+1}. The critical property of z_{t+1}, given by equation (10), is that its expected future values are always zero.

The general solution to equation (9), for any date t, $t \geqslant 0$, is

$$B_t = \beta^{-t} B_0 + \sum_{\tau=1}^{t} \beta^{\tau-t} z_\tau, \tag{11}$$

where date zero denotes the first date of trading of the stock. Equation (11) expresses the rational-bubbles component at date t as composed of two terms. The first term is the product of the eigenvalue raised to the power t and the value of the rational-bubbles component at date zero. The second term is a weighted sum of realisations of z_τ from $\tau = 1$ to $\tau = t$. The weights are powers of the eigenvalue such that the contribution of z_τ to B_t increases exponentially with the difference between t and τ. For example, a past realisation z_τ, $1 \leqslant \tau < t$, contributed only the amount z_τ to B_τ, but contributes $\beta^{\tau-t} z_\tau$ to B_t.

II. THE INCEPTION OF RATIONAL BUBBLES

The fact that a negative rational-bubbles component cannot exist means that, in addition to satisfying equation (9), the rational-bubbles component of a stock price at date $t+1$ satisfies $B_{t+1} \geqslant 0$. Taken together, equation (9) and this nonnegativity condition imply that realisations of z_{t+1} must satisfy

$$z_{t+1} \geqslant -\beta^{-1} B_t \quad \text{for all} \quad t \geqslant 0. \tag{12}$$

Equation (12) says that the realisation z_{t+1} must be large enough to insure that equation (9) implies a nonnegative value for B_{t+1}.

Suppose that B_t equals zero. In that case, equation (12) implies that z_{t+1} must be nonnegative. But, equation (10) says that the expected value of z_{t+1} is zero. Thus, if B_t equals zero, then z_{t+1} equals zero with probability one.

This result says that if a rational bubble does not exist at date t, $t \geqslant 0$, a rational bubble cannot get started at data $t+1$, nor, by extension, at any subsequent date. Therefore, if a rational bubble exists at present, it must have started at date zero, the first date of trading of the stock, and, hence, this stock must have been overvalued relative to market fundamentals at every past date. The essential idea underlying this line of argument is that, because the inception of a rational bubble at any date after the first date of trading would involve an innovation in the stock price, the expected initial values of a positive rational bubble and a negative rational bubble would have to be equal. Accordingly, because free disposal rules out a negative rational-bubbles component, a positive rational-bubbles component also cannot start after the first date of trading.

Suppose that, prior to the first date of trading, the issuer of the stock and potential stockholders anticipate the introduction of trading and form an expectation about the initial stock price. Suppose further that this expectation coincides with market fundamentals – that is,

$$\mathrm{E}_{-1} B_0 = \mathrm{E}_{-1} q_0 - \mathrm{E}_{-1} F_0 = 0. \tag{13}$$

Equation (13) would imply that B_0 is a random variable with mean zero. Accordingly, given the nonnegativity condition $B_0 \geqslant 0$, B_0 would equal zero with probability one. This observation implies that if a positive rational bubble exists, the issuer of the stock and potential stockholders who, prior to the first date of trading, anticipated the initial pricing of this stock expected it to be overvalued relative to market fundamentals.

III. CAN POSITIVE RATIONAL BUBBLES BURST AND RESTART?

As mentioned above, the existence of a positive rational-bubbles component is empirically plausible only if, despite explosive conditional expectations, the probability is small that the rational bubble component will ever become large enough to dominate the stock price. This observation suggests the following model of the innovation z_{t+1}:

$$z_{t+1} = (\theta_{t+1} - \beta^{-1}) B_t + \epsilon_{t+1}, \tag{14}$$

where θ_{t+1} and ϵ_{t+1} are mutually and serially independent random variables. If the processes generating θ_{t+1} and ϵ_{t+1} satisfy

$$\mathrm{E}_{t-j}\,\theta_{t+1} = \beta^{-1} \quad \text{for all} \quad j \geqslant 0 \tag{15}$$

and

$$\mathrm{E}_{t-j}\,\epsilon_{t+1} = 0 \quad \text{for all} \quad j \geqslant 0, \tag{16}$$

then z_{t+1} as given by equation (14) satisfies equation (10).

