Signed Measures

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Signed measures, or charges

If $(X, \mathcal{M})$ is a measurable set, and μ_1, μ_2 are two measures, then it is easy to show that $\alpha\mu_1$, $\forall\alpha > 0$, and $\mu_1 + \mu_2$ are measures on the same $(X, \mathcal{M})$.

How about $\nu = \mu_1 - \mu_2$?

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Definition

A **signed measure** *(or "charge") ν on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \mathbb{R} \cup \{-\infty \text{ or } +\infty\}$ such that*

1. $\nu(\emptyset) = \mathbf{0}$;
2. *For any countable disjoint union $E = \bigcup_{n\in\mathbb{N}} E_n$, $E_n \in \mathcal{M}$, $\forall n \in \mathbb{N}$, we have*

$$\nu(E) = \sum_{n=1}^{\infty} \nu(E_n), \quad \text{converging absolutely if } \nu(E) \text{ is finite.}$$

Ex: If $f(x) \in L^1(X, \mu)$, and $E \in \mathcal{M}$,

$$\nu(E) = \int_E f(x)\, d\mu$$

When $f \geq 0$, we used the fact that $\nu(E)$ defined this way is a measure on $(X, \mathcal{M})$

When f changes sign, write $f = f^+ - f^-$, $f^\pm \geq 0$,

so
$$\nu(E) = \underbrace{\int_E f^+ d\mu}_{\nu^+(E)} - \underbrace{\int_E f^- d\mu}_{\nu^-(E)}$$

(pos) measure ν^+ (pos) measure ν^-

If we call $A = \{x \mid f(x) \geq 0\}$, $B = \{x \mid f(x) < 0\}$,

then $A \cap B = \emptyset$, $A \cup B = X$, and

$$\nu^+(E) = \nu(E \cap A), \quad \nu^-(E) = -\nu(E \cap B).$$

This signed measure splits X into complementary sets A, B, and each of ν^+, ν^- is supported on one of the two sets.

We'll prove this is <u>always</u> the case!

Positive and negative sets

Definition

Let ν *be a signed measure on* $(X, \mathcal{M})$.

- *We say that* $A \in \mathcal{M}$ *is a positive set for* ν *if every measurable* $E \subset A$ *has* $\nu(E) \geq 0$.
- *We say that* $B \in \mathcal{M}$ *is a negative set for* ν *if every measurable* $E \subset B$ *has* $\nu(E) \leq 0$.
- *We say* N *is a null set for* ν *if every measurable subset* E *of* N *has* $\nu(E) = 0$.

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Hahn Decomposition Theorem

Let ν be a signed measure on $(X, \mathcal{M})$. Then $\exists$ positive set A and negative set B for ν, with $A \cap B = \emptyset$ and $X = A \cup B$.

Note: the decomposition is not unique, as null sets can be part of A or B.

Proof of the Hahn Decomposition

Lemma

Let ν be a signed measure, and assume $E \in \mathcal{M}$ with $0 < \nu(E) < \infty$. Then E contains a positive set $A \subset E$ with $\nu(A) > 0$.

If E is a positive set, we're done.

If not, then E contains subsets $F \subset E$, with $\nu(F) < 0$.

Let $m_1 = \inf\{ m \in \mathbb{N} \mid \exists\, E_1 \in \mathcal{M},\ E_1 \subset E, \text{ and } \nu(E_1) < -\frac{1}{m} \}$.

Choose $E_1 \in \mathcal{M}$ with $\nu(E_1) < -\frac{1}{m_1} < 0$.

If $E \setminus E_1$ is a positive sets, let $A = E \setminus E_1$, and stop.

If not, $E \setminus E_1$ contains subsets with $\nu(F) < 0$.

Let $m_2 = \inf\{ m \in \mathbb{N} \mid \exists\, E_2 \in \mathcal{M},\ E_2 \subset (E \setminus E_1), \text{ and } \nu(E_2) < -\frac{1}{m} \}$.

Choose $E_2 \in \mathcal{M}$ with $\nu(E_2) < \frac{1}{m_2} < 0$, and $E_2 \subset E \setminus E_1$.

