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ON THE POSSIBILITY OF SPECULATION UNDER RATIONAL EXPECTATIONS

BY JEAN TIROLE[1]

This paper considers the possibility of static and dynamic speculation when traders have rational expectations. Its central theme is that, unless traders have different priors or are able to obtain insurance in the market, speculation relies on inconsistent plans, and thus is ruled out by rational expectations. Its main contribution lies in the integration of the rational expectations equilibrium concept into a model of dynamic asset trading and in the study of the speculation created by potential capital gains. Price bubbles and their martingale properties are examined. It is argued that price bubbles rely on the myopia of traders and that they disappear if traders adopt a truly dynamic maximizing behavior.

1. INTRODUCTION

SPECULATION IS GENERALLY DEFINED as a process for transferring price risks. Given this admittedly vague definition, there is considerable disagreement about the conditions which allow a speculative market to arise. The *Working theory* (see (Hirshleifer [**15**, **17**], Feiger [7]) makes differences in beliefs the key to speculative behavior: in particular the degree of traders' risk aversion affects only the size of their gamble. Associated with this theory and (as we shall see below) potentially at the root of its internal inconsistency is the idea that better informed traders are able to make money on the average. On the other hand, the *Keynes–Hicks theory* of speculation emphasizes not differences in beliefs, but differences in willingness to take risk or in initial positions as the foundation of a speculative market. The social function of speculation is thus to shift price risks from more to less risk averse traders or from traders with riskier positions to those with less risky positions. In other words, speculation in the Keynes–Hicks tradition is a substitute for insurance markets.

In markets with sequential trading (e.g., a stock market), the prospect of capital gains introduces a new motive for speculation: Harrison and Kreps [**14**], following Kaldor and Keynes, say that "investors exhibit speculative behavior if the right to resell [an] asset makes them willing to pay more for it than they would pay if obliged to hold it forever."

In this paper, we investigate the possibility of speculative behavior when traders have rational expectations. The general idea is fairly simple: unless traders have different priors about the value of a given asset or are able to use the corresponding market for insurance purposes, this market does not give rise to gains from trade. Thus speculation relies on inconsistent plans and is ruled out by rational expectations.

[1]This work arose from discussions I had with Drew Fudenberg and Eric Maskin on speculation. I am also very grateful to them as well as to Margaret Bray, Peter Diamond, Frank Hahn, David Levine, David Kreps, and James Mirrlees for helpful comments. Eric Maskin provided very helpful comments on the current version.

We start by recalling Kreps' **[17]** result on the impossibility of pure speculation in the static model (see Milgrom–Stokey **[20]** for an alternative approach). This theorem provides insight on the rest of the paper, and moreover has important consequences for the theory of speculation. In particular, it definitely contradicts the Working theory for markets with traders having the same prior (but differential private information) and deriving information from the price. Indeed one might ask what is needed in order to observe speculative behavior. In Section 3, we state the four conditions giving rise to static speculation.

The main contribution of this paper lies in the integration of the rational expectations equilibrium (REE) concept into a model of dynamic speculation. We distinguish between myopic and fully dynamic concepts of rational expectations. We first characterize myopic REE and demonstrate the martingale properties of "price bubbles." We then argue that the refined concept of fully dynamic REE is more reasonable if one assumes rationality of the traders. We conclude by proving that in a fully dynamic REE, price bubbles do not exist.

Before turning to a formal treatment of static (Section 3) and dynamic (Section 4) speculation, we give an informal analysis of the ideas and results of the paper (Section 2).

2. RATIONAL EXPECTATIONS AND SPECULATION

The idea behind a rational expectations equilibrium (REE) is that each trader is able to make inferences from the market price about the profitability of his trade. Traders know the statistical relationship between the market price and the realized value of their trade (the "forecast function") and use the information conveyed by the price as well as their private information to choose their demands.

In Section 3 we consider the consequences of rational expectations for static speculation. We observe that, *contrary to the Working–Hirshleifer–Feiger view, rational and risk averse traders never trade solely on the basis of differences in information*. Risk neutral traders may trade, but do not expect any gain from their trade. Consider a purely speculative market (i.e., a market where the aggregate monetary gain is zero and insurance plays no role). Assume that it is common knowledge that traders are risk averse, rational, have the same prior and that the market clears. Then it is also common knowledge that a trader's expected monetary gain given his information must be positive in order for him to be willing to trade. The market clearing condition then requires that no trader expect a monetary gain from his trade. This process can be illustrated by the following elementary example: At the beginning of a seminar the speaker states a proposition. Suppose that the validity of the proposition is in question, and that each member of the audience but the speaker either has no information about its validity or else has some counter-example in mind (which is correct with certainty or with a high probability). In the first case, the member will not be willing to bet with the speaker, who, after all, having worked on the topic before the seminar, is endowed with superior information. In the second case, he will be willing to bet that the proposition is incorrect. The speaker can therefore deduce that only members of the audience having a counter-example in mind will be willing to bet with him.[2] Consequently, the speaker will not be willing to bet at all.

Section 4 considers a model of a sequential stock market similar to that of Harrison and Kreps **[14]** in order to focus on the Kaldor–Keynes definition of speculation. To this end we describe a stock market as a sequence of rational expectations equilibria. The dividends of a given firm $(d_0, d_1, \ldots, d_t, \ldots)$ are assumed to follow an exogenously given stochastic process. Each trader, who is assumed to be risk neutral, will have in each period some information (signal) about the process. This information differs among traders. It is often assumed that in markets with homogeneous information traders base their behavior on the comparison between the current price and (the probability distribution of) next period's price; the corresponding REE for a stock market with heterogeneous information will be named "*myopic REE*." We show that, for any given period, even if short sales are prohibited, a trader will not expect a gain from his trade, regardless of what information he may possess (of course, the price expectation is taken relative to the trader's own information and the information he can infer from the market). This does not mean that the price of the stock has to be equal to any *market fundamental* (i.e., the expected present discounted value of dividends). The right to resell the asset in general makes traders willing to pay more for it than they would pay if obliged to hold it forever, i.e., more than their market fundamental. Indeed, in an equilibrium of a stock market with an infinite horizon, the market fundamentals of different traders are not generally equal. Each active trader's price bubble is defined to be the difference between the market price and his market fundamental. Price bubbles are shown to follow discounted martingales. This differs from a finite-horizon stock market, in which the price is equal to the market fundamental of any active trader (of any trader, if short sales are allowed).

