Density in the L^p spaces

Prof. S. Alama

McMaster University

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Density in a metric space

Let (M, d) be a metric space. We say $A \subset M$ is dense in M if the closure $\overline{A} = M$.

Equivalently,

- $\forall f \in M$ and $\forall \epsilon > 0$, $B_\epsilon(f) \cap A \neq \emptyset$.

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Equivalently,

- $\forall f \in M$ and $\forall \epsilon > 0$, $B_\epsilon(f) \cap A \neq \emptyset$.
- or, $\forall f \in M$, $\exists$ sequence $f_n \in A$ with $f_n \xrightarrow[n\to\infty]{} f$.

The spaces $L^p(\mathbb{R}^d)$

Specialize to the Lebesgue measure on $\mathbb{R}^d$.

Theorem

Let $1 \leq p < \infty$. The following are all dense in $L^p(\mathbb{R}^d)$:

1. *The (integrable) simple functions;*
2. *The step functions;*
3. *$C_0(\mathbb{R}^d)$, the class of all continuous functions of compact support in $\mathbb{R}^d$.*

① Holds in other measure spaces $(X, \mathcal{M}, \mu)$ when μ is σ-finite, i.e. $X = \bigcup_{n=1}^{\infty} X_n$ with $X_n \in \mathcal{M}$ and $\mu(X_n) < \infty$ $\forall n$

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None of the above are dense in $L^\infty(\mathbb{R}^d)$.

Idea: if $E, F \in \mathcal{L}$, $m(E \Delta F) > 0$, then

$$\|\chi_E - \chi_F\|_\infty = 1.$$

In $\ell^p = \{ (x(n))_{n\in\mathbb{N}} \mid \|x\|_p = (\sum_{n=1}^\infty |x(n)|^p)^{1/p} < \infty \}$, $1 \le p < \infty$

Then $D = \{ x \in \ell^p \mid x(n) = 0$ for all but finitely many $n \in \mathbb{N}\}$ is dense in ℓ^p

$\ell^\infty = \{ (x(n))_{n\in\mathbb{N}} \mid \sup_n |x(n)| = \|x\|_\infty < \infty \}$, then D is not dense in ℓ^∞.

We're going to use old results on measurable functions.

By Thm I.4.2 in SS, for any $f \in L^p(\mathbb{R}^d)$, since f is measurable, $\exists$ sequence $\{\varphi_n\}$ of simple functions (each with bounded support) with $|\varphi_n(x)| \le |f(x)|$ $\forall x \in \mathbb{R}^d$ and $\varphi_n(x) \xrightarrow[n\to\infty]{} f(x)$ pointwise in $\mathbb{R}^d$.

$$|\varphi_n(x) - f(x)|^p \le (|\varphi_n(x)| + |f(x)|)^p \le (2|f(x)|)^p = 2^p |f(x)|^p.$$

Since $f \in L^p$, $2^p|f|^p \in L^1(\mathbb{R}^d)$, so by LDCT

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |\varphi_n(x) - f(x)|^p \, dm = 0.$$

ie $\lim_{n\to\infty} \|\varphi_n - f\|_p = 0.$ ① ✓

For ②, use step ①, $\forall \varepsilon > 0$ $\exists$ simple φ, bounded support with $\|f - \varphi\|_p < \varepsilon/2$. By Thm I.4.3 $\exists$ sequence of step functions $\{\psi_n\}$ with $\psi_n \longrightarrow \varphi$ a.e. in $\mathbb{R}^d$. By construction $|\psi_n(x)| \le |\varphi(x)|$. Since φ has bounded support, by the Bounded Convergence Thm,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |\varphi - \psi_n|^p \, dm = \lim_{n\to\infty} \int_K |\varphi - \psi_n|^p \, dm = 0.$$

compact set, $\psi_n = 0$, $\varphi = 0$ outside K

$\therefore$ For n suff. large $\|\varphi - \psi_n\|_p < \varepsilon/2$, so

$$\|f - \psi_n\|_p \le \|f - \varphi\|_p + \|\varphi - \psi_n\|_p < \varepsilon. \quad ② ✓$$

③ is the same as ②, using the Exercise which gave continuous g_n with bounded support, $g_n(x) \longrightarrow \psi(x)$ a.e., $x \in \mathbb{R}^d$.

step function $\|\psi - f\|_p < \varepsilon/2$.

Again, by construction, $|g_n(x)| \le |\psi(x)|$, so the same argument applies $\square$.

An application

Let $f \in L^p(\mathbb{R}^d)$, with $1 \le p < \infty$, and for $h \in \mathbb{R}^d$ define the translation operator

$$\tau_h f(x) = f(x+h).$$

Then,

$$\lim_{h \to 0} \left\| \tau_h f - f \right\|_p = 0.$$

Use density of continuous functions of compact support!