Substituting for z_{t+1} in equation (9) from equation (14) gives

$$B_{t+1} = \theta_{t+1} B_t + \epsilon_{t+1}. \tag{17}$$

Equation (17) says that, with z_{t+1} given by equation (14), an existing rational-bubbles component, B_t, will burst next period if the event $\theta_{t+1} = 0$ occurs. If this event has positive probability, then any rational-bubbles component would burst at a random, but almost surely finite, future date. Specifically, if the probability associated with $\theta_{t+1} = 0$ is Π, $0 < \Pi < 1$, then the expected duration of a rational-bubbles component is Π^{-1} periods and the probability that B_t will not burst by date T ($T > t$) is $(1-\Pi)^{T-t}$, which tends to zero as T approaches infinity.

Given that realisations of θ_{t+1} and ϵ_{t+1} are mutually and serially independent and also independent of B_0, ϵ_{t+1} is independent of B_t for all $t \geqslant 0$. In this case, if the event $\theta_{t+1} = 0$ were by chance to coincide with a positive realisation of ϵ_{t+1}, then, according to equation (17), as an existing rational-bubbles component bursts, a new rational-bubbles component, which is independent of all existing and past rational-bubbles components, would simultaneously start.

Quah (1986) suggests this model as a generalisation of a model of rational bubbles that could burst and restart proposed by Blanchard and Watson (1982). Quah argues that the property that any existing rational-bubbles component will almost surely burst at a date in the finite future, in addition to implying a small probability that the rational-bubbles component would become large enough to dominate the stock price, also makes these models of rational bubbles immune to the arguments of Brock (1982) and Tirole (1982) that a transversality condition precludes rational bubbles. Quah's presumption is that stockholders ignore the possibility, which has zero probability, that the rational-bubbles component will never burst.

The result derived in Section II that, given the impossibility of a negative rational-bubbles component, a rational-bubbles component can start only on the first date of trading directly implies that a rational-bubbles component that burst could not restart at a later date. The essential property that a negative rational-bubbles component cannot exist follows directly from equation (8) and, hence, obtains whatever the process that generates the innovation z_{t+1}.

This property means that in the present model, in addition to satisfying equation (17), the rational-bubbles component satisfies $B_{t+1} \geqslant 0$. Therefore, the event $\theta_{t+1} = 0$ cannot coincide with a negative realisation of ϵ_{t+1}. Accordingly, given that the event $\theta_{t+1} = 0$ has positive probability and that the random variables ϵ_{t+1} and θ_{t+1} are independent, ϵ_{t+1} must be nonnegative. But, equation (16) says that the expected value of ϵ_{t+1} is zero. Therefore, ϵ_{t+1} equals zero with probability one and the chance coincidence of $\theta_{t+1} = 0$ and $\epsilon_{t+1} > 0$ has zero probability.

This result says that the impossibility of a negative rational-bubbles component also precludes the possibility that a new independent positive rational-bubbles component simultaneously starts when an existing positive rational-bubbles component bursts. In sum, the analysis in sections II and III has shown that, if a positive rational-bubbles component exists, then it must have started on the first date of trading of the stock, it has not yet burst, and it will not restart if it bursts. Together with the assumption that the event $\theta_{t+1} = 0$ has positive probability, these properties correspond to Blanchard's (1979) specification of a rational-bubbles component that exists from the first date of trading, eventually bursts, and does not restart.

IV. CONCLUDING REMARKS

Free disposal of equity, which directly rules out the existence of negative rational bubbles in stock prices, also imposes theoretical restrictions on the possible existence of positive rational bubbles. The analysis has showed that a positive rational bubble can start only on the first date of trading of a stock. By implication, a rational bubble that bursts cannot restart. Moreover, the existence of a rational bubble at any date would imply that the stock has been overvalued relative to market fundamentals since the first date of trading and that prior to the first date of trading the issuer of the stock and potential stockholders who anticipated the initial pricing of the stock must have expected that the stock would be overvalued relative to market fundamentals. A further question, which we do not attempt to answer here, is whether the issuer in this case would want to sell only a finite quantity of this stock.

