IAA3 Lecture 03-19-2020

Clairaut 1713–1765

Clairaut's Theorem Let $U = \{(x,y) \mid (x-a)^2 + (y-b)^2 < r^2\}$, the open disk of radius r centered at (a,b).
If $f(x,y)$ is a real valued function defined on U such that f_{xy} and f_{yx} both exist and are continuous on U then $f_{xy} = f_{yx}$ on U. [or say U is an open set]

Take-away. For most functions encountered in practice $f_{xy} = f_{yx}$ (may not be equally easy to calculate)
e.g., $f(x,y) = h(x) + y^2$, $h(x)$ complicated

3rd and higher derivatives.

$f(x,y)$ has eight 3rd derivatives

$$f_{xxx}, \underbrace{f_{xxy}, f_{xyx}, f_{yxx}}_{\text{same by Clairaut}}, \underbrace{f_{yyx}, f_{yxy}, f_{xyy}}_{\text{same by Clairaut}}, f_{yyy}$$

$f(x,y) = x^2 y^3$ $\quad f_x = 2xy^3, \quad f_y = 3x^2y^2$

$f_{xx} = 2y^3, \quad f_{yy} = 6x^2y, \quad f_{xy} = f_{yx} = 6xy^2$

$f_{xxx} = 0, \quad f_{yyy} = 6x^2$

$f_{xxy} = f_{xyx} = f_{yxx} = 6y^2$

$f_{yyx} = f_{yxy} = f_{xyy} = 12xy$

A <u>partial differential equation</u> (PDE) is an equation involving a function of several variables and its partial derivatives.

<u>Laplace's equation</u> $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ a second order equation

Find all $f(x,y)$ satisfying equation and perhaps additional conditions, e.g. boundary conditions

(i) $f(x,y) = x^2 - y^2$ $f_x = 2x$, $f_y = -2y$, $f_{xx} = 2$, $f_{yy} = -2$

$= \mathrm{Re}(z^2)$ $f_{xx} + f_{yy} = 2 - 2 = 0$

(ii) $f(x,y) = x^3 - 3xy^2$ another solution

$= \mathrm{Re}(z^3)$

(iii) $f(x,y) = e^x \cos y$ another solution

$= \mathrm{Re}(e^z)$

<u>Wave Equation</u> $f(x,t)$ x space variable, t time

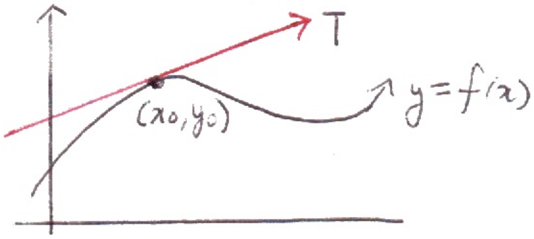
$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

$f(x,y) = \sin(x+ct)$ is a solution.

<u>Heat Equation</u> $\frac{\partial f}{\partial t} = k\frac{\partial^2 f}{\partial x^2}$ where $k > 0$.

14.4

recall the equation for a tangent line to a curve



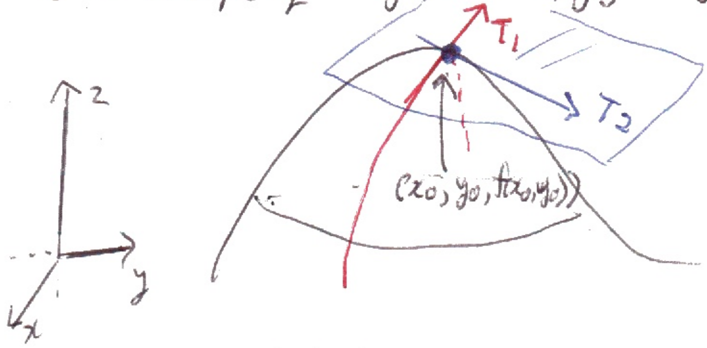
slope of $T = f'(x_0)$
equation for T:
$y = f(x_0) + f'(x_0)(x-x_0)$
$= T_1(x)$ (Taylor polynomial)

Let $P(x_0,y_0,z_0)$ be a point in space. A plane passing through P has an equation of the form
$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0 \quad (A, B, C \text{ not all zero})$$

For a plane not parallel to the z-axis, we have $C \neq 0$ and so we can solve the above equation for z
$$z = a(x-x_0) + b(y-y_0) + z_0 \quad \text{where } a = -A/C,\ b = -B/C$$
Note. a = slope of xz-slice, b = slope of yz-slice
Consider the graph of $f(x,y)$ near $(x_0,y_0) \in$ domain of f

slicing parallel to the xz-plane (through $(0,y_0,0)$) we see
a = slope of $g(x) = f(x,y_0)$ at x_0 $= f_x(x_0,y_0)$

slicing parallel to the yz-plane (through $(x_0,0,0)$ we see
b = slope of $h(y) = f(x_0,y)$ at y_0 $= f_y(x_0,y_0)$



The tangent plane at $(x_0,y_0,f(x_0,y_0))$

slope of T_1 is $f_x(x_0,y_0)$
slope of T_2 is $f_y(x_0,y_0)$

So the equation for the tangent plane to the graph of $f(x,y)$ at $(x_0, y_0, f(x_0,y_0))$ is

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$$

EXAMPLE Find an equation for the tangent plane to $z = 2x^2 - y^3$ at $(1,1,1)$ (note that $1 = 2\cdot1^2 - 1^3$)

$f(x,y) = 2x^2 - y^3$ $\quad f_x = 4x, \quad f_y = -3y^2$

$\qquad\qquad\qquad\qquad f_x(1,1) = 4, \quad f_y(1,1) = -3$

$$\begin{aligned} z &= 4(x-1) + (-3)(y-1) + 1 \\ &= 4x - 3y. \end{aligned}$$

WARNING an example of a badly behaved function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

The graph of this function is very jagged near $(0,0)$ and there is no tangent plane at $(x,y) = (0,0)$. Note $f(x,y)$ is not continuous at $(0,0)$. Nevertheless, f_x and f_y both exist at $(0,0)$ and $f_x(0,0) = f_y(0,0) = 0$.

Sufficient condition for the existence of a tangent plane.
If f_x and f_y exist at and near (x_0,y_0) and both are continuous at and near (x_0,y_0) then f has a tangent plane at (x_0,y_0) given by $z = f_x(x_0,y_0)(x-x_0) + f_y(y-y_0) + f(x_0,y_0)$