One may nevertheless dislike the concept of myopic REE, especially in an economy with a finite number of traders. Indeed, a sequence of myopic REE does not necessarily lead to a well defined (i.e., converging) expected gain function for each trader. In Section 4a, we exhibit an elementary example of myopic REE where any optimal strategy (i.e., maximizing a trader's expected payoff over the whole time horizon) requires the trader to realize his profits in finite time (i.e., quit the market). This is inconsistent, as the set of traders is finite. We are thus led to define a *fully dynamic REE* as a sequence of self-fulfilling forecast functions such that there exists for each agent a sequence of (information contingent) stock holdings, called a "strategy", satisfying the following properties: (i) in each period t, and for any information a trader i may have at

[2] This point may remind the readers familiar with the literature on auctions of the winner's curse.

time t, the corresponding strategy maximizes i's expected present discounted gain from t on (i's posterior being computed from the common prior, and i's information, whether acquired individually or inferred from the market price); (ii) the market clears in each period and for any information traders have in this period.

As one might expect, the definition of a fully dynamic REE puts very strong restrictions on the type of price and expectation functions that can arise in equilibrium. In fact, Section 4b shows that *in a fully dynamic REE, price bubbles disappear and every trader's market fundamental equals the price of the stock, regardless of whether short sales are allowed or not*. This implies that speculative behavior in the Kaldor–Keynes–Harrison–Kreps sense cannot be observed in a fully dynamic REE.

3. STATIC SPECULATION

Let us formalize the notion of a purely speculative market. Consider a market with I risk averse or risk neutral traders: $i = 1, \ldots, I$. The traders exchange at price p claims for an asset with random value $\tilde{p}$. Trader i's *ex-post* ("realized") gain is: $G^i = (\tilde{p} - p)x^i$, where x^i is his transaction on the market. Trader i's utility is a concave function of G^i, and he is assumed to maximize his expected utility given his information. The market clears when $\sum_i x^i = 0$.

Let E be the set of payoff-relevant environments, i.e., the set of potential realizations of $\tilde{p}$. Each trader receives a private signal s^i belonging to a set S^i. The vector of all signals is: $s = (\ldots, s^i, \ldots)$ belonging to a set S (contained in $\times_i S^i$). Then $\Omega \equiv E \times S$ is the set of states of nature, and we assume that all the traders have the *same prior* ν on Ω. Let T be a set contained in S; we denote by $\nu^i(s^i \mid T)$ the marginal probability of signal s^i conditional on $\{s \in T\}$. $\nu^i(s^i)$ denotes the prior probability of signal s^i. We assume that all signals have a positive probability:

$$\forall i, \forall s^i \in S^i: \qquad \nu^i(s^i) > 0.$$

It will be clear that the result holds for much more general probability spaces.

DEFINITION 1: A REE is a forecast function[3] Φ which associates with each set of signals s a price $p = \Phi(s)$, and a set of trades $x^i(p, s^i, S(p))$ for each agent i, relative to information s^i and $s \in S(p) \equiv \Phi^{-1}(p)$, such that:

1. $x^i(p, s^i, S(p))$ maximizes i's expected utility conditional on i's private information s^i, and the information conveyed by the price $S(p)$.
2. The market clears: $\sum_i x^i(p, s^i, S(p)) = 0$.

[3]One might more generally define forecast correspondences in the case of multiple equilibria. The results of this paper would not be affected.

We already know that the total monetary gain in such a market is zero: $\sum_i G^i = 0$. We shall say that the market is purely speculative if moreover the participants' initial positions (corresponding to no trade on the market) are uncorrelated with the return on the asset.[4]

Since trader i has a concave utility function, has no insurance motive in the market, and has the option not to trade, he must expect a nonnegative gain:

$$(1)\qquad E\big(G^i \mid s^i, S(p)\big) \geq 0.$$

This has to be true for any single s^i belonging to the projection $S^i(p)$ of $S(p)$ on S^i. This implies:

$$(2)\qquad E\big(G^i \mid S(p)\big) = \sum_{s^i \in S^i(p)} E\big(G^i \mid s^i, S(p)\big)\nu^i\big(s^i \mid S(p)\big)$$

$$= E\Big(E\big(G^i \mid s^i, S(p)\big) \mid S(p)\Big) \geq 0.$$

From the market clearing condition, $\sum_i G^i = 0$,

$$(3)\qquad \sum_i E\big(G^i \mid S(p)\big) = 0.$$

This implies in turn that

$$\forall i: \qquad E\big(G^i \mid S(p)\big) = 0,$$

and consequently

$$(4)\qquad \forall i: \qquad E\big(G^i \mid s^i, S(p)\big) = 0.$$

In other words, in a REE no trader can expect a gain. We can now state the following proposition:

PROPOSITION 1: *In a REE of a purely speculative market with risk-averse or risk-neutral traders, risk-averse traders do not trade; risk-neutral traders may trade, but they do not expect any gain from their trade.*

Proposition 1 shows that one must relax at least one of the previous assumptions if static speculation is to occur:

(a) One may introduce risk-loving traders.

[4] "Uncorrelated" is relative to the information of the trader. This definition is more stringent than the condition that the initial position be uncorrelated with the return of the asset and the signal received by the trader. Kreps [17] observes that the information conveyed by the price may introduce some correlation between the initial position and the return of the asset and thus create an insurance motive for speculation. Hence, we assume that the initial positions of all traders are uncorrelated with the return on the asset and the set of signals.

