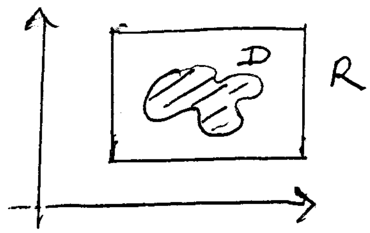
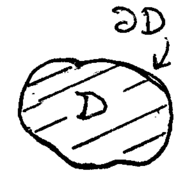
Integration over general regions



$f(x,y)$ whose domain is the bounded set $D \subset R = [a,b]\times[c,d]$

Define $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$

If $F(x,y)$ is integrable over R, we define

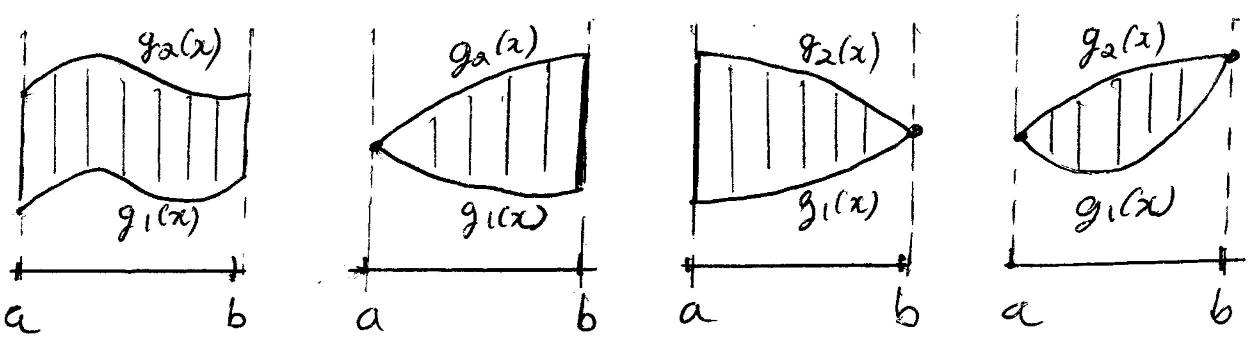


$$\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA$$

Typically, $F(x,y)$ will be discontinuous at points on ∂D (the <u>boundary</u> of D). However if f is continuous on D and ∂D consists of a finite union of smooth curves then $F(x,y)$ is integrable

Type I Regions

D lies between the graphs of two continuous functions of x, that is, $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$



THE FOUR KINDS OF TYPE I REGIONS

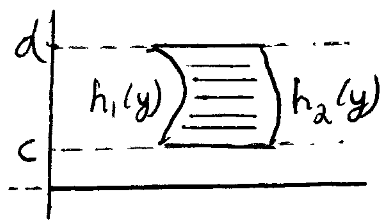
$$\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

because $F(x,y)=0$ if $y>g_2(x)$ or $y<g_1(x)$.

So for a Type I region $D=\{(x,y)\,|\,a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$ Fubini's Theorem gives

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

<u>Type II region</u>



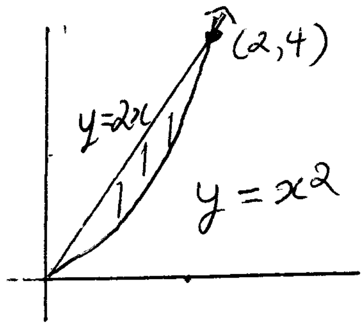
$$D=\{(x,y)\,|\,c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$$

For a Type II region

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

<u>Remark</u> If $f(x,y)\ge 0$ then $\iint_D f(x,y)\,dA$ can be interpretted as the volume of the solid lying under the graph of $z=f(x,y)$ and over the region D.

Also, $\iint_D 1\,dA$ = area of D



$D = \{(x,y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}$

expressed as a Type I region

EXAMPLE Calculate $\iint_D x^2+y^2\, dA$ (D as above)

$$\iint_D x^2+y^2\, dA = \int_0^2 \int_{x^2}^{2x} x^2+y^2\, dy\, dx$$

$$\int_{x^2}^{2x} x^2+y^2\, dy = x^2 y + \tfrac{1}{3} y^3 \Big|_{y=x^2}^{y=2x}$$

$$= \left(x^2(2x) + \tfrac{1}{3}(2x)^3\right) - \left(x^2(x^2) + \tfrac{1}{3}(x^2)^3\right)$$

$$= \frac{14}{3}x^3 - x^4 - \frac{1}{3}x^6$$

$$\int_0^2 \int_{x^2}^{2x} x^2+y^2\, dy\, dx = \int_0^2 \frac{14}{3}x^3 - x^4 - \frac{1}{3}x^6\, dx$$

$$= \frac{14}{12}x^4 - \frac{x^5}{5} - \frac{1}{21}x^7 \Big|_0^2 = \frac{216}{35}$$ (ANSWER)

The same region $D=\{(x,y) \mid 0\le x\le 2,\ x^2\le y\le 2x\}$ can also be described as a Type II region as follows

$$D=\{(x,y) \mid 0\le y\le 4,\ \ y/2\le x\le \sqrt{y}\},\ \text{so}$$

$$\iint_D x^2+y^2\,dA = \int_0^4 \int_{y/2}^{\sqrt{y}} x^2+y^2\,dx\ dy$$

Let $B(y)=\displaystyle\int_{y/2}^{\sqrt{y}} x^2+y^2\,dx = \frac{x^3}{3}+y^2x \Big|_{x=y/2}^{x=\sqrt{y}}$

$$= \frac{-13}{24}y^3 + y^{5/2} + \frac{1}{3}y^{3/2}$$

$$\int_0^4 B(y)\,dy = \int_0^4 \frac{-13}{24}y^3 + y^{5/2}+\frac{1}{3}y^{3/2}\,dy$$

$$= \frac{-13}{96}y^4 + \frac{2}{7}y^{7/2} + \frac{2}{15}y^{5/4}\Big|_0^4$$

$$= \frac{216}{35}$$

as predicted by Fubini's theorem which says in this case

$$\int_0^2\int_{x^2}^{2x} x^2+y^2\,dy\,dx$$

$$= \int_0^4\int_{y/2}^{\sqrt{y}} x^2+y^2\,dx\,dy$$

both equal to $\displaystyle\iint_D x^2+y^2\,dA$

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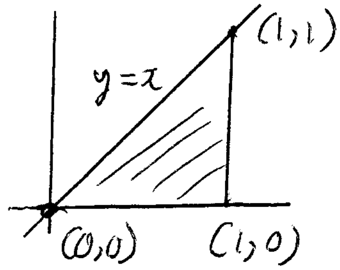

D = the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$

$$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\} \text{ as Type I}$$
$$= \{(x,y) \mid 0 \le y \le 1,\ y \le x \le 1\} \text{ as Type II}$$

Let $f(x,y) = e^{x^2}$. Evaluate $\iint_D f(x,y)\,dA$

$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^x e^{x^2}\,dy\ dx \quad (D \text{ as Type I})$$
$$= \int_0^1 x e^{x^2}\,dx$$
$$= \tfrac{1}{2} e^{x^2} \Big|_0^1 = \tfrac{1}{2}(e-1)$$

$$\iint_D f(x,y)\,dA = \int_0^1 \int_y^1 e^{x^2}\,dx\ dy \quad (D \text{ as Type II})$$

$\int_y^1 e^{x^2}\,dx$ is a <u>non-elementary</u> integral

It can still be evaluated using power series but this is much harder.
(So don't do it this way.)