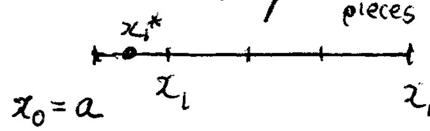
Stewart 15.1 – Double integrals over rectangles

Review of the definite integral for functions of one variable

$f(x)$ defined on $[a, b] = \{x \mid a \le x \le b\}$. Divide $[a, b]$ into n equal pieces of length $\Delta x = (b-a)/n$



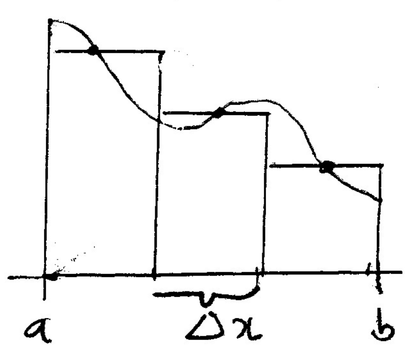
$[x_{i-1}, x_i]$ $i = 1, \dots, n$

choose sample points $x_i^* \in [x_{i-1}, x_i]$ and form the Riemann sum $\sum_{i=1}^{n} f(x_i^*) \Delta x$

DEFINITE INTEGRAL $\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$

If f is continuous then this limit exists and is independent of the choice of sample points. More generally, it is still OK if f is bounded and continuous except perhaps at finitely many points in $[a, b]$.

If $f(x) \ge 0$ then $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b and the Riemann sums are sums of areas of approximating rectangles.



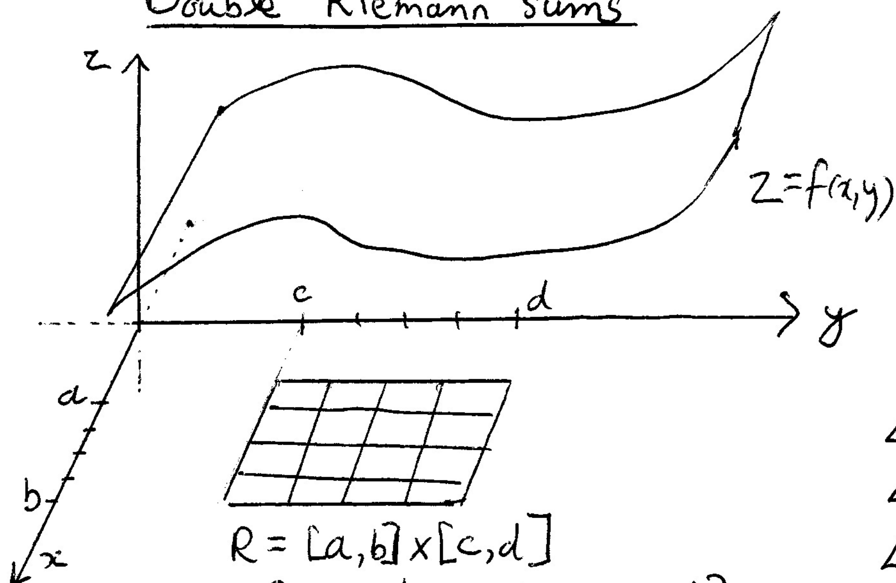
$n=3$ in this picture so

$\Delta x = (b-a)/3$

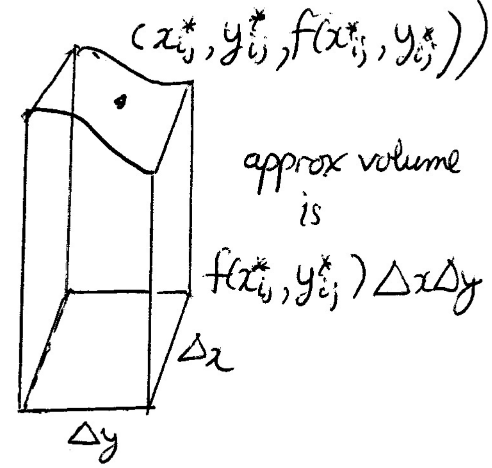
1AA3 Lecture

03-30-2020

Double Riemann sums



$R = [a,b] \times [c,d]$
$= \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$



approx volume is $f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$

$\Delta x = (b-a)/m$
$\Delta y = (d-c)/n$
$\Delta A = (\Delta x)(\Delta y)$ are of the base rectangle

If $f(x,y) \ge 0$ the volume under $f(x,y)$ and over R is approximately

$$Vol \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \quad \text{a double Riemann sum}$$

$$\text{exact volume} = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

In general, if $f(x,y)$ is a function then the double integral of f over R is

$$\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

provided the limit exists in which case we say f is <u>integrable</u>
Continuous functions are integrable. More generally, bounded function for which the set of discontinuities is "small" (in a sense to be described later) are integrable
The abbreviated notation $\iint_R f$ is sometimes used for $\iint_R f(x,y)\,dA$

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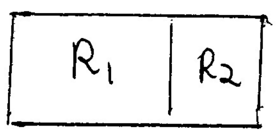
General properties of double integrals that one can see from the definition as a limit of Riemann sums

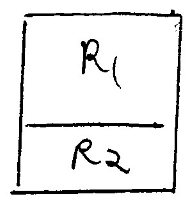
(1) $$\iint_R f(x,y) + g(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

(2) $$\iint_R c f(x,y)\,dA = c\iint_R f(x,y)\,dA \qquad c \text{ a constant}$$

(3) $$f(x,y) \geq g(x,y) \text{ in } R \Rightarrow \iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

(4) 

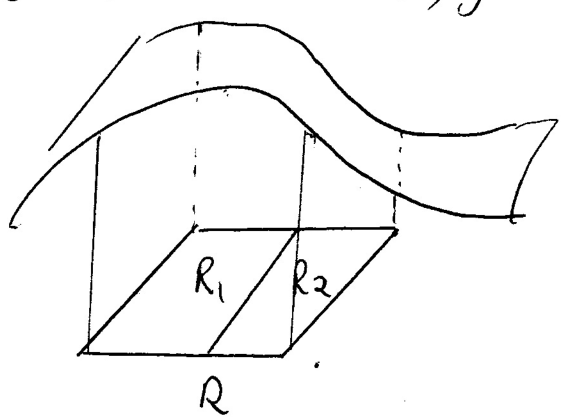
or



$R = R_1 \cup R_2$

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

We believe these formulas (which can be proved mathematically) because in the case $f, g \geq 0$ "volumes add".



Visualization of Property (4)

Volume over R = Volume over R_1 + Volume over R_2