Measurable Functions

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Measurable functions

Let $(X, \mathcal{M}, \mu)$ be a measure space, $E \in \mathcal{M}$, and $f: E \subset X \to \mathbb{R}$ a real-valued function.

Definition

We say f is **measurable** *if* $\forall a \in \mathbb{R}$,

$$\{x \in E \mid f(x) < a\} = f^{-1}((-\infty, a)) \in \mathcal{M}.$$

We can allow $f(x) = \pm\infty$ as values – "extended real valued" functions.

$\mathcal{M} = \mathcal{L}$, "Lebesgue measurable functions"

$\mathcal{M} = \underset{\text{Borel}}{\mathcal{B}}$, "Borel meas. functions"

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We can also define measurability for extended real-valued functions, $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$, provided we assume the sets $\{f(x) = \pm\infty\}$ are measurable.

Lemma

Each of the following is equivalent:

1. $\{x \in E \mid f(x) < a\} \in \mathcal{M}, \forall a \in \mathbb{R}.$
2. $\{x \in E \mid f(x) \geq a\} \in \mathcal{M}, \forall a \in \mathbb{R}.$
3. $\{x \in E \mid f(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}.$
4. $\{x \in E \mid f(x) \leq a\} \in \mathcal{M}, \forall a \in \mathbb{R}.$

Pf: Assume ①. Then

$$\{x \mid f(x) \leq a\} = \bigcap_{n=1}^{\infty} \underbrace{\{x \mid f(x) < a + \tfrac{1}{n}\}}_{\in \mathcal{M}\ \forall n,\ \text{by } ①.} \in \mathcal{M} \quad ④$$

Assume ④. Then $\{x \mid f(x) > a\} = \{x \mid f(x) \leq a\}^c \in \mathcal{M}$ ③

Assume ③. Then

$$\{x \mid f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) > a - \tfrac{1}{n}\} \in \mathcal{M} \quad ②$$

Assume ②. Then $\{x \mid f(x) < a\} = \{x \mid f(x) \geq a\}^c \in \mathcal{M}$, ①

Properties of measurable functions

1. f is measurable if and only if
   - $f^{-1}(\mathcal{O}) \in \mathcal{M}$, for all open $\mathcal{O} \subset \mathbb{R}$.
   - $f^{-1}(F) \in \mathcal{M}$, for all closed $F \subset \mathbb{R}$.

Recall any $\mathcal{O} \subset \mathbb{R}$ is a countable disjoint union of open intervals, $\mathcal{O} = \bigcup_{j=1}^{\infty} (a_j, b_j)$

And $f^{-1}\left(\bigcup_{j=1}^{\infty} (a_j, b_j) \right) = \bigcup_{j=1}^{\infty} f^{-1}((a_j, b_j))$, and

$\mathcal{M}$ is a σ-algebra.

If F is closed, since $f^{-1}(F) = \left(f^{-1}(\underset{\text{open}}{F^c}) \right)^c$, so again this follows from the σ-algebra property of $\mathcal{M}$.

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2. If f is continuous then f is Lebesgue measurable (and Borel measurable). If Φ is continuous and f is measurable, then $\Phi \circ f$ is measurable.

- f continuous $\Longrightarrow$ $f^{-1}(\mathcal{O})$ is open.
  $\mathcal{O}$ open
  $\therefore f^{-1}(\mathcal{O}) \in \mathcal{L}$ or $\mathcal{B}$
  $\forall$ open sets $\mathcal{O}$

  So continuous $\Longrightarrow$ Lebesgue and Borel measurable.

- $$(\Phi \circ f)^{-1}[(-\infty, a)) = \{x \mid \Phi(f(x)) < a\}$$
  $$= \{x \mid f(x) \in \underbrace{\Phi^{-1}[(-\infty, a))}_{\text{open}}\} \in \mathcal{M}. \quad \forall a \in \mathbb{R}.$$

Careful: It is not true if Φ is only measurable!

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2. If f is continuous then f is Lebesgue measurable (and Borel measurable). If Φ is continuous and f is measurable, then $\Phi \circ f$ is measurable.
3. Assume f_n is a sequence of measurable functions. Then each is measurable:

$$\sup_n f_n(x), \quad \inf_n f_n(x), \quad \limsup_{n\to\infty} f_n(x), \quad \liminf_{n\to\infty} f_n(x)$$

Pf:

$$\{x \mid \sup_n f_n(x) > a\} = \bigcup_{n=1}^{\infty} \underbrace{\{x \mid f_n(x) > a\}}_{\in \mathcal{M}\ \forall n.} \ \therefore \in \mathcal{M}.$$

$$\inf_n f_n(x) = -\sup_n(-f_n(x))$$

$$\limsup_{n\to\infty} f_n(x) = \inf_{k\in\mathbb{N}}\left(\sup_{n\geq k} f_n(x)\right), \quad \liminf_{n\to\infty} f_n(x) = \sup_{k\in\mathbb{N}}\left(\inf_{n\geq k} f_n(x)\right)$$

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4. If f_n is measurable $\forall n$, and $f(x) = \lim_{n\to\infty} f(x)$, then f is measurable.
5. If f, g are measurable, then so are $f+g$ and fg.

To show $(f+g)$ is measurable, we claim

$$\{x \mid f(x) + g(x) > a\} = \bigcup_{r \in \mathbb{Q}} \{x \mid f(x) > a - r\} \cap \{x \mid g(x) > r\}$$

First, if $f(x) > a-r$, $g(x) > r$, then $f(x)+g(x) > a$, so $\supset$.

