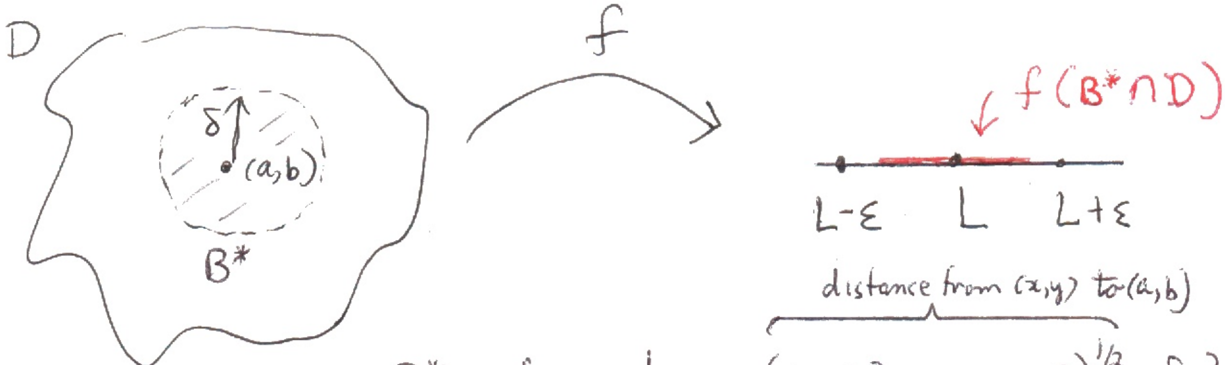
14.2 Limits

Let $f(x,y)$ be a function with domain $D \subset \mathbb{R}^2$

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$ (Note: we do not assume $(a,b) \in D$)

"the limit of $f(x,y)$ as (x,y) approaches (a,b) is equal to L" means

We can make $f(x,y)$ as close to L as we like by taking (x,y) sufficiently close to (a,b)



$$B^* = \{(x,y) \mid 0 < \underbrace{((x-a)^2 + (y-b)^2)^{1/2}}_{\text{distance from }(x,y)\text{ to }(a,b)} < \delta\}$$

punctured open disk of radius δ centered at (a,b)

In order for $\lim_{(x,y)\to(a,b)} f(x,y)$ to exist the answer should be independent of how (x,y) approaches the point (a,b)

EXAMPLE Show $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

approach $(0,0)$ along the x-axis, that is $y=0$:

$$\lim_{x\to 0} \frac{x^2-0^2}{x^2+0^2} = \lim_{x\to 0} 1 = 1$$

approach $(0,0)$ along the y-axis, that is $x=0$:

$$\lim_{y\to 0} \frac{0^2-y^2}{0^2+y^2} = \lim_{y\to 0} -1 = -1$$

$1 \neq -1$ so the limit does not exist

EXAMPLE Show $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

First we check the limit along any non-vertical line $y = mx$

$$\lim_{x\to 0} \frac{x(mx)^2}{x^2+(mx)^4} = \lim_{x\to 0} \frac{m^2x}{1+m^4x^2} = 0$$

So the function has the same limiting value along any line through $(0,0)$.

Let $(x,y)\to(0,0)$ along the parabola $x=y^2$

$$\lim_{y\to 0} \frac{y^2y^2}{(y^2)^2+y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

$0 \neq 1/2$ so the limit does not exist.

EXAMPLE Show $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$. (In particular the limit exists)

We use the <u>Squeeze Theorem:</u>

Let $h(x,y)$ and $f(x,y)$ and $g(x,y)$ be functions such that $h(x,y) \le f(x,y) \le g(x,y)$ and $\lim_{(x,y)\to(a,b)} h(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = L$

Then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

$$0 \le \frac{3x^2|y|}{x^2+y^2} = 3|y|\frac{x^2}{x^2+y^2} \le 3|y|$$

$\lim_{(x,y)\to(0,0)} 3|y| = 0$ $\qquad h(x,y)=0, \; g(x,y)=3|y|$

So by the Squeeze Theorem $\lim_{(x,y)\to(0,0)} \frac{3x^2|y|}{x^2+y^2} = 0$

This implies $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

(because if $|u|$ is close to 0 then so is u)

<u>Continuity</u> $f(x,y)$ is <u>continuous at (a,b)</u> $\in D$ (the domain of $f(x,y)$) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

$f(x,y)$ is <u>continuous</u> if it is continuous at each $(a,b) \in D$

<u>Examples of continuous functions</u> (two variable case)

1. <u>Polynomial functions</u>

$$f(x,y) = \sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij} x^i y^j \qquad D = \mathbb{R}^2 \text{ (domain)}$$

2. <u>Rational functions</u>

$$f(x,y) = \frac{g(x,y)}{h(x,y)}$$ $h(x,y), g(x,y)$ polynomials functions, $h(x,y)$ not identically 0

domain $D = \{(x,y) \in \mathbb{R}^2 \mid h(x,y) \neq 0\}$
$f(x,y)$ is continuous at points of D

3.
$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \end{cases}$$

domain $D = \mathbb{R}^2$

For $(a,b) \neq (0,0)$ $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

because $f(x,y)$ restricted to $\mathbb{R}^2 - \{(0,0)\}$ is a rational function

$$\lim_{(x,y)\to(0,0)} f(x,y) \overset{\checkmark \text{ previously shown}}{=} 0 = f(0,0)$$

So $f(x,y)$ is also continuous at $(0,0)$.