(b) One may depart from the strict Bayesian assumption that priors are identical for everybody and that differences in beliefs are simply the result of differences in information.

(c) One other way of transforming the market into a "positive-sum game" from the point of view of the set of rational agents is to introduce a non-rational agent. A related method consists in introducing traders whose (possibly stochastic) demand or supply is independent of the market price (see Grossman **[11]** and Grossman–Stiglitz **[13]**), although one must be cautious and give a more complete description of the model before calling these traders irrational. The set of all rational players is then able to take advantage of this type of player, who, roughly speaking, faces an unfair bet.

(d) The absence of correlation between the initial position of the traders and the market outcome (and the corresponding impossibility of anyone using the market to hedge) is a central condition for the nonexistence of a "pure betting market." If this condition fails to hold, the market can be seen as a means of supplying insurance to traders with risky positions. This view vindicates the Keynes–Hicks position and is the essence of Danthine's **[5]** and Bray's **[3, 4]** models of a futures market.[5]

Let us now examine where the previous argument breaks down when one of the assumptions is relaxed. First, if a trader either is risk-loving or has an initially risky position on the market (cases (a) and (d)), he may in equilibrium expect a negative gain. Thus (1) does not hold. Relation (1) also fails to hold when one introduces irrational agents or fixed supplies or demands into the market (case (c)); to illustrate this simply, assume that there is a fixed supply $\bar{x}$ of a risky asset, so that the market clearing condition is: $\sum_i x^i = \bar{x}$. Assume further that all traders have the same information and the same constant absolute risk-aversion utility function, and that the distribution of the future price of the asset is normal. It is well known[6] that the demand of the rational traders is proportional to $(E(\tilde{p}) - p)$ where $E(\tilde{p})$ denotes the expectation of the price relative to the common information. Thus in equilibrium: $k(E(\tilde{p}) - p) = \bar{x}$ $(k > 0)$. The aggregate expected gain of the rational traders is then: $\bar{x}^2/k > 0$, whereas the traders with the fixed supplies expect an aggregate loss $(-\bar{x}^2/k)$ relative to their not selling the asset. Finally, if traders have different priors[7] (case (b)), the sum of the expected gains may well be strictly positive: Since the posteriors have to be computed from different priors, (3) does not hold.

[5]The distinction between (c) and (d) is not as clear-cut as it might seem, if one considers the examples of REE which can be found in the literature. Consider, for example, Grossman's **[10]** one-period stock market; there is a fixed supply $\bar{x}$ of the stock. If, following Grossman, one assumes that traders have constant absolute risk aversion utility functions, the demands are independent of wealth and thus one does not have to specify who owns the initial stock in order to compute the equilibrium price. However, the stock market equilibrium may be interpreted in terms of (c) if the holders of the initial stock $\bar{x}$ sell the whole stock to the set of rational buyers whatever the price or in terms of (d) if the rational traders also own the initial stock and thus try to hedge (or speculate) on the market.

[6]See, for example, Grossman **[10, 11]**.

[7]See, for example, Harrison–Kreps **[14]**, Hirshleifer **[15, 16]** and Miller **[21]**.

4. DYNAMIC SPECULATION: A MODEL OF A STOCK MARKET WITH HETEROGENEOUS INFORMATION

This section is particularly concerned with the Kaldor–Keynes–Harrison–Kreps definition of speculation, according to which investors exhibit speculative behavior if the right to resell an asset makes them willing to pay more for it than they would pay if obliged to hold it forever. To this end, we describe the market for a given stock as a sequence of REE.

The stock may be traded at dates $t = 0, 1, 2, \ldots$. The (nonnegative) dividend d_t is declared immediately prior to trading at time t, and paid to traders who hold the stock at $(t-1)$. As in Harrison and Kreps [**14**], we assume that the sequence of dividends $\{d_0, d_1, \ldots, d_t, \ldots\}$ is an exogenously given stochastic process (for example driven by the demand in the market of the firm's output, . . .). At time t, the stock is traded at price p_t.

There is a finite set of traders $i = 1, \ldots, I$. Trader i is assumed to be risk-neutral and to discount the future with the discount factor γ.[8] The traders' risk neutrality implies that speculation would not exist if the market were not to reopen after the first period, and thus allow us to focus on the dynamic features of asset markets. His holding of the stock at time t is x_t^i and, given an aggregate stock $\bar{x}$, the market clearing condition is $\sum_i x_t^i = \bar{x}$ $(= \sum_i x_{t-1}^i)$. If short sales are prohibited, we impose that $x_t^i \geq 0$. In this case we shall say that trader i is *active* at time t if either $x_t^i \neq x_{t-1}^i$ or $0 < x_t^i = x_{t-1}^i < \bar{x}$. If short sales are allowed, the convention will be that every trader is active at every period. The motivation for this definition will become clear later. The market is active at time t if some traders are active.

Information: Let E be the set of pay off-relevant environments. E is here taken to be the set of potential processes governing the sequence of dividends. At each period t, trader i has some private information about the underlying stochastic process; this may include past dividends, past prices, market studies, tips, etc. We represent trader i's information at time t as an element (event) s_t^i of a partition F_t^i of a set S^i. It is natural to assume that the partition F_t^i becomes finer and finer over time: $F_t^i \subseteq F_{t+1}^i$. The vector of all signals at time t is $s_t = (\ldots, s_t^i, \ldots)$. s_t is a subset of a set S contained in $X_i S^i$. Let $\Omega \equiv E \times S$ denote the set of states of nature. We shall assume that all traders have the same prior ν on Ω. Let ν^i denote the marginal probability distribution on S^i; for simplicity we assume that all signals have positive probability: $\forall i, \forall t, \forall s_t^i \in F_t^i : \nu^i(s_t^i) > 0$ (this assumption can easily be relaxed).

