1AA3 Lecture

03-31-2020

<u>Last time:</u> $R = [a,b] \times [c,d]$, $R \xrightarrow{f} \mathbb{R}$

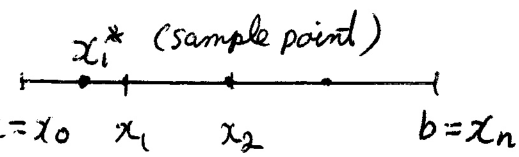
$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A \quad \text{(limit of double Riemann sums)}$$

f is <u>integrable</u> if this limit exists.

important application to probability and statistics: <u>AVERAGES</u>

<u>one variable warm-up</u> Recall the <u>average</u> of $[a,b] \xrightarrow{f} \mathbb{R}$ is

$$f_{av} = \lim_{n\to\infty} \frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$



$$= \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \quad \text{where } \Delta x = (b-a)/n$$

$$= \frac{1}{b-a} \int_a^b f(x)\,dx$$

<u>Note</u> $b-a$ is the length of $[a,b]$

We can do something similar with a function $R = [a,b] \times [c,d] \xrightarrow{f} \mathbb{R}$

The <u>average</u> of $f(x,y)$ over R is

$$f_{av} = \frac{1}{\text{area}(R)} \iint_R f(x,y)\,dA$$

note that $\text{area}(R) = (b-a)(d-c)$.

Let $f(x,y)$ be a function defined on the rectangle $[a,b]\times[c,d] = \{(x,y) \mid a \le x \le b,\quad c \le y \le d\}$

<u>Partial integration</u> $\quad \int_c^d f(x,y)\,dy$

Hold x fixed and integrate the corresponding function of y in the usual way.

<u>EXAMPLE</u> $\quad \int_0^1 2xy\,dy = xy^2\Big|_{y=0}^{y=1} = x(1)^2 - x(0)^2 = x$

Observe that $\frac{\partial}{\partial y}\, xy^2 = 2xy$

Let $A(x) := \int_c^d f(x,y)\,dy$, for $a \le x \le b$, yielding a function of x

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right]dx \quad \underline{\text{Iterated Integral}}$$

We can also partially integrate w.r.t. x first (holding y constant) and then integrate the resulting function of y

$$\int_c^d \left[\int_a^b f(x,y)\,dx\right]dy$$

$$\int_0^2 \int_0^1 2xy\,dy\,dx = \int_0^2 x\,dx = \frac{x^2}{2}\Big|_0^2 = 2$$

$$\int_0^1 \int_0^2 2xy\,dx\,dy = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2$$

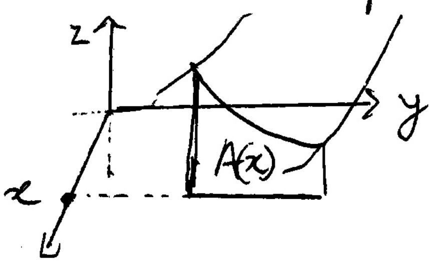
SAME!

<u>Fubini's Theorem</u> If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$

then $\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$.

More generally, this is true if $f(x,y)$ is bounded on R and the <u>exceptional set</u> $E=\{(x,y)\in R \mid f \text{ is } \underline{\text{not}} \text{ continuous at } (x,y)\}$ is a finite union of smooth curves.

We won't attempt to prove this theorem, but observe that in the case $f(x,y)\geq 0$, the three expressions all represent the volume of the solid under $z=f(x,y)$ and over R



$A(x)=\int_c^d f(x,y)\,dy$ = area under $e(y)=f(x,y)$ fixed x

"Two ways to slice a loaf"

<u>EXAMPLE</u> $R=[1,2]\times[0,\pi]$. Evaluate $\iint\limits_R y\sin(xy)\,dA$

$$\iint\limits_R y\,\text{sen}(xy)\,dA = \int_0^\pi \int_1^2 y\sin(xy)\,dx\,dy \quad \text{by Fubini}$$

$$\int_1^2 y\sin(xy)\,dx = -\cos(xy)\Big|_{x=1}^{x=2} = -\cos(2y)+\cos(y)$$

$$\int_0^\pi [\quad '' \quad]\,dy = \int_0^\pi -\cos(2y)+\cos(y)\,dy$$

$$= -\tfrac{1}{2}\sin(2y)+\sin(y)\Big|_0^\pi = 0$$

Note that the integral $\int_0^\pi y\sin(xy)\,dy$ is harder to evaluate but can be done by integration by parts

A useful special case of Fubini's Theorem

$f(x,y) = U(x)V(y)$ on the domain $R = [a,b]\times[c,d]$

$$\iint_R f(x,y)\,dA = \left(\int_a^b U(x)\,dx\right)\left(\int_c^d V(y)\,dy\right)$$

Justification: By Fubini,

$$\iint_R f(x,y)\,dA = \int_a^b\left[\int_c^d U(x)V(y)\,dy\right]dx$$

$$\int_c^d U(x)V(y)\,dy = U(x)\int_c^d V(y)\,dy \quad \text{and so}$$

$$\int_a^b\left[\quad \downarrow \quad\right]dx = \int_a^b U(x)\left(\int_c^d V(y)\,dy\right)dx$$

$$= \left(\int_a^b U(x)\,dx\right)\left(\int_c^d V(y)\,dy\right)$$

EXAMPLE $R = [0,\pi/2]\times[0,\pi/2]$, $f(x,y) = (\cos x)(\sin y)$

$$\iint_R (\cos x)(\sin y)\,dA = \left(\int_0^{\pi/2}\cos(x)\,dx\right)\left(\int_0^{\pi/2}\sin(y)\,dy\right)$$

$$= \left(\sin(x)\Big|_{x=0}^{x=\pi/2}\right)\left(-\cos(y)\Big|_{y=0}^{y=\pi/2}\right) = 1\cdot 1 = 1$$

WARNING Most functions $f(x,y)$ are not of the form $U(x)V(y)$.

The fact that the domain R is a rectangle (as opposed to more complicated shape) is also important here.