$\forall \, \varepsilon > 0 \quad \exists \; g$ continuous, $K = \operatorname{supp}(g) = \overline{\{x \mid g(x) \neq 0\}}$ compact, with $\|f - g\|_p < \varepsilon/3$

Compact support $\Longrightarrow$ g is uniformly continuous, so $\exists \, \delta > 0$ with $|g(x+h) - g(x)| < \frac{\varepsilon}{3m(K)}$ $\forall \, h \in \mathbb{R}^d$ with $|h| < \delta$.

$\therefore \int_{\mathbb{R}^d} |g(x+h) - g(x)|^p \, dm < \frac{\varepsilon^p}{3^p}$, $\forall \, |h| < \delta$. So $\|\tau_h g - g\|_p < \frac{\varepsilon}{3}$

$\therefore \|f - \tau_h f\|_p \le \|f - g\|_p + \|g - \tau_h g\|_p + \|\tau_h g - \tau_h f\|_p$

$< \frac{\varepsilon}{3}$ $\quad < \varepsilon/3$

$$\|\tau_h g - \tau_h f\|_p^p = \int_{\mathbb{R}^d} |g(x+h) - f(x+h)|^p \, dm$$
$$= \int_{\mathbb{R}^d} |g(x) - f(x)|^p \, dm,$$

because Lebesgue measure is translation invariant,
$m(\tau_h E) = m(E)$, $\quad \tau_h E = \{x+h \mid x \in E\}$,

Lemma: If φ is Lebesgue integrable on $\mathbb{R}^d$, $h \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} f(x+h) \, dm = \int_{\mathbb{R}^d} f(x) \, dm$$

Start with $\varphi(x) = \chi_E$, $\quad m(E) < \infty$.

$$\int_{\mathbb{R}^d} \chi_E \, dm = m(E), \quad \int_{\mathbb{R}^d} \tau_h \chi_E \, dm = \int_{\mathbb{R}^d} \chi_{\tau_h E} \, dm$$
$$= m(\tau_h E) = m(E) \checkmark$$

Then if φ is simple, $\int_{\mathbb{R}^d} \tau_h \varphi \, dm = \int_{\mathbb{R}^d} \varphi \, dm.$

Then, if $\varphi \in L^1(\mathbb{R}^d)$, approx. φ by simple φ_n,
$\varphi_n \to \varphi$ pointwise, $|\varphi_n| \leq |\varphi|$ $\forall x \in \mathbb{R}^d$, use
LDCT to get the identity we need.

Separability

A metric space (M, d) is separable if it contains a **countable** dense subset.

Ex: $\cdot$ $\mathbb{Q}$ in $\mathbb{R}$

$\cdot$ in $C([a,b])$ with $\underbrace{\text{sup norm}}_{\|f\| = \sup_{x \in [a,b]} |f(x)|}$, {all polynomials} is a dense set (Weierstrass Approx. Thm)

{all polynomials with rational coeffs} is countable & dense.

$\cdot$ In ℓ^p, $1 \le p < \infty$, $\{x \in \ell^p \mid x(n) \in \mathbb{Q} \ \forall n,\ x(n) = 0$ all but finitely many $n \in \mathbb{N}\}$

is a countable, dense set.

Each is a separable metric space.

$\cdot$ ℓ^∞ is not separable.

$B = \{$binary seq $x \in \ell^\infty \mid x(n) = 0$ or 1, $\forall n\}$ uncountable

$x, y \in B$, $x \ne y$, $\|x - y\|_\infty = 1$.

Separability

A metric space (M, d) is separable if it contains a **countable** dense subset.

Theorem

1. *For all $p \in [1, \infty)$, $L^p(\mathbb{R}^d)$ is separable.*
2. *$L^\infty(\mathbb{R}^d)$ is not separable.*

① We already know step functions are dense in $L^p(\mathbb{R}^d)$.

Given step function $\psi(x) = \sum_{j=1}^{n} c_j \chi_{R_j}$, R_j rectangles,

approximate c_j by $\tilde{c}_j \in \mathbb{Q}$ and R_j by $\tilde{R}_j$,

rectangles with $\tilde{R}_j \subset R_j$, $\tilde{R}_j$ with rational intervals.

$\tilde{\psi}(x) = \sum_{j=1}^{n} \tilde{c}_j \chi_{\tilde{R}_j}$, $|\tilde{\psi}(x) - \psi(x)|$ small except on

a set of arbitrarily small measure (and $|\tilde{\psi}(x)| \le 2|\psi(x)|$).

From here one can estimate $\|\tilde{\psi} - \psi\|_p$, show $< \varepsilon$. (leaving out some details)