At each time t, trader i can derive some information in addition to his private signal s_t^i simply by observing the current price p_t. Anticipating ourselves a bit, a REE at time t is characterized by a forecast function Φ_t, which associates with any set of signals s_t a price $p_t = \Phi_t(s_t)$. Conversely, the observation of price p_t indicates that s_t belongs to $S_t(p_t) \equiv \Phi_t^{-1}(p_t)$ ($S_t(p_t)$ is an element of $X_i F_t^i$). For

[8] Traders do not have a budget constraint at each period. They can borrow and lend at the rate $(1/\gamma - 1)$.

notational simplicity, we shall often use the shorthand S_t for $S_t(p_t)$. To summarize, at time t trader i has information (s_t^i, S_t) based on his private signal (s_t^i) and the information conveyed by the price (S_t).

Consider a trader i having at time t information (s_t^i, S_t). This information can be regarded as a probability distribution on S (which takes zero values except on $s_t^i \times S_t^{-i}$, where S_t^{-i} denotes the projection of S_t on $X_{j\neq i}S^j$) and thus on Ω. This in turn induces a conditional probability distribution on $X_j F_{t+\tau}^j$, $\forall \tau \geq 1$. Trader i assigns a probability to any set of signals $s_{t+\tau} = (\ldots, s_{t+\tau}^j, \ldots)$ in $X_j F_{t+\tau}^j$. With a set of signals $s_{t+\tau}$ received by the traders at time $(t+\tau)$, there will be associated a price $p_{t+\tau} = \Phi_{t+\tau}(s_{t+\tau})$, so that trader i will have information $(s_{t+\tau}^i, S_{t+\tau} = \phi_{t+\tau}^{-1}(p_{t+\tau}))$. To summarize, with each information (s_t^i, S_t) at time t, trader i associates a probability of having at time $(t+\tau)$ information $(s_{t+\tau}^i, S_{t+\tau})$ and facing price $p_{t+\tau}$.

4a. *Myopic REE*

It is often assumed in the literature on sequential trading that traders choose their trades on the basis of short run considerations;[9] more precisely, in each period they compare their current trading opportunities with the expected trading opportunities in the following period. The application of this concept to a stock market with heterogeneous information leads to the following definition:

DEFINITION 2: A myopic REE is a sequence of self-fulfilling forecast functions $s_t = (\ldots, s_t^i, \ldots) \to p_t = \Phi_t(s_t)$, such that there exists a sequence of associated stock holdings $\{x_t^i(s_t^i, p_t)\}$[10] for each trader, satisfying:

(1) Market clearing: $\forall t, \forall s_t : \sum_i x_t^i(s_t^i, p_t) = \bar{x}$.

(2) Short-run optimizing behavior:

(i) If short sales are allowed,

$$\forall t, \forall s_t, \forall i: \qquad p_t = E\left[\gamma d_{t+1} + \gamma p_{t+1} \mid s_t^i, S_t\right].$$

(ii) If short sales are prohibited,

$$\begin{aligned} &\text{if } p_t = E\left[\gamma d_{t+1} + \gamma p_{t+1} \mid s_t^i, S_t\right], &&\text{then } x_t^i(s_t^i, p_t) \in [0, \bar{x}],\\ &\text{if } p_t > E\left[\gamma d_{t+1} + \gamma p_{t+1} \mid s_t^i, S_t\right], &&\text{then } x_t^i(s_t^i, p_t) = 0,\\ &\text{if } p_t < E\left[\gamma d_{t+1} + \gamma p_{t+1} \mid s_t^i, S_t\right], &&\text{then } x_t^i(s_t^i, p_t) = \bar{x}. \end{aligned}$$

The interpretation of (2) is that each trader maximizes his expected short-run gain.

We now prove that even if short sales are prohibited, the price p_t must be equal to the expectation of the sum of the discounted dividend and the discounted next

[9] See, for example, Sargent–Wallace [25], Flood–Garber [9], Blanchard [2], as well as some of the literature on growth with heterogeneous capital goods.

[10] We shall often use the shorthand $x_t^i(s_t^i, p_t)$ for $x_t^i(s_t^i, S_t(p_t), p_t)$.

period price for any trader active at time t, i.e., no trader expects a short-run gain from his trade. This result is the analog of Proposition 1 in this model.[11]

PROPOSITION 2: *Even if short sales are prohibited, for any trader i active at time t:*

$$p_t = E\left(\gamma d_{t+1} + \gamma p_{t+1} \mid s_t^i, S_t\right).$$

PROOF: Let $g_t^i \equiv -p_t \Delta x_t^i$ and ${}_t g_{t+1}^i \equiv [p_{t+1} + d_{t+1}]\Delta x_t^i$ denote the changes in i's cash flows at t and $(t+1)$ resulting from his trade $\Delta x_t^i \equiv x_t^i - x_{t-1}^i$ at time t.

From the market clearing condition at time t: $\forall s_{t+1}$,

$$\sum_i g_t^i = 0 \qquad \text{and} \qquad \sum_i {}_t g_{t+1}^i = 0.$$

This implies

$$\sum_i \left[g_t^i + \gamma\, {}_t g_{t+1}^i \right] = 0.$$

Taking the expectation relative to the set of signals consistent with p_t,

$$\sum_i E\left[g_t^i + \gamma\, {}_t g_{t+1}^i \mid S_t \right] = 0.$$

From the maximizing behavior of agent i,

$$\forall s_t \in S_t: \qquad E\left[g_t^i + \gamma\, {}_t g_{t+1}^i \mid s_t^i, S_t \right] \geq 0,$$

and thus, by integrating over s_t^i,

$$E\left(g_t^i + \gamma\, {}_t g_{t+1}^i \mid S_t \right) = E\Big(E\left(g_t^i + \gamma\, {}_t g_{t+1}^i \mid s_t^i, S_t \right) \mid S_t \Big) \geq 0.$$

Thus if for some information s_t and some trader i_0, we had

$$E\left(g_t^{i_0} + \gamma\, {}_t g_{t+1}^{i_0} \mid s_t^{i_0}, S_t \right) > 0,$$

we would conclude that

$$\sum_i E\left(g_t^i + \gamma\, {}_t g_{t+1}^i \mid S_t \right) > 0,$$

a contradiction. Proposition 2 then follows for all active traders. *Q.E.D.*

[11] Proposition 3 can be seen as a generalization to heterogeneous information of Samuelson's [24] Theorem 3 (note also that Samuelson hypothesizes that a stock's present price is set at the expected discounted value of its future dividends, which we do not impose here). It also shows that, even with differential information, no trader can gain from his trade, which restricts the validity of Samuelson's conjecture (p. 373) that "there is no incompatibility in principle between behavior of stock prices that behave like random walk at the same time that there exists subsets of investors who can do systematically better than the average investors" to subsets of traders with "measure zero."

