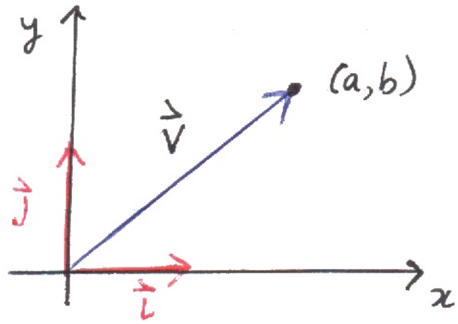
IAA3 Lecture

03-26-2020

Vectors in $\mathbb{R}^2$ (the plane)

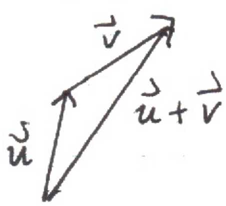
a *vector* represents a quantity with magnitude and direction



The vector $\vec{v}$ (pictured) has *components* (a,b) and we write $\vec{v}=(a,b)$

$\vec{i}=(1,0)$ and $\vec{j}=(0,1)$ are called the *standard unit basis vectors*. The *length* of $\vec{v}=(a,b)$ is $\|\vec{v}\| = (a^2+b^2)^{1/2}$. $\vec{v}$ is called a *unit vector* if $\|\vec{v}\|=1$, which we view as specifying a direction

Vector addition



$\vec{u}=(u_1,u_2)$, $\vec{v}=(v_1,v_2)$

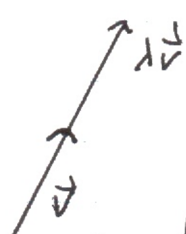
$$\vec{u}+\vec{v}=(u_1+v_1,\ u_2+v_2)$$

geometrically

algebraically

Scalar multiplication

λ a scalar

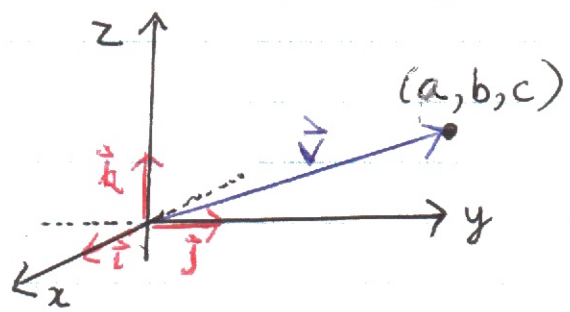


$$\vec{v}=(v_1,v_2)$$
$$\lambda\vec{v}=(\lambda v_1,\lambda v_2)$$

geometrically

algebraically

Vectors in $\mathbb{R}^3$ (space)



$\vec{v} = (a, b, c)$

$\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$

standard unit basis vectors

note that $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$

The <u>length</u> of $\vec{v} = (a, b, c)$ is $\|\vec{v}\| = (a^2 + b^2 + c^2)^{1/2}$

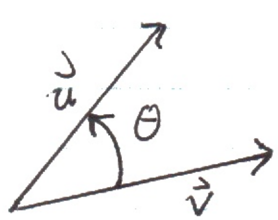
$\vec{v}$ is a <u>unit vector</u> if $\|\vec{v}\| = 1$

The Dot Product

$\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$

algebraically, $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ <u>dot product</u>

geometrically, $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$



θ is the angle between $\vec{u}$ and $\vec{v}$ (pictured)

The vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$

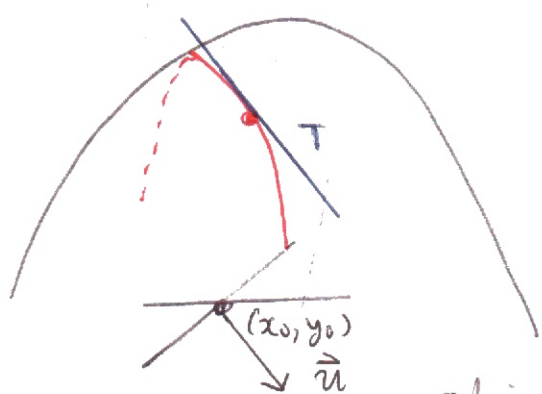
If $\vec{u}, \vec{v}$ are non-zero this occurs precisely when $\theta = \frac{\pi}{2}$ (a right angle)

We may view a function $f(x,y)$ as a function on vectors, that is, $f(\vec{v})$ where $\vec{v} = (x,y)$.