Continue like this, defining disjoint sets $E_1, E_2, \dots, E_n \subset E$ and $m_1, m_2, \dots, m_n \in \mathbb{N}$, with $E_n \subset E \setminus \bigcup_{j=1}^{n-1} E_j$ and $\nu(E_j) < -\frac{1}{m_j}$. If $E \setminus \bigcup_{j=1}^{m} E_j$ is a positive set, stop (call it A, done.) If not, this iteration continues $\forall\, n \in \mathbb{N}$.

Call $A = E \setminus \bigcup_{j=1}^{\infty} E_j$, so $E = A \cup \left(\bigcup_{j=1}^{\infty} E_j\right)$ is a disjoint union.

Since $0 < \nu(E) < \infty$,

$$-\infty < \nu\left(\bigcup_{j=1}^{\infty} E_j\right) \underset{\text{disjoint}}{=} \sum_{j=1}^{\infty} \nu(E_j) < -\sum_{j=1}^{\infty} \frac{1}{m_j}.$$

$$\Rightarrow \sum_{j=1}^{\infty} \frac{1}{m_j} < \infty \implies \lim_{j \to \infty} m_j = \infty.$$

Let $F \subset A$, $F \in \mathcal{M}$. Then $\forall\, k \in \mathbb{N}$,

$$F \subset A \subset E \setminus \bigcup_{j=1}^{k} E_j.$$ By choice of E_j

$$\nu(F) \geq \frac{-1}{m_{k+1}},$$ true $\forall\, k \in \mathbb{N}$.

$\therefore\ \nu(F) \geq 0$, so A is a positive set.

Since $\nu(E \setminus A) = \nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \underbrace{\nu(E_j)}_{< -\frac{1}{m_j} < 0} < 0$,

by additivity, $\nu(A) = \underbrace{\nu(E)}_{> 0 \text{ hyp}} - \underbrace{\nu(A \setminus E)}_{< 0} > 0$, so A is the desired set ✓

Proof of Hahn Decomp. Assume ν cannot take the value $+\infty$,

Let $\mathcal{P} = \{$ all positive subsets of $X \}$. By Lemma $\mathcal{P}$ contains sets with $\nu(A) > 0$. Define

$$\lambda = \sup \{ \nu(E) \mid E \in \mathcal{P} \}.$$

Then, $\exists$ sequence $(A_k)_{k \in \mathbb{N}}$ in $\mathcal{P}$, with

$$\nu(A_k) \xrightarrow[k \to \infty]{} \lambda.$$

Let $A = \bigcup_{k=1}^{\infty} A_k$. First, claim $A \in \mathcal{P}$.

Pf of claim: $\forall E \subset A$, let $E_1 = E \cap A_1$, $E_2 = (E \cap A_2) \setminus E_1$, $\dots$, $E_k = (E \cap A_k) \setminus (A_1 \cup \dots \cup A_{k-1})$, so the $(E_k)_{k \in \mathbb{N}}$ is disjoint, $E_k \subset A_k$ $\forall k$, so $\nu(E_k) \geq 0$ ($A_k \in \mathcal{P}$)

$$\therefore \nu(E) = \nu\left(\bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \nu(E_k) \geq 0, \text{ so } A \in \mathcal{P}.$$

Since $A \in \mathcal{P}$, $\nu(A) \leq \lambda$. On the other hand,

$A \setminus A_k \subset A$ so $\nu(A \setminus A_k) \geq 0$, and

$$\underbrace{\nu(A) = \nu(A_k) + \nu(A \setminus A_k)}_{\text{disjoint union}} \geq \nu(A_k) \xrightarrow{k \to \infty} \lambda$$

$\therefore \nu(A) \geq \lambda$, and hence

$\nu(A) = \lambda$, we attain the max in $\mathcal{P}$.

Finally, let $B = X \setminus A$. Need to show B is negative set.

For contradiction, assume $\exists\, E \in \mathcal{M}$, $E \subset B$ with $\nu(E) > 0$. By Lemma, $\exists\, E_0 \subset E$, $E_0 \in \mathcal{M}$, with $\nu(E_0) > 0$ and E_0 a positive set.

$A \cup E_0 \in \mathcal{P}$, and

Then, $\nu(A \cup E_0) = \nu(A) + \nu(E_0) > \nu(A) = \lambda$,

but $\lambda = \sup\limits_{\mathcal{P}} \nu(E)$. $\Rightarrow\!\Leftarrow$

$\therefore$ B is negative set, and we're done $\square$.