It is often thought that the price of an asset in a speculative market may reflect speculative attributes as well as the asset's basic value, that is the price of an asset is the sum of its fundamental value and its speculative value ("price bubble"). Sargent and Wallace [**25**] and Flood and Garber [**9**] show in a monetary model with homogeneous information that price bubbles are not inconsistent with rational expectations; they are not even inconsistent with a positive probability in any period that the bubble "bursts" and the market "crashes down" to the market fundamental (see Blanchard [**2**]; in this case the price has to grow faster during the duration of the bubble than in the previous case in order for the asset holders to be compensated for the probability of a crash). We study those price bubbles in our stock market with heterogeneous information.

Given information (s_t^i, S_t), one can define a *market fundamental* as the expectation of the present discounted value of future dividends:

$$F(s_t^i, S_t) \equiv E\left(\sum_{\tau=1}^{\infty} \gamma^{\tau} d_{t+\tau} \mid s_t^i, S_t \right).$$

For the price p_t consistent with S_t, the *price bubble* as seen by an individual with information (s_t^i, S_t) is defined by

$$B(s_t^i, p_t) \equiv p_t - F(s_t^i, S_t).$$

Note that the price bubble depends on the information, and thus generally differs among individuals. Note also that the Kaldor–Keynes definition of speculative behavior of trader i amounts to

$$B(s_t^i, p_t) > 0.$$

PROPOSITION 3: *In a stock market with finite horizon $\overline{T}$, whether short sales are allowed or not, the price bubbles are all equal to zero for the traders active in the market. Thus a market fundamental can be uniquely defined as the common market fundamental of all active traders, and is equal to the price*:

$$\forall t, \forall i \textit{ active at } t: \qquad p_t = E\left(\sum_{u=t+1}^{\overline{T}} \gamma^{u-t} d_u \mid s_t^i, S_t \right).$$

PROOF: The price of the stock at $\overline{T}$ is 0. Consider a trader i who is active at $(\overline{T}-1)$. Proposition 2 implies that: $p_{\overline{T}-1} = E(\gamma d_{\overline{T}} \mid s_{\overline{T}-1}^i, S_{\overline{T}-1})$. This means that at $(\overline{T}-1)$ an active trader is indifferent between selling and holding the stock until the end period $\overline{T}$. Consider now an active trader i at time $(\overline{T}-2)$. According to his information at $(\overline{T}-1)$, he will hold the stock until $\overline{T}$ or trade at $(\overline{T}-1)$. But we saw that if he is active, he is indifferent between trading and holding. Thus: $\forall i, \forall s_{\bar{T}-2}$ such that i is active at $(\bar{T}-2)$:

$$p_{\bar{T}-2} = E\left(\gamma d_{\bar{T}-1} + \gamma^2 d_{\bar{T}} \mid s^i_{\bar{T}-2}, S_{\bar{T}-2}\right).$$

Proposition 3 is then proven by induction. *Q.E.D.*

Thus in a finite horizon stock market, backward induction from the final "crash" leads to the absence of price bubbles. The picture changes dramatically in the finite horizon case.

PROPOSITION 4: (a) *If short sales are allowed, then price bubbles are (discounted) martingales*: $\forall i, \forall (s^i_t, S_t), \forall T \geq 1$,

$$B(s^i_t, p_t) = \gamma^T E\left(B(s^i_{t+T}, p_{t+T}) \mid s^i_t, S_t\right).$$

(b) *If short sales are prohibited, the price bubble of trader i endowed with information (s^i_t, S_t) satisfies the preceding martingale property between t and $(t+T)$ if, conditionally on his information at t, trader i is active in each period t, $t+1, \ldots, t+T-1$.*

PROOF: The proof is a simple application of the law of iterated projections.

(a) By definition of an equilibrium: $\forall t, \forall s_t, \forall i$,

$$\begin{aligned} p_t &= E\left(\gamma d_{t+1} + \gamma p_{t+1} \mid s^i_t, S_t\right) \\ &= E\left[\gamma d_{t+1} + \gamma\left[E\left[\gamma d_{t+2} + \gamma p_{t+2} \mid s^i_{t+1}, S_{t+1}\right]\right] \mid s^i_t, S_t\right] \\ &= E\left[\gamma d_{t+1} + \gamma^2 d_{t+2} + \gamma^2 p_{t+2} \mid s^i_t, S_t\right]. \end{aligned}$$

By induction,

$$\begin{aligned} p_t &= E\left(\sum_{\tau=1}^{T} \gamma^\tau d_{t+\tau} + \gamma^T p_{t+T} \mid s^i_t, S_t\right) \\ &= E\left(\sum_{\tau=1}^{T} \gamma^\tau d_{t+\tau} \mid s^i_t, S_t\right) + \gamma^T E\left(E\left(\sum_{\tau=1}^{\infty} \gamma^\tau d_{t+T+\tau} \mid s^i_{t+T}, S_{t+T}\right) \mid s^i_t, S_t\right) \\ &\quad + \gamma^T E\left(B(s^i_{t+T}, p_{t+T}) \mid s^i_t, S_t\right). \end{aligned}$$