In permitting the inception of a rational bubble only at the first date of trading, the rational expectations model of equity with free disposal is like a perfect foresight model and unlike the general linear rational-expectations model analysed, for example, by Shiller (1978). As in the case of perfect foresight, a single initial condition stating that the stock price conforms to market fundamentals at the first date of trading would guarantee that both the rational expectations of the stock price and the actual realisations of the stock price conform to market fundamentals at all dates.

The analysis in this paper focused on an asset (equity) that pays a real dividend. The case of a real asset that directly yields utility – for example, gold or tulips – is identical. The case of a pure fiat money, however, is different in that free disposal does not necessarily rule out negative (that is, inflationary) rational bubbles – see, for example, Flood and Garber (1980), and Obstfeld and

Rogoff (1983). We can, however, rule out rational deflationary bubbles by appealing to the arguments of Brock or Tirole against the possibility of positive rational bubbles or by assuming in the overlapping generations setting that the relevant interest rate exceeds the growth rate of the economy. In either case, an argument analogous to that of the present paper, with some technical modifications having to do with the nonlinear structure of the model involving a fiat money, would limit the possible inception of rational inflationary bubbles – see Diba and Grossman (1988).

Federal Reserve Bank of Philadelphia
Brown University

Date of receipt of final typescript: March 1988

References

Benveniste, L. M. and Scheinkman, J. A. (1982). 'Duality theory for dynamic optimization models of economics: the continuous time case.' *Journal of Economic Theory*, vol. 27, June, pp. 1–19.

Blanchard, O. J. (1979). 'Speculative bubbles, crashes, and rational expectations.' *Economics Letters*, vol. 3, pp. 387–9.

—— and Watson, M. W. (1982). 'Bubbles, rational expectations, and financial markets.' In *Crises in the Economic and Financial Structure* (ed. P. Wachtel). Lexington: Lexington Books.

Brock, W. A. (1982). 'Asset prices in a production economy.' In *The Economics of Information and Uncertainty* (ed. J. J. McCall). University of Chicago Press.

Diba, B. T. and Grossman, H. I. (1988). 'Rational inflationary bubbles.' *Journal of Monetary Economics*, vol. 21, January, pp. 35–46.

Flood, R. and Garber, P. (1980). 'Market fundamentals versus price level bubbles: the first tests.' *Journal of Political Economy*, vol. 88, August, pp. 745–70.

Gray, J. A. (1984). 'Dynamic instability in rational expectations models: an attempt to clarify.' *International Economic Review*, vol. 25, February, pp. 93–122.

Lucas, R. E. (1978). 'Asset prices in an exchange economy.' *Econometrica*, vol. 46, November, pp. 1429–45.

Mussa, M. (1984). 'Rational expectations models with a continuum of convergent solutions.' NBER Technical Working Paper No. 41.

Obstfeld, M. and Rogoff, K. (1983). 'Speculative hyperinflations in maximizing models: can we rule them out?' *Journal of Political Economy*, vol. 91, August, pp. 675–87.

O'Connell, S. A. and Zeldes, S. P. (1988). 'Rational Ponzi games.' *International Economic Review* (forthcoming).

Quah, D. (1986). 'Stationary rational bubbles in asset prices.' Unpublished.

Shiller, R. (1978). 'Rational expectations and the dynamic structure of macroeconomic models: a critical review.' *Journal of Monetary Economics*, vol. 4, January, pp. 1–44.

Tirole, J. (1982). 'On the possibility of speculation under rational expectations.' *Econometrica*, vol. 50, September, pp. 1163–81.

—— (1985). 'Asset bubbles and overlapping generations.' *Econometrica*, vol. 53, September, pp. 1071–100.