On the other hand, if $f(x)+g(x) > a$, take $\varepsilon > 0$ with

$f(x)+g(x) > a+\varepsilon$. Let $r \in \mathbb{Q}$ with

$$r < g(x) < r+\varepsilon.$$

Then, $f(x) > a+\varepsilon - g(x) > a+\varepsilon-(r+\varepsilon) = a-r$ ✓

$\therefore \subset$.

Since $\mathbb{Q}$ is countable, f, g meas $\Rightarrow$ $f+g$ meas.

For fg, first notice f meas $\Rightarrow$ $\Phi(f) = f^2$

is measurable (Φ is continuous.).

Then $fg = \dfrac{(f+g)^2-(f-g)^2}{4}$ is measurable.

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4. If f_n is measurable $\forall n$, and $f(x) = \lim_{n\to\infty} f(x)$, then f is measurable.
5. If f, g are measurable, then so are $f + g$ and fg.
   We say two functions $f(x) = g(x)$ almost everywhere (or a.e.) if

$$m(\{x \mid f(x) \neq g(x)\}) = 0.$$

6. If f is measurable and $f = g$ a.e., then g is measurable.

Recall – a set of measure zero is measurable

Simple and step functions

Here are some kinds of functions which are easy to deal with.

- Let $E \in \mathcal{M}$. Its characteristic function

chi

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

χ_E is measurable $\Longleftrightarrow$ $E \in \mathcal{M}$.

$$\{x \mid \chi_E(x) \geq a\} = \begin{cases} \phi, & \text{if } a > 1 \\ E, & \text{if } 0 < a \leq 1 \\ \mathbb{R}^d, & \text{if } a \leq 0. \end{cases}$$

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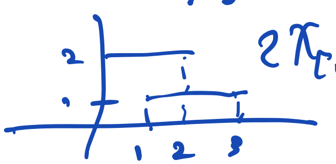
$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

- A simple function is a finite linear combination of characteristic functions,

$2\chi_{[0,2]} + \chi_{[1,3]} =$

$$\varphi(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x),$$



$2\chi_{[0,1]} + 3\chi_{[1,2]} + \chi_{[2,3]}.$

for $E_k \in \mathcal{M}$, $a_k \in \mathbb{R}$, of [finite measure $\mu(E_k) < \infty$], $k = 1, \dots, N$.

in SS this is required, not in Bartle!

Remark: The same simple function can have different representations with different sets E_k, values a_k. But there is a representation of any $\varphi(x)$ with disjoint E_k, and distinct a_k values (see Bartle...)

Simple and step functions

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finite measures, call it Integrable simple function.

for $E_k \in \mathcal{M}$, $a_k \in \mathbb{R}$, of [finite measure $\mu(E_k) < \infty$,] $k = 1, \ldots, N$.

- a step function is a simple function whose sets $E_k = R_k$ are rectangles $\forall k = 1, \ldots, N$.

Approximation by simple functions

We do this for $f: \mathbb{R}^k \to \mathbb{R}$, but the same construction works for any σ–finite measure space.

Theorem

Let $f: \mathbb{R}^d \to \mathbb{R}$ be a nonnegative measurable function. Then $\exists$ an increasing sequence φ_n of simple functions, $0 \le \varphi_k(x) \le \varphi_{k+1}(x)$ with $f(x) = \lim_{k\to\infty} \varphi_k(x)$ pointwise on $\mathbb{R}^d$.

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ *be any measurable function. Then* $\exists$ *a sequence* φ_n *of simple functions with* $|\varphi_k(x)| \leq |\varphi_{k+1}(x)|$ *with* $f(x) = \lim_{k \to \infty} \varphi_k(x)$ *pointwise on* $\mathbb{R}^d$.

Approximation by step functions

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ *be any measurable function. Then* $\exists$ *a sequence* ψ_n *of step functions with* $f(x) = \lim_{k\to\infty} \psi_k(x)$ *almost everywhere on* $\mathbb{R}^d$.

Approximation by step functions

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be any measurable function. Then $\exists$ a sequence ψ_n of step functions with $f(x) = \lim_{k\to\infty} \psi_k(x)$ almost everywhere on $\mathbb{R}^d$.

Exercise: any measurable $f : \mathbb{R}^d \to \mathbb{R}$ is the pointwise a.e. limit of a sequence of continuous functions with compact support.

Littlewood's Principles

1. "Every measurable set is nearly a finite union of rectangles."

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2. "A measurable function is nearly continuous."

Theorem (Lusin's Theorem)

Let $E \in \mathcal{L}$ *and* $m(E) < \infty$. *Then* $\forall \epsilon > 0$, $\exists$ *closed* $F_\epsilon \subset E$ *with* $m(E \setminus F_\epsilon) < \epsilon$ *such that* $f|_{F_\epsilon}$ *is continuous.*

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Let $E \in \mathcal{L}$ and $m(E) < \infty$. Then $\forall \epsilon > 0$, $\exists$ closed $F_\epsilon \subset E$ with $m(E \setminus F_\epsilon) < \epsilon$ such that $f|_{F_\epsilon}$ is continuous.

3. "Any pointwise convergent sequence of measurable functions is nearly uniformly convergent.]]

Theorem (Egorov's Theorem)

Assume $E \in \mathcal{L}$ with $m(E) < \infty$, and f_k is a sequence of measurable functions with $f_k(x) \to f(x)$ a.e. For all $\epsilon > 0$, $\exists$ closed set $A_\epsilon \subset E$ for which $m(E \setminus A_\epsilon) < \epsilon$ and $f_k \to f$ uniformly on A_ϵ.