Measurable Functions

Prof. S. Alama

McMaster University

Math 721

Measurable functions

Let $(X, \mathcal{M}, \mu)$ be a measure space, $E \in \mathcal{M}$, and $f : E \subset X \to \mathbb{R}$ a real-valued function.

Definition

We say f *is* **measurable** *if* $\forall a \in \mathbb{R}$,

$$\{x \in E \mid f(x) < a\} = f^{-1}((-\infty, a)) \in \mathcal{M}.$$

Measurable functions

Let $(X, \mathcal{M}, \mu)$ be a measure space, $E \in \mathcal{M}$, and $f : E \subset X \to \mathbb{R}$ a real-valued function.

Definition

We say f *is* **measurable** *if* $\forall a \in \mathbb{R}$,

$$\{x \in E \mid f(x) < a\} = f^{-1}((-\infty, a)) \in \mathcal{M}.$$

We can also define measurability for extended real-valued functions, $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$, provided we assume the sets $\{f(x) = \pm\infty\}$ are measurable.

Lemma

Each of the following is equivalent:

1. $\{x \in E \mid f(x) < a\} \in \mathcal{M}, \forall a \in \mathbb{R}$.
2. $\{x \in E \mid f(x) \geq a\} \in \mathcal{M}, \forall a \in \mathbb{R}$.
3. $\{x \in E \mid f(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$.
4. $\{x \in E \mid f(x) \leq a\} \in \mathcal{M}, \forall a \in \mathbb{R}$.

Properties of measurable functions

1. f is measurable if and only if
   - $f^{-1}(\mathcal{O}) \in \mathcal{M}$, for all open $\mathcal{O} \subset \mathbb{R}$.
   - $f^{-1}(F) \in \mathcal{M}$, for all closed $F \subset \mathbb{R}$.

Properties of measurable functions

1. f is measurable if and only if
   - $f^{-1}(\mathcal{O}) \in \mathcal{M}$, for all open $\mathcal{O} \subset \mathbb{R}$.
   - $f^{-1}(F) \in \mathcal{M}$, for all closed $F \subset \mathbb{R}$.
2. If f is continuous then f is Lebesgue measurable (and Borel measurable). If Φ is continuous and f is measurable, then $\Phi \circ f$ is measurable.

Properties of measurable functions

1. f is measurable if and only if
   - $f^{-1}(\mathcal{O}) \in \mathcal{M}$, for all open $\mathcal{O} \subset \mathbb{R}$.
   - $f^{-1}(F) \in \mathcal{M}$, for all closed $F \subset \mathbb{R}$.
2. If f is continuous then f is Lebesgue measurable (and Borel measurable). If Φ is continuous and f is measurable, then $\Phi \circ f$ is measurable.
3. Assume f_n is a sequence of measurable functions. Then each is measurable:

$$\sup_n f_n(x), \quad \inf_n f_n(x), \quad \limsup_{n\to\infty} f_n(x), \quad \liminf_{n\to\infty} f_n(x)$$

Properties of measurable functions

1. f is measurable if and only if
   - $f^{-1}(\mathcal{O}) \in \mathcal{M}$, for all open $\mathcal{O} \subset \mathbb{R}$.
   - $f^{-1}(F) \in \mathcal{M}$, for all closed $F \subset \mathbb{R}$.
2. If f is continuous then f is Lebesgue measurable (and Borel measurable). If Φ is continuous and f is measurable, then $\Phi \circ f$ is measurable.
3. Assume f_n is a sequence of measurable functions. Then each is measurable:

$$\sup_n f_n(x), \quad \inf_n f_n(x), \quad \limsup_{n\to\infty} f_n(x), \quad \liminf_{n\to\infty} f_n(x)$$

4. If f_n is measurable $\forall n$, and $f(x) = \lim_{n\to\infty} f(x)$, then f is measurable.

Properties of measurable functions

1. f is measurable if and only if
   - $f^{-1}(\mathcal{O}) \in \mathcal{M}$, for all open $\mathcal{O} \subset \mathbb{R}$.
   - $f^{-1}(F) \in \mathcal{M}$, for all closed $F \subset \mathbb{R}$.
2. If f is continuous then f is Lebesgue measurable (and Borel measurable). If Φ is continuous and f is measurable, then $\Phi \circ f$ is measurable.
3. Assume f_n is a sequence of measurable functions. Then each is measurable:

$$\sup_n f_n(x), \quad \inf_n f_n(x), \quad \limsup_{n\to\infty} f_n(x), \quad \liminf_{n\to\infty} f_n(x)$$

4. If f_n is measurable $\forall n$, and $f(x) = \lim_{n\to\infty} f(x)$, then f is measurable.
5. If f, g are measurable, then so are $f + g$ and fg.