Using the law of iterated projections,

$$p_t = E\left(\sum_{\tau=1}^{\infty} \gamma^\tau d_{t+\tau} \mid s^i_t, S_t\right) + \gamma^T E\left(B(s^i_{t+T}, p_{t+T}) \mid s^i_t, S_t\right)$$

or

$$B(s^i_t, p_t) = \gamma^T E\left(B(s^i_{t+T}, p_{t+T}) \mid s^i_t, S_t\right).$$

(b) It is clear that the proof still holds without short sales, if trader i is active in every intermediate period for any state of information which can occur, given that i's information at t is (s_t^i, S_t). *Q.E.D.*

SPECIAL CASE: *Homogeneous Information* (myopic REE version of Radner's [22] equilibrium of plans, prices and price expectations): Assume that all traders have at each period the same information, i.e., receive the same signal $s_t \in F_t$. The price p_t conveys no extra information, and traders base their expectations on s_t. The following proposition is trivial:

PROPOSITION 5: *In a stock market with homogeneous information, whether short sales are allowed or not, the price bubble is the same for every trader, and has the martingale property.*

Note that if a heterogeneous-information REE is fully revealing, i.e., $S_t(p_t)$ is a "sufficient statistic" for the set of signals in each period, we are in a situation analogous to the special case.

As explained in the introduction, a myopic REE exhibits some rather unattractive features. This can be illustrated by a simple stock market with no uncertainty. Assume there is one unit of a stock, whose price at time t is p_t. A constant dividend $d_t = 1$ $(t \geq 1)$ is distributed just before trading. If traders have a discount factor $\frac{1}{2}$, the market fundamental is: $(\frac{1}{2} + \frac{1}{4} + \cdots) \times 1 = 1$.

A myopic REE is simply a price function p_t such that

$$p_t = \tfrac{1}{2}(1 + p_{t+1}).$$

The general solution is: $p_t = 1 + \alpha 2^t$, where $\alpha 2^t$ represents a price bubble. Assume there are two individuals (or two types) A and B, and consider the following sequence of trades (trader A is the initial owner of the stock):

At time 0,	trader A	sells the stock to trader B	at price 2,	
... 1	... B	... A	... 3,	
... 2	... A	... B	... 5,	
... 3	... B	... A	... 9,	*etc.*

This is a myopic REE. The first thing to observe is that, if we try to compute the discounted gains of the traders, they do not converge:

$$G^A = 2 - \frac{1}{2}(3) + \frac{1}{4}(5+1) - \frac{1}{8}(9) + \frac{1}{16}(17+1)\ldots$$
$$- \frac{1}{2^{2k-1}}(1 + 2^{2k-1}) + \frac{1}{2^{2k}}(1 + 2^{2k} + 1)\ldots,$$

$$G^B = -2 + \frac{1}{2}(3+1) - \frac{1}{4}(5) + \frac{1}{8}(9+1) - \frac{1}{16}(17) + \cdots$$
$$+ \frac{1}{2^{2k-1}}(1 + 2^{2k-1} + 1) - \frac{1}{2^{2k}}(1 + 2^{2k})\ldots.$$

Thus, it is not possible to define present discounted gains associated with the myopic REE strategies. Nevertheless, we may observe that A (resp. B) can always guarantee himself 2 (resp. 0) by leaving the market just after selling. In fact, if a trader wants to maximize his present discounted gain, he has to "realize his profits" by refusing to repurchase the stock at some date; this strategy can also be viewed as a dominant strategy in that the trader avoids running the risk of getting stuck with a devalued stock if the other trader switches to a "finite time strategy." Thus it would be natural to assume that A's payoff is 2 and B's payoff is 0. But those payoffs are inconsistent since they must add up to the market fundamental which is 1.

To summarize, in a myopic REE, each trader must (i) believe that he will be able to sell the asset, (ii) realize his profits in finite time. These two conditions are inconsistent with the assumption that the number of traders is finite.[12]

4b. *Fully Dynamic REE*

Requiring that the strategy of each trader maximizes his expected present discounted gain leads to the definition of a fully dynamic REE:

DEFINITION 3: A fully dynamic REE is a sequence of self-fulfilling forecast functions: $s_t = (\ldots, s_t^i, \ldots) \to p_t = \Phi_t(s_t) \Leftrightarrow s_t \in S_t(p_t) \equiv \Phi_t^{-1}(p_t)$, such that there exists a sequence of (information contingent) stock holdings (strategies) $x_t^i(s_t^i, p_t)$ satisfying: (i) market clearing: $\forall t, \forall s_t : \sum_i x_t^i(s_t^i, p_t) = \bar{x}$; (ii) maximizing behavior: at each time t, and for any information (s_t^i, S_t) trader i may possess, i's strategy (restricted to the information sets reachable from (s_t^i, S_t) at t) maximizes i's expected present discounted gain from t on—i's posterior being computed from the common prior and i's information (s_t^i, S_t).

As the following proposition shows, long-run maximizing behavior considerably restricts the eligible set of price and forecast functions:

PROPOSITION 6: *Whether short sales are allowed or not, price bubbles do not exist in a fully dynamic REE*:

$$\forall t, \forall s_t, \forall i: \qquad F(s_t^i, S_t) = p_t, \qquad i.e., \qquad B(s_t^i, p_t) = 0.$$

PROOF: We prove Proposition 6 in the case where short sales are prohibited. Let $\{x_t^i(s_t^i, p_t)\}$ be a set of optimal strategies. Let $G_t^i \equiv \sum_{\tau=1}^{\infty} \gamma^\tau d_{t+\tau} x_{t+\tau-1}^i + \sum_{\tau=1}^{\infty} \gamma^\tau p_{t+\tau}(x_{t+\tau-1}^i - x_{t+\tau}^i)$ be the discounted sum of realized dividends and capital receipts associated with i's optimal strategy.

[12]This is not true with an infinite number of traders. For example, in an overlapping generation model, a price bubble is consistent with each generation leaving the market after realizing its profit.

