Recall that if the graph of $f(x,y)$ has a tangent plane at (a,b) in its domain, then the equation for this plane (which passes through $(a,b,f(a,b))$) is

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

WARNING An example of a badly behaved function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

The graph of this function is very jagged near $(0,0)$ (think Jagged Mountain or Leviathan Peak in Colorado) and there is no tangent plane at $(0,0)$. Nevertheless, f_x and f_y both exist at $(0,0)$ and $f_x(0,0) = f_y(0,0) = 0$.

A sufficient condition for the existence of a tangent plane

If f_x and f_y exist at and near (a,b) and both are continuous at and near (a,b) then $f(x,y)$ has a tangent plane at (a,b) given by

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Note. This sufficient condition is not a necessary condition

For a function $f(x,y)$ and (a,b) in its domain, define the <u>linearization of f at (a,b)</u> to be the function

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Note that $z = L(x,y)$ is the equation of the tangent plane to f at (a,b)

<u>Idea</u> $f(x,y) \approx L(x,y)$ for (x,y) <u>near</u> (a,b)

<u>EXAMPLE</u> Use the linearization of $f(x,y) = (x^2+y^2)^{1/2}$ to estimate $f(3.01, 4.02)$.

<u>linearize at (3,4)</u> $f(3,4) = (3^2+4^2)^{1/2} = 5$

$f_x = \frac{\partial}{\partial x}(x^2+y^2)^{1/2} = 1/2\,(x^2+y^2)^{-1/2}\frac{\partial}{\partial x}(x^2+y^2) = x\,(x^2+y^2)^{-1/2}$

$f_x(3,4) = \frac{3}{5} = 0.6$

$f_y = y\,(x^2+y^2)^{-1/2}$, $f_y(3,4) = \frac{4}{5} = 0.8$

$L(x,y) = 0.6\,(x-3) + 0.8\,(y-4) + 5$

$f(x,y) \approx L(x,y)$ near $(3,4)$ and so

$$\begin{aligned} f(3.01, 4.02) &\approx L(3.01, 4.02) \\ &= 0.6\,(0.01) + 0.8\,(0.02) + 5 \\ &= 5.022 \end{aligned}$$

a calculator gives $f(3.01, 4.02) = 5.02200159$ so the approximation is good to 5 decimal places

Let $f(x_1,\dots,x_n)$ be a function of n-variables

The linearization of f at $(a_1,\dots,a_n)$ in the domain of f is the function

$$L(x_1,\dots,x_n) \doteq \sum_{j=1}^{n} f_{x_j}(a_1,\dots,a_n)(x_j-a_j) \quad + f(a_1,\dots,a_n)$$

$w = L(x_1,\dots,x_n)$ is the equation for the tangent hyperplane to f at $(a_1,\dots,a_n)$

EXAMPLE. $f(x,y,z) = x\,e^y \cos(z)$
The linearization of f at $(1,0,0)$ is:

$$f_x = e^y \cos(z) \qquad f_x(1,0,0) = e^0\cos(0) = 1$$

$$f_y = x\,e^y\cos(z) \qquad f_y(1,0,0) = 1\cdot e^0\cos(0) = 1$$

$$f_z = x\,e^y(-\sin(z)) \qquad f_z(1,0,0) = 1\cdot e^0(-\sin(0)) = 0$$

$$L(x,y,z) = f_x(1,0,0)(x-1) + f_y(1,0,0)(y-0) + f_z(1,0,0)(z-0) + f(1,0,0)$$

$$= 1(x-1) + 1(y-0) + 0(z-0) + 1$$
$$= x+y$$

EXAMPLE $f(x,y,z) = x^2y^3z^4$ at $(1,1,1)$

$$L(x,y,z) = 2(x-1) + 3(y-1) + 4(z-1) + 1 = 2x+3y+4z-8.$$

<u>Idea:</u> Let L be the linearization of $f(x_1, \ldots, x_n)$ at $(a_1, \ldots, a_n)$. Then

$$f(x_1, \ldots, x_n) \approx L(x_1, \ldots, x_n) \text{ \underline{near} } (a_1, \ldots, a_n)$$

When is this valid?

f is <u>differentiable at $(a_1, \ldots, a_n)$</u> if

$$\lim_{\vec{h} \to \vec{0}} \frac{|f(\vec{a}+\vec{h}) - L(\vec{a}+\vec{h})|}{\|\vec{h}\|} = 0$$

here $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{h} = (h_1, \ldots, h_n)$

and $\|\vec{h}\| = (h_1^2 + \cdots + h_n^2)^{1/2}$

This is precisely the condition that assures the existence of the tangent hyperplane.

<u>differentials for function of several variables</u>

For us, a formal way of dealing with linear approximation :

$$z = f(x,y)$$

$$dz := f_x\,dx + f_y\,dy$$

dx, dy viewed as independent variables. "1-forms" in advanced math

<u>EXAMPLE</u> $z = f(x,y) = x^2y - 3y^2$

$f_x = 2xy \qquad f_y = x^2 - 6y$

$$dz = 2xy\,dx + (x^2 - 6y)\,dy$$

$$w = f(x_1, \dots, x_n). \qquad dw := \sum_{j=1}^{n} f_{x_j}\,dx_j$$

<u>EXAMPLE</u> $w = f(x_1, \dots, x_n) = x_1^2 + \cdots + x_n^2$, $\quad \frac{\partial f}{\partial x_j} = 2x_j$

$$dw = \sum_{j=1}^{n} 2x_j\,dx_j$$

$$\Delta w := f(x_1 + \Delta x_1, \dots, x_n + \Delta x_n) - f(x_1, \dots, x_n)$$

<u>increment</u> of w

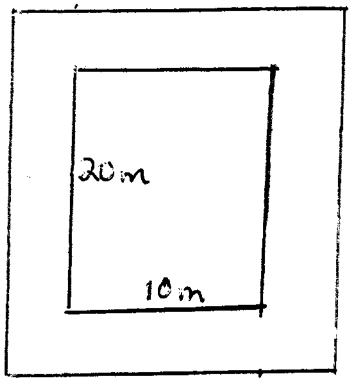
Idea: $\quad \Delta w \approx dw$ for $dx_j = \Delta x_j$, Δx_j small

$$\Delta w \approx L(x_1 + \Delta x_1, \dots, x_n + \Delta x_n) - f(x_1, \dots, x_n)$$

$$= \sum_{j=1}^{n} f_{x_j}\,\Delta x_j = dw \quad (\text{where } dx_j = \Delta x_j)$$

note $L(x_1 + \Delta x_1, \dots, x_n + \Delta x_n) = f(x_1, \dots, x_n) + \sum f_{x_j}\,\Delta x_j$

not to scale



billboard 10 m x 20 m

boundary strip is 5 cm wide
(5 cm = 0.05 m)

Use differentials to estimate the area of the strip

area $A = xy$ x = width, y = height

$\Delta x = 0.05 + 0.05 = 0.1$, $\Delta y = 0.05 + 0.05 = 0.1$

$$dA = \frac{\partial A}{\partial x}\, dx + \frac{\partial A}{\partial y}\, dy = y\,dx + x\,dy$$

$dx = \Delta x = 0.1$, $dy = \Delta y = 0.1$

$$dA = 20(0.1) + 10(0.1) = 3\ m^2$$

<u>comparison with exact value</u>

$$(10+0.1)(20+0.1) - (10)(20) = 3.01\ m^2$$