Properties of measurable functions

1. f is measurable if and only if
   - $f^{-1}(\mathcal{O}) \in \mathcal{M}$, for all open $\mathcal{O} \subset \mathbb{R}$.
   - $f^{-1}(F) \in \mathcal{M}$, for all closed $F \subset \mathbb{R}$.
2. If f is continuous then f is Lebesgue measurable (and Borel measurable). If Φ is continuous and f is measurable, then $\Phi \circ f$ is measurable.
3. Assume f_n is a sequence of measurable functions. Then each is measurable:

$$\sup_n f_n(x), \quad \inf_n f_n(x), \quad \limsup_{n\to\infty} f_n(x), \quad \liminf_{n\to\infty} f_n(x)$$

4. If f_n is measurable $\forall n$, and $f(x) = \lim_{n\to\infty} f(x)$, then f is measurable.
5. If f, g are measurable, then so are $f + g$ and fg.
   We say two functions $f(x) = g(x)$ almost everywhere (or a.e.) if
   $$m(\{x \mid f(x) \neq g(x)\}) = 0.$$
6. If f is measurable and $f = g$ a.e., then g is measurable.

Simple and step functions

Here are some kinds of functions which are easy to deal with.

- Let $E \in \mathcal{M}$. Its characteristic function

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

Simple and step functions

Here are some kinds of functions which are easy to deal with.

- Let $E \in \mathcal{M}$. Its characteristic function

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

- A simple function is a finite linear combination of characteristic functions,

$$\varphi(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x),$$

for $E_k \in \mathcal{M}$, $a_k \in \mathbb{R}$, of finite measure $\mu(E_k) < \infty$, $k = 1, \dots, N$.

Simple and step functions

Here are some kinds of functions which are easy to deal with.

- Let $E \in \mathcal{M}$. Its characteristic function

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

- A simple function is a finite linear combination of characteristic functions,

$$\varphi(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x),$$

  for $E_k \in \mathcal{M}$, $a_k \in \mathbb{R}$, of finite measure $\mu(E_k) < \infty$, $k = 1, \ldots, N$.
- a step function is a simple function whose sets $E_k = R_k$ are rectangles $\forall k = 1, \ldots, N$.

Approximation by simple functions

We do this for $f : \mathbb{R}^k \to \mathbb{R}$, but the same construction works for any σ-finite measure space.

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative measurable function. Then $\exists$ an increasing sequence φ_n of simple functions, $0 \leq \varphi_k(x) \leq \varphi_{k+1}(x)$ with $f(x) = \lim_{k\to\infty} \varphi_k(x)$ pointwise on $\mathbb{R}^d$.

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ *be any measurable function. Then* $\exists$ *a sequence* φ_n *of simple functions with* $|\varphi_k(x)| \leq |\varphi_{k+1}(x)|$ *with* $f(x) = \lim\limits_{k\to\infty} \varphi_k(x)$ *pointwise on* $\mathbb{R}^d$.

Approximation by step functions

Theorem

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be any measurable function. Then $\exists$ a sequence ψ_n of step functions with $f(x) = \lim_{k\rightarrow\infty} \psi_k(x)$ almost everywhere on $\mathbb{R}^d$.

Approximation by step functions

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be any measurable function. Then $\exists$ a sequence ψ_n of step functions with $f(x) = \lim_{k\to\infty} \psi_k(x)$ almost everywhere on $\mathbb{R}^d$.

Exercise: any measurable $f : \mathbb{R}^d \to \mathbb{R}$ is the pointwise a.e. limit of a sequence of continuous functions with compact support.

Littlewood's Principles

1. "Every measurable set is nearly a finite union of rectangles."

Littlewood's Principles

1. "Every measurable set is nearly a finite union of rectangles."
2. "A measurable function is nearly continuous."

Theorem (Lusin's Theorem)

Let $E \in \mathcal{L}$ and $m(E) < \infty$. Then $\forall \epsilon > 0$, $\exists$ closed $F_\epsilon \subset E$ with $m(E \setminus F_\epsilon) < \epsilon$ such that $f|_{F_\epsilon}$ is continuous.

Littlewood's Principles

1. "Every measurable set is nearly a finite union of rectangles."
2. "A measurable function is nearly continuous."

Theorem (Lusin's Theorem)

Let $E \in \mathcal{L}$ and $m(E) < \infty$. Then $\forall \epsilon > 0$, $\exists$ closed $F_\epsilon \subset E$ with $m(E \setminus F_\epsilon) < \epsilon$ such that $f|_{F_\epsilon}$ is continuous.

3. "Any pointwise convergent sequence of measurable functions is nearly uniformly convergent.]]

Theorem (Egorov's Theorem)

Assume $E \in \mathcal{L}$ with $m(E) < \infty$, and f_k is a sequence of measurable functions with $f_k(x) \to f(x)$ a.e. For all $\epsilon > 0$, $\exists$ closed set $A_\epsilon \subset E$ for which $m(E \setminus A_\epsilon) < \epsilon$ and $f_k \to f$ uniformly on A_ϵ.