# 1AA3 Lecture

03-24-2020

The <u>Chain Rule</u> (1-variable)

(CR) $\frac{d}{dt} f(g(t)) = f'(g(t))\, g'(t)$

$y = f(x), \quad x = g(t)$

(CR) can be written as $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

<u>EXAMPLE.</u> $\frac{d}{dt}\cos(e^t) = -\sin(e^t)\frac{d}{dt}e^t = -\sin(e^t)e^t$

## <u>Chain Rule</u> (2-variables) first case

$z = f(x,y), \quad x = g(t), \quad y = h(t)$

<u>Technical condition on f.</u> We require $f(x,y)$ to be <u>differentiable as</u> function of 2 variables

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f(x+h_1, y+h_2) - (f_x(x,y)h_1 + f_y(x,y)h_2 + f(x,y))|}{(h_1^2+h_2^2)^{1/2}} = 0$$

<u>Take-away:</u> f has a tangent plane at $(x,y)$ approximating f

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

<u>EXAMPLE</u> $z = f(x,y) = x^3y^5 \quad x = \cos(t),\ y = \sin(t)$

$dx/dt = -\sin(t),\ dy/dt = \cos(t)$

$dz/dt = (\partial z/\partial x)(dx/dt) + (\partial z/\partial y)(dy/dt) =$

$= 3x^2y^5(-\sin(t)) + 5x^3y^4(\cos(t)) = -3\cos^2 t\sin^6 t + 5\cos^4 t\sin^4 t$

## Chain Rule (2-variables) second case

$z = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

## Chain Rule general case

$u(x_1, \dots, x_n)$ function of $n$ variables

each $x_j(t_1, \dots, t_m)$ function of $m$ variables $t_1, \dots, t_m$

$$\frac{\partial u}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial t_j} \qquad j = 1, \dots, m$$

## Matrix formulation

$$\begin{bmatrix} \frac{\partial u}{\partial t_1} & \cdots & \frac{\partial u}{\partial t_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_m} \\ \vdots & & \\ \frac{\partial x_n}{\partial t_1} & & \frac{\partial x_n}{\partial t_m} \end{bmatrix}$$

size $1 \times m$ size $1 \times n$ size $n \times m$

EXAMPLE Let $u(s,t) = f(s^2-t^2,\ t^2-s^2)$
where $f(x,y)$ is any differentiable function

Show that $u$ satisfies the PDE

$$t\frac{\partial u}{\partial s} + s\frac{\partial u}{\partial t} = 0$$

let $x = s^2-t^2$, $y = t^2-s^2$

① $$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}2s + \frac{\partial f}{\partial y}(-2s)$$

② $$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

$t$ ① $+$ $s$ ② $=$ cancellation of the terms on the right of $t$① and $s$②

$= 0$

## Implicit differentiation

The equation $F(x,y) = C$ (a constant) can be solved in principle (but maybe not explicitly) under reasonable conditions (The Implicit Function Theorem) for $y$ as a function of $x$, $y = y(x)$

$$\underbrace{\frac{d}{dx} F(x, y(x))}_{\frac{\partial F}{\partial x} \underbrace{\frac{dx}{dx}}_{1} + \frac{\partial F}{\partial y}\frac{dy}{dx}} = \frac{d}{dx} C = 0$$

by the 2-variable Chain Rule

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}$$

$$F(x,y) = y^5 - 3x^2y^3 + x^5 = 0$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = -\frac{(-6xy^3 + 5x^4)}{5y^4 - 9x^2y^2}$$

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$F(x,y,z) = C$, solve for $z$ as a function of $x$ and $y$, $z = z(x,y)$

$$\underbrace{\frac{\partial}{\partial x} F(x,y,z)} = \frac{\partial}{\partial x} C = 0$$

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \frac{-F_x}{F_z}$$ and similarly

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

$$F(x,y,z) = \sin(x+y) + \sin(x+z) + \sin(y+z) = 0$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(\cos(x+y) + \cos(x+z))}{\cos(x+z) + \cos(y+z)}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(\cos(x+y) + \cos(y+z))}{\cos(x+z) + \cos(y+z)}$$

<u>Implicit Function Theorem.</u> $F(x,y,z)$ function of 3-variables $F(a,b,c) = K$ and $F_z(a,b,c) \neq 0$. Assume $F_x, F_y, F_z$ are continuous near $(a,b,c)$. Then there exists a differentiable function $z(x,y)$ defined near $(a,b)$ such that $F(x,y,z(x,y)) = K$ for all $(x,y)$ near $(a,b)$.