14.6 The <u>directional derivative</u> of $f(x,y)$ at (x_0,y_0) in the direction of the unit vector $\vec{u}$ is

$$D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f((x_0,y_0)+h\vec{u}) - f(x_0,y_0)}{h}$$

rate of change in the direction $\vec{u}$



slope of T is $D_{\vec{u}} f(x_0,y_0)$

slice in the direction of $\vec{u}$

Let $\vec{u} = (a,b)$ and
$$g(t) = f((x_0,y_0)+t\vec{u}) = f(x_0+ta,\ y_0+tb)$$

$$g'(0) = \lim_{h\to 0} \frac{g(h)-g(0)}{h} = \lim_{h\to 0} \frac{f((x_0,y_0)+h\vec{u}) - f(x_0,y_0)}{h} = D_{\vec{u}} f(x_0,y_0)$$

We may also calculate $g'(0)$ using the 2-variable chain rule

$$g'(0) = f_x(x_0,y_0)\underbrace{\frac{d}{dt}(x_0+ta)\Big|_{t=0}}_{a} + f_y(x_0,y_0)\underbrace{\frac{d}{dt}(y_0+tb)\Big|_{t=0}}_{b}$$

$$\boxed{\begin{aligned} &\vec{u} = (a, b), \quad f \text{ differentiable at } (x_0, y_0) \\ &D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b \end{aligned}}$$

Note that $D_{\vec{i}} f(x_0, y_0) = f_x(x_0, y_0)$, $D_{\vec{j}} f(x_0, y_0) = f_y(x_0, y_0)$

3 variables. $\vec{u} = (a, b, c)$ unit vector $f(x, y, z)$

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f((x_0, y_0, z_0) + h\vec{u}) - f(x_0, y_0, z_0)}{h}$$

$$D_{\vec{u}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)\, a + f_y(x_0, y_0, z_0)\, b + f_z(x_0, y_0, z_0)\, c$$

Gradient vector (2 dimensions), $f(x, y)$

$$\nabla f(x, y) := \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$$ gradient of f

∇ is "nabla"

$$D_{\vec{u}} f(x, y) = \nabla f(x, y) \bullet \vec{u}$$ (dot product)

Gradient vector (3 dimensions) $f(x, y, z)$

$$\nabla f(x, y, z) := \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

$$D_{\vec{u}} f(x, y, z) = \nabla f(x, y, z) \bullet \vec{u}$$

1AA3 Lecture

Theorem "Crazy mountain biker theorem"
Let f be a function of two or three variables. The maximum value of the directional derivative $D_{\vec{u}} f(\vec{x})$ is $\|\nabla f(\vec{x})\|$ occuring when $\vec{u}$ has the same direction as ∇f.

Proof $D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos\theta = \|\nabla f\| \cos\theta$
where θ is the angle between ∇f and $\vec{u}$. The maximum value of $\cos\theta$ is 1 occuring at $\theta = 0$. Hence the maximum value of $D_{\vec{u}} f$ is $\|\nabla f\|$ occuring when $\theta = 0$, that is, $\vec{u}$ is in the same direction as ∇f. So $\vec{u} = \nabla f / \|\nabla f\|$.

Example 1. Let $f(x,y) = xe^y$. At the point $(2,0)$, in what direction does f have the maximum rate of change? What is the maximum rate of change?

$$\nabla f = (f_x, f_y) = (e^y, xe^y), \quad \nabla f(2,0) = (e^0, 2e^0) = (1,2)$$

maximum rate of change is $\|\nabla f\| = (1^2+2^2)^{1/2} = \sqrt{5}$
direction of maximum change is $\vec{u} = \nabla f / \|\nabla f\| = (1/\sqrt{5}, 2/\sqrt{5})$

Example 2. $f(x,y,z) = (x^2+y^2+z^2)^{1/2}$ at $(3,6,-2)$

$$\nabla f(x,y,z) = \left(x(x^2+y^2+z^2)^{-1/2}, y(x^2+y^2+z^2)^{-1/2}, z(x^2+y^2+z^2)^{-1/2}\right)$$

Note that $\|\nabla f\| = 1$ for all (x,y,z).
at $(3,6,-2)$, $\nabla f = (3/7, 6/7, -2/7)$ (already a unit vector) is the direction of maximum change.