Clearly the G_t^i's add up to the market fundamental times the quantity of the stock:[13]

$$\sum_i G_t^i = \left(\sum_{\tau=1}^{\infty} \gamma^\tau d_{t+\tau} \right) \bar{x} = f_t \bar{x} \tag{5}$$

where f_t denotes the "realized market fundamental", i.e., the discounted sum of the realized dividends per unit of stock from t on. The proof uses the following lemmas:

LEMMA 1: *The market fundamental relative to the market information exceeds the price*: $\forall s_t : F(S_t) \geq p_t$.

PROOF OF LEMMA 1: Since trader i optimizes, he can not gain by selling x_t^i and leaving the market at time t:

$$E\left(G_t^i \mid s_t^i, S_t\right) \geq x_t^i p_t.$$

This inequality will act as a transversality condition for trader i's stochastic dynamic programming problem. Thus:

$$\begin{aligned} E\left(G_t^i \mid S_t\right) &= \sum_{s_t^i \in S_t^i} E\left(G_t^i \mid s_t^i, S_t\right) \nu^i\left(s_t^i \mid S_t\right) \\ &\geq p_t \left(\sum_{s_t^i \in S_t^i} x_t^i\left(s_t^i, p_t\right) \nu^i\left(s_t^i \mid S_t\right) \right) \end{aligned}$$

where S_t^i denotes the projection of S_t on F_t^i. The last expression in brackets is nothing but the statistical average of i's stockholding at price p_t. This implies

$$\sum_i E\left(G_t^i \mid S_t\right) \geq p_t \bar{x}$$

or

$$E\left(f_t \bar{x} \mid S_t\right) \geq p_t \bar{x} \Rightarrow F(S_t) \geq p_t \qquad \textit{Q.E.D.}$$

[13]With an infinite number of traders, the adding-up in (5) may make no sense. Consider the perfect information stock market described at the end of Section 4a. Assume now that there exists a countable number of infinitely-lived traders $\{A_0, A_1, \ldots, A_t, \ldots\}$. Consider the following sequence of trades (A_0 holds the stock initially):

At time 0,	trader A_0	sells the stock to trader A_1	at price 2,
... 1,	... A_1	... A_2	... 3,
... t,	... A_t	... A_{t+1}	... $(1+2^t)$.

Then the present discounted pay off for all traders but A_0 is 0; for A_0, it is 2. But the market fundamental is 1. This may remind the reader of the familiar paradoxes on infinity. Note that this example does not depend on non-maximizing behaviors of the traders.

LEMMA 2: *No trader expects a gain from his trade at time t:*

$$\forall t, \forall s_t, \forall i: \qquad E\left(G_t^i \mid s_t^i, S_t\right) = E\left(G_t^i\left(x_{t-1}^i\right) \mid s_t^i, S_t\right)$$

where $G_t^i(x_{t-1}^i)$ is defined as G_t^i except that at t, i holds x_{t-1}^i instead of $x_t^i(s_t^i, p_t)$ (the holding strategies being unchanged after t).

PROOF OF LEMMA 2: From (5), the trading game at t, given the holding strategies beyond t is a zero-sum game:

$$\forall t, \forall s_t: \qquad \sum_i \left(G_t^i - G_t^i\left(x_{t-1}^i\right)\right) = 0.$$

Hence

$$\sum_i E\left(G_t^i - G_t^i\left(x_{t-1}^i\right) \mid S_t\right) = 0.$$

The optimizing trader i cannot improve upon $\{x_t^i\}$ by holding x_{t-1}^i at t, and following the same strategy beyond t:

$$\forall t, \forall s_t, \forall i: \qquad E\left(G_t^i - G_t^i\left(x_{t-1}^i\right) \mid s_t^i, S_t\right) \geq 0.$$

Now one can apply the same argument as in the proofs of Propositions 1 and 2 to the functions $\{G_t^i - G_t^i(x_{t-1}^i)\}$. *Q.E.D.*

Using Lemma 1:

$$\forall t, \forall s_t, \forall i: \qquad \sum_{s_t^i \in S_t^i} F\left(s_t^i, S_t\right) \nu^i\left(s_t^i \mid S_t\right) \geq p_t.$$

Imagine now that the market fundamental of some agent i_0 who does not hold the whole stock at the start of the period $(x_{t-1}^{i_0} < \bar{x})$ were to strictly exceed the price: $F(s_t^{i_0}, S_t) > p_t$. Then i_0 could buy and make a strictly positive expected profit, contradicting Lemma 2. Thus, for all i such that $x_{t-1}^i \neq \bar{x}, \forall s_t: F(s_t^i, S_t) = p_t$. Integrating the previous equality gives $F(S_t) = p_t$. Now if i holds the whole stock at the beginning of the period, his market fundamental cannot be lower than p_t without contradicting Lemma 2. Thus $\forall s_t^i: F(s_t^i, S_t) \geq p_t$. But then $F(s_t^i, S_t) = p_t$. *Q.E.D.*

REMARK 1: Proposition 6—the law of equalization of market fundamentals—does not imply that the price p_t fully reveals the complete signal s_t. Consider the following example. There are two traders A and B and two Bernouilli processes independent and uncorrelated over time:

$$s_t^A = \begin{cases} 0 & \text{with probability } \tfrac{1}{2}, \\ 1 & \text{with probability } \tfrac{1}{2}; \end{cases} \qquad s_t^B = \begin{cases} 0 & \text{with probability } \tfrac{1}{2}, \\ 1 & \text{with probability } \tfrac{1}{2}. \end{cases}$$

Assume that the dividend depends on the signals in the following way:

$$d_{t+1} = \left(s_t^A + s_t^B\right) \bmod 2$$

(i.e., $d_{t+1}(0,0) = d_{t+1}(1,1) = 0$, $d_{t+1}(0,1) = d_{t+1}(1,0) = 1$). With a discount factor $\frac{1}{2}$, the market fundamental corresponding to the absence of information is $\frac{1}{2}$. It is easy to see that the following noninformative price function is a fully dynamic REE:

$s_t^A \backslash s_t^B$	0	1
0	$\frac{1}{2}$	$\frac{1}{2}$
1	$\frac{1}{2}$	$\frac{1}{2}$

Remark 2: Harrison and Kreps [**14**] have shown that in a stock market in which priors differ, yet are common knowledge, and are never updated, the market price strictly exceeds the market fundamental of the traders. Thus the right to resell the stock gives traders the incentive to pay more for it than if they were obliged to hold it forever.

