Lebesgue Measurable Sets

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Math 721

Review: Observations on the exterior measure

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|, \quad E \subset \bigcup_{j=1}^{\infty} Q_j \text{ covered by cubes}$$

1. (Monotonicity) If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.
2. (Countable sub-additivity) If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.
3. $m_*(E) = \inf m_*(\mathcal{O})$, with infimum taken over all open sets $\mathcal{O} \supset E$.
   - Any open set $\mathcal{O} \subset \mathbb{R}^d$ can be written as an almost-disjoint union of cubes; see Theorem 1.4.
4. If $E_1, E_2 \subset \mathbb{R}^d$ are disjoint and $\text{dist}(E_1, E_2) > 0$, then $m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$.
5. If $E = \bigcup_{j=1}^{\infty} Q_j$ is a countable union of almost disjoint cubes, then $m_*(E) = \sum_{j=1}^{\infty} |Q_j|$.

Measurable Sets

Recall: the exterior (outer) measure of $E \subset \mathbb{R}^d$,

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|, \quad E \subset \bigcup_{j=1}^{\infty} Q_j \text{ covered by cubes,}$$

Definition

A set $E \subset \mathbb{R}^d$ is (Lebesgue) measurable if $\forall \epsilon > 0$, $\exists$ an open set $\mathcal{O}$ with

$$E \subset \mathcal{O} \text{ and } m_*(\mathcal{O} \setminus E) < \epsilon.$$

If E is measurable, its Lebesgue measure $m(E) = m_*(E)$.

Following Bartle, we denote by $\mathcal{L}$ the class of all Lebesgue measurable sets in $\mathbb{R}^d$.

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5. If $E \in \mathcal{L}$, then the complement $E^c \in \mathcal{L}$.
6. A countable intersection $\bigcap_{j=1}^{\infty} E_j$ of measurable $E_j \in \mathcal{L}$ is measurable.

We call $\mathcal{L}$ a σ-algebra of sets.

m is countably additive on $\mathcal{L}$.

Theorem

Let $E_j \in \mathcal{L}$, $\forall j \in \mathbb{N}$ be a disjoint collection of measurable sets. Then:

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j).$$

Monotone sequences of sets

Definition

We say the sequence $E_1, E_2, \ldots$ *of sets* *increases* *to* E, $E_j \nearrow E$, *if*

$$E_j \subset E_{j+1} \text{ and } E = \bigcup_{j=1}^{\infty} E_j.$$

Similarly, $E_1, E_2, \ldots$ *of sets* *decreases* *to* E, $E_j \searrow E$, *if*

$$E_{j+1} \subset E_j \text{ and } E = \bigcap_{j=1}^{\infty} E_j.$$

Corollary

Assume $E_j \in \mathcal{L}, \forall j \in \mathbb{N}$.

1. *If* $E_j \nearrow E$, *then* $E \in \mathcal{L}$ *and* $m(E) = \lim_{k\to\infty} m(E_k)$.
2. *If* $E_j \searrow E$ *and* $m(E_j) < \infty$ *for some* $j \in \mathbb{N}$, *then* $m(E) = \lim_{k\to\infty} m(E_k)$.

Littlewood's First Principle

"A measurable set is nearly a finite union of cubes."

Theorem

Let $E \in \mathcal{L}$ with $m(E) < \infty$. Then $\forall \epsilon > 0$,

1. *$\exists K$ compact, $K \subset E$, with $m(E \setminus K) < \epsilon$.*
2. *$\exists$ a finite union of nearly disjoint closed cubes $F = \bigcup_{j=1}^{N} Q_j$ for which $m(E \triangle F) < \epsilon$.*

Recall: symmetric difference $E \triangle F = (E \cap F^c) \cup (F \cap E^c)$.