14.3 PARTIAL DERIVATIVES

The partial derivatives of $f(x,y)$ at (a,b) in the domain of f are

$$f_x(a,b) = \lim_{t\to 0} \frac{f(a+t,b) - f(a,b)}{t}$$ partial derivative with respect to x

$$f_y(a,b) = \lim_{t\to 0} \frac{f(a,b+t) - f(a,b)}{t}$$ partial derivative with respect to y

Let $g(x) := f(x,b)$ $\quad g'(a) = \lim_{t\to 0} \frac{g(a+t) - g(a)}{t} = f_x(a,b)$

Let $h(y) := f(a,y)$ $\quad h'(b) = \lim_{t\to 0} \frac{h(b+t) - h(b)}{t} = f_y(a,b)$

RULE:

1. To calulate f_x regard y as a constant and differentiate $f(x,y)$ with respect to x the usual way
2. To calculate f_y regard x as a constant and differentiate $f(x,y)$ with respect to y the usual way

ALTERNATE NOTATION. Let $z = f(x,y)$

$$f_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$$ also $D_1 f$ or $D_x f$

first of two variables

$$f_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}$$ also $D_2 f$ or $D_y f$

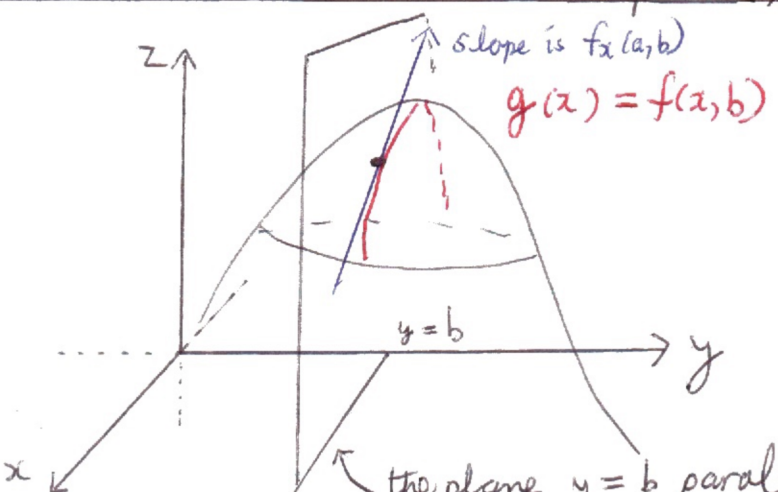
symbol "∂" is called a "del"

<u>EXAMPLE</u> $f(x,y) = \cos(x^2y^3)$

$$\frac{\partial f}{\partial x} = -\sin(x^2y^3)\,\frac{\partial}{\partial x}(x^2y^3) = -2xy^3\sin(x^2y^3)$$

$$\frac{\partial f}{\partial y} = -\sin(x^2y^3)\,\frac{\partial}{\partial y}(x^2y^3) = -3x^2y^2\sin(x^2y^3)$$

GEOMETRIC INTERPRETATION of f_x, f_y



the plane $y=b$ parallel to the xz-coordinate plane

$f_x(a,b) = g'(a) =$ slope of tangent to $g(x)=f(x,b)$ at $x=a$

$f_y(a,b) = h'(b) =$ slope of tangent to $h(y) = f(a,b)$ at $y=b$

take slice at $x=a$, parallel to the yz-coordinate plane

IMPLICIT DIFFERENTIATION

$x^3 + y^3 + z^3 + xyz = 1$ implicitly defines z as a function of x and y

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} + \left(yz + xy\frac{\partial z}{\partial x}\right) = 0$$ from product rule.

so $\dfrac{\partial z}{\partial x} = \dfrac{-3x^2 - yz}{3z^2 + xy}$, Similarly $\dfrac{\partial z}{\partial y} = \dfrac{-3y^2 - xz}{3z^2 + xy}$

Functions of more than two variables

$w = f(x_1, \ldots, x_n)$ $f_{x_1}, \ldots, f_{x_n}$ partial derivatives

$$f_{x_i} = \frac{\partial f}{\partial x_i} = \frac{\partial w}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

(also denoted $D_{x_i} f$, $D_i f$)

differentiate the usual way with respect to x_i regarding the other variables as constants

EXAMPLE $f(x,y,z) = x e^{yz}$

$$\frac{\partial f}{\partial x} = e^{yz}$$

$$\frac{\partial f}{\partial y} = x \frac{\partial}{\partial y} e^{yz} = x e^{yz} \left(\frac{\partial}{\partial y} yz \right) = xz e^{yz}$$

$$\frac{\partial f}{\partial z} = x \frac{\partial}{\partial z} e^{yz} = xy e^{yz}$$

EXAMPLE $f(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j^2 = x_1^2 + \cdots + x_n^2$

$$\frac{\partial f}{\partial x_i} = 2x_i \qquad i = 1, \ldots, n$$

Higher derivatives

$f(x,y)$ has four kinds of 2nd derivatives

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

f_{xy}: left to right; $\frac{\partial^2 f}{\partial y \partial x}$: right to left

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

f_{xy} and f_{yx}: mixed partial derivative

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

EXAMPLE $\quad f(x,y) = x^5 + 3x^2y^3 + y^7$

$f_x = 5x^4 + 6xy^3, \qquad f_y = 9x^2y^2 + 7y^6$

$f_{xx} = 20x^3 + 6y^3, \qquad f_{yy} = 18x^2y + 42y^5$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(5x^4 + 6xy^3\right) = 18xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(9x^2y^2 + 7y^6\right) = 18xy^2$$

Note that $f_{xy} = f_{yx}$. This is not a coincidence but rather a consequence of <u>Clairaut's Theorem</u> which we will discuss in the next lecture