Their result may still hold with identical priors, differential information and updating, if one takes a *self-fulfilling equilibrium*, i.e., an equilibrium in which traders use the information conveyed by the price efficiently, but do not necessarily extract information from their trade if they have a demand correspondence. (See Kreps [**17**] for a formal definition. This equilibrium concept is used by Feiger [**8**].) Consider the following example (due to David Kreps). The model is the same as in the previous remark, except for the dividend process: $d_{t+1} = s_t^A + s_t^B$.

The following stationary price function leads to a self-fulfilling equilibrium:

$s_t^A \backslash s_t^B$	0	1
0	$\frac{9}{16}$	$\frac{21}{16}$
1	$\frac{21}{16}$	$\frac{21}{16}$

For example, when $s_t^B = 1$, B believes that the next dividend will be 1 with probability $\frac{1}{2}$ and 2 with probability $\frac{1}{2}$, since he cannot infer anything from the price. Thus he is willing to pay

$$\tfrac{1}{2}\left(\left(\tfrac{1}{2} \times 1 + \tfrac{1}{2} \times 2\right) + \left(\tfrac{1}{4} \times \tfrac{9}{16} + \tfrac{3}{4} \times \tfrac{21}{16}\right)\right) = \tfrac{21}{16}.$$

Now assume that $(s_t^A, s_t^B) = (0,1)$. A is fully informed (and is willing to pay $17/16$); thus he does not want to hold the stock (short sales are assumed to be prohibited). B, who holds the stock, has for a market fundamental:

$$\tfrac{1}{2}\left(\tfrac{1}{2} \times 1 + \tfrac{1}{2} \times 2\right) + \tfrac{1}{4}\left(\tfrac{1}{2} \times 1 + \tfrac{1}{4} \times 0 + \tfrac{1}{4} \times 2\right) + \cdots = \tfrac{20}{16} < \tfrac{21}{16} = p_t.$$

Thus the Harrison–Kreps result holds.

The explanation of this result is the following: B anticipates that he will be able to sell when information is (1,0); since his market fundamental (16/16) will then be lower than the price (21/16), he is now willing to pay more (21/16) than his market fundamental (20/16).

When B observes $s_t^B = 1$, he ought to realize that he is playing against a better informed trader. Of course if the quantity x_t^B were "measurable" (i.e., depended only on B's information), B would be willing to trade even if he realized that A is better informed, but then equilibrium would be destroyed. Even if B's information makes him indifferent to all feasible trades, it makes a difference whether B lets an auctioneer (or the market) pick his trade, or if he chooses it himself![14]

Note that the kind of price bubble arising in the previous example can not be observed in a myopic REE (and of course not in a fully dynamic REE, where no price bubble exists). Proposition 4 tells us that in a myopic REE prices have to grow "on average" at the rate $(1/\gamma)$. On the contrary the price function of the previous example is constant over time, as are the price bubbles.[15]

4c. *Discussion*

This section has been concerned with the relative evaluation of two assets over time. We have assumed that traders are never constrained by wealth in the amount they purchase. One may wonder whether the presence of wealth constraints would not be an alternative reason to rule out price bubbles. Let us first note that wealth constraints can be relaxed by the availability of credit. However we would like to argue that, even if credit schemes are not designed to enable the traders to buy the asset, price bubbles may still exist with an infinite number of traders, e.g., with overlapping generations. We know from growth theory that the economy may grow at a rate equal to or exceeding the rate of interest. Thus it is not clear that exponentially growing price bubbles would in general be ruled out by the presence of wealth constraints. Casual observation suggests that assets such as stamps, coins, paintings, diamonds, some land, etc., are consistently priced above their market fundamental.

5. CONCLUDING REMARKS

Two basic principles underlie the mathematics. First, one should not count on differences in information in order to achieve a speculative gain. This result is best understood by using the familiar common-knowledge interpretation of a REE, and by observing that not everyone can possess "better than average" information. Of course, in a market where some other traders do rely on the belief that they have superior information, it might pay to do so as well. We then

[14] This point has already been recognized by David Kreps [**17**, section on "information from quantity"].

[15] The price bubbles, for trader A for example, are: 1/16, 5/16, 1/16, when information is (0,0), (0,1), {(1,0) or (1,1)}.

face a recursive problem. The question is: Can rational traders expect in equilibrium a speculative gain based on their allegedly superior information or their information concerning the other traders' behavior? The common-knowledge interpretation of a REE would require the answer to be no.

Second, in a dynamic framework with a finite number of agents, a rational trader will not enter a market where a bubble has already grown, since some traders have already realized their gains and left a negative-sum game to the other traders. Again, if one is able to find a "sucker," it may pay to participate. The point is that in an equilibrium with a finite number of traders, it is not possible for everyone to find a buyer and avoid "getting stuck with a hot potato." This is not to deny the positive relevance of Keynes' "Castles in the Air" theory, which undoubtedly explains a number of speculative phenomena. More research should be devoted to the explanation of actual price bubbles by non-rational behavior[16] as well as to the study of the manipulability and controllability of speculative markets. But Section 4 vindicates the "Firm Foundation" asset pricing theory as a normative concept for the kind of markets we have considered; moreover, the views developed above have some counterparts in the investment literature (see, e.g., Malkiel [**19**]).

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[16] For an example of a behavioral theory of price bubbles, see Levine [**18**].

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