The Jordan Decomposition

Let ν be a signed measure, with Hahn Decomposition sets A, B. Then

$$\nu^+(E) := \nu(E \cap A), \quad \nu^-(E) := -\nu(E \cap B), \quad E \in \mathcal{M},$$

are measures on $(X, \mathcal{M})$.

(positive)

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Note that $\nu^+(B) = 0$ and $\nu^-(A) = 0$, with $X = A \cup B$ and $A \cap B = \emptyset$.
We say the measures $\nu^\pm$ are mutually singular.

Examples

$\mathcal{M} = \mathcal{B}$, Borel sets

If ν is a signed Borel measure on $[a,b] \subset \mathbb{R}$,

then by the Hahn/Jordan Decomposition

$$\nu(E) = \nu^+(E) - \nu^-(E),$$ and each

of $\nu^\pm$ are (positive) Borel measures on $[a,b] \subset \mathbb{R}$.

$\therefore$ $\exists$ monotone non-decreasing $F^\pm(x)$, (right-continuous) with

$$\nu^\pm((\alpha,\beta]) = F^\pm(\beta) - F^\pm(\alpha) \qquad \otimes$$

Conversely, if $F^\pm$ are non-decreasing (right continuous) functions on $[a,b]$, we define a ~~signed~~ measure using $\otimes$ on the algebra $\mathcal{A}$ of intervals, as before.

For a given non-monotone $F(x)$, can we associate a Borel $\overset{\text{signed}}{\wedge}$ measure $\nu((\alpha,\beta]) = F(\beta) - F(\alpha)$ with that F?

<u>Need</u>: F is a difference of monotone increasing functions.

The total variation measure

Define the measure on $(X, \mathcal{M})$,

$$|\nu|(E) = \nu^+(E) + \nu^-(E), \quad E \in \mathcal{M}.$$

This is called the total variation of the signed measure ν.

For example $\nu(E) = \int_E f(x)\, d\mu(x)$,

$$= \underbrace{\int_E f^+(x)\, d\mu}_{\nu^+(E)} - \underbrace{\int_E f^-(x)\, d\mu}_{\nu^-(E)}$$

Then $|\nu|(E) = \int_E |f|\, d\mu$

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This is called the total variation of the signed measure ν.
Exercise: Show that

$$|\nu|(E) = \sup \sum_{k=1}^{n} |\nu(E_k)|,$$

where the sup is taken over all disjoint finite collections $\{E_k\}_{k=1,\dots,n}$ of measurable subsets of E.

Example: Signed Borel measures and BV functions.

If we have a Borel signed measure, ν on $[a,b] \subset \mathbb{R}$, define

$$F(x) = \nu((a,x])$$

so $\nu((\alpha,\beta]) = F(\beta) - F(\alpha)$

Partition $[a,b]$ into $a = x_0 < x_1 < \cdots < x_N = b$

$$(*) \quad \sum_{j=1}^{N} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{N} |\nu((x_{j-1}, x_j])|$$

$$\leq |\nu|([a,b]),$$

(since the Total Variation is the sup)

taken over any partition of $[a,b]$.

We say $F:[a,b] \to \mathbb{R}$ is of Bounded Variation if $\circledast$ is finite.

Theorem: F is of Bounded Variation on $[a,b] \subset \mathbb{R}$ iff $\exists$ monotone, non-decreasing functions $F^{\pm}(x)$ with $F(x) = F^{+}(x) - F^{-}(x)$.

Finally, finite Borel signed measures appear naturally as the dual space to $\underbrace{C([a,b])}_{\text{continuous functions on } [a,b], \text{ with sup norm}}$

Riesz Representation Theorem: ($C([a,b])$ version)

Let $\ell : C([a,b]) \to \mathbb{R}$ be a bounded linear functional, that is

- $\exists\, M \geq 0$ with $|\ell(f)| \leq M \|f\|_\infty$ $\forall f \in C([a,b])$.

- $\ell(\alpha f + \beta g) = \alpha \ell(f) + \beta \ell(g)$

  $\forall f, g \in C([a,b])$ and $\forall \alpha, \beta \in \mathbb{R}$.

Then, $\exists$ signed finite Borel measure ν with

$$\ell(f) = \int_{[a,b]} f(x)\, d\nu(x).$$

(Bartle – 1D. In $\mathbb{R}^\ell$, $C(K)$, in Royden-Fitzpatrick)

compact