1. Exercise 3.1.2 on Page 146:

The mgf of a random variable $X$ is $(\frac{2}{3} + \frac{1}{3}e^t)^9$. Show that

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = \sum_{x=1}^{5} \binom{9}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}.$$  

Answer:

Since the $M(t)$ of $X$ is $(\frac{2}{3} + \frac{1}{3}e^t)^9$, we have that $X$ is a binomial distribution with $n = 9$, $p = \frac{1}{3}$, $\mu = np = 3$, $\sigma^2 = np(1-p) = 2$. Therefore, $\mu - 2\sigma = 3 - 2\sqrt{2}, \mu + 2\sigma = 3 + 2\sqrt{2}$. Thus

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(X = 1, 2, 3, 4, 5) = \sum_{x=1}^{5} \binom{9}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}.$$  

2. Exercise 3.1.16 on Page 148:

Show that the moment generating function of the negative binomial distribution is $M(t) = p / [1 - (1-p)e^t]^r$. Find the mean and the variance of this distribution.

Hint: In the summation representing $M(t)$, make use of the Maclaurin’s series for $(1 - w)^{-r}$.

Answer:

Let $Y$ has a negative binomial distribution, so we have that

$$P(Y = y) = \binom{y + r - 1}{r - 1} p^r (1-p)^y, \quad y = 0, 1, 2, \ldots$$

$$M(t) = E e^{ty} = \sum_{y=0}^{\infty} \binom{y + r - 1}{r - 1} e^{ty} p^r (1-p)^y, \quad y = 0, 1, 2, \ldots$$

$$= p^r \sum_{y=0}^{\infty} \binom{y + r - 1}{y} [(1-p)e^t]^y, \quad y = 0, 1, 2, \ldots$$

$$= p^r [1 - (1-p)e^t]^{-r}$$

Because

$$(1 - w)^{-r} = \sum_{y=0}^{\infty} \binom{y + r - 1}{y} w^y,$$

where $w = (1-p)e^t$.

$$E(Y) = \frac{r(1-p)}{p}, \quad Var(Y) = \frac{r(1-p)}{p^2}.$$
3. Exercise 3.2.1 on Page 154:

If the random variable $X$ has a Poisson distribution such that $P(X = 1) = P(X = 2)$, find $P(X = 4)$.

Answer:
Since $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$, so

\[
e^{-\lambda} \frac{\lambda}{1!} = e^{-\lambda} \frac{\lambda^2}{2!}
\]

Then, we have that $\lambda = 2$.
Thus,

\[
P(X = 4) = \frac{2^4 e^{-2}}{4!} = \frac{2}{3} e^{-2}.
\]

4. Exercise 3.2.13 on Page 155:

Let $X$ and $Y$ have the joint pmf $p(x, y) = e^{-2}/[x!(y - x)!], y = 0, 1, 2, \ldots, x = 0, 1, \ldots, y$, zero elsewhere.

(a) Find the mgf $M(t_1, t_2)$ of this joint distribution.
(b) Compute the means, the variances, and the correlation coefficient of $X$ and $Y$.
(c) Determine the conditional mean $E(X|y)$.

Hint: Note that

\[
\sum_{x=0}^{y} \frac{[\exp(t_1 x)]y!}{[x!(y - x)!]} = [1 + \exp(t_1)]^y.
\]

Answer: for (a),

\[
M(t_1, t_2) = E^{(t_1 X + t_2 Y)}
= \sum_{y=0}^{\infty} \sum_{x=0}^{y} e^{(t_1 x + t_2 y)} \frac{e^{-2}}{x!(y - x)!}
= \sum_{y=0}^{\infty} \frac{e^{t_2 y} e^{-2}}{y!} \sum_{x=0}^{y} \frac{e^{t_1 x} y!}{x!(y - x)!}
= \sum_{y=0}^{\infty} \frac{e^{t_2 y} e^{-2}}{y!} (1 + e^{t_1})^y
= \sum_{y=0}^{\infty} \frac{[e^{t_2}(1 + e^{t_1})]^y e^{-2}}{y!}
= e^{-2} \exp\{(1 + e^{t_1}) e^{t_2}\}.
\]
For (b),
\[ E(X) = 1, \ E(Y) = 2, \ E(X^2) = 2, \ E(Y^2) = 6, \ Var(X) = 1, \ Var(Y) = 2, \ \rho = \frac{1}{\sqrt{2}}. \]

For (c),
\[
P(Y = y) = \sum_{x=0}^{y} \frac{e^{-2}}{x!(y-x)!} = \frac{e^{-2}y!}{y!} \sum_{x=0}^{y} \frac{y}{x!(y-x)!} = \frac{2ye^{-2}}{y!}
\]
\[
P(X|Y = y) = \frac{e^{-2}/[x!(y-x)!]}{(2ye^{-2})/y!} = \frac{y!}{x!(y-x)!} \cdot \frac{1}{2^y}
\]
\[
E(X|Y = y) = \sum_{x=0}^{y} x \cdot \frac{y!}{x!(y-x)!} \cdot \frac{1}{2^y} = \frac{y}{2^y} \sum_{x=1}^{y} \frac{(y-1)!}{(x-1)!(y-x)!} = \frac{y}{2}
\]

5. Exercise 3.3.24 on Page 166:

Let \(X_1, X_2\) be two independent random variables having gamma distributions with parameters \(\alpha_1 = 3, \beta_1 = 3\) and \(\alpha_2 = 5, \beta_2 = 1\), respectively.
(a) Find the mgf of \(Y = 2X_1 + 6X_2\).
(b) What is the distribution of \(Y\)?

Answer:
\[
f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot x^{\alpha-1}e^{-x/\beta}, \quad 0 < x < \infty.
\]

For (a), since
\[
Ee^{2tx} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{2tx} \frac{1}{\beta^\alpha} x^{\alpha-1}e^{-x/\beta} dx
\]
\[
= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} \left( \frac{\beta y}{1 - 2t\beta} \right)^{\alpha-1}e^{-y/\beta} \left( \frac{\beta}{1 - 2t\beta} \right) dy, \quad \text{(Let } y = \frac{x(1 - 2t\beta)}{\beta})
\]
\[
= \frac{1}{(1 - 2t\beta)^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1}e^{-y} dy
\]
\[
= \frac{1}{(1 - 2t\beta)^\alpha}
\]

Similarly, we obtain that
\[
Ee^{6tx} = \frac{1}{(1 - 6t\beta)^\alpha}
\]

So we have that
\[
M(t) = Ee^{t(2X_1+6X_2)} = Ee^{2tx_1} \cdot Ee^{6tx_2} = (1 - 6t)^{-8}, \quad (t < 1/6).
\]

For (b), by the mgf of \(Y\) in (a), we have that \(Y\) has a Gamma distribution and \(\alpha = 8, \ \beta = 6\).
6. Exercise 3.4.28 on Page 178:

Let \(X_1\) and \(X_2\) be independent with normal distributions \(N(6, 1)\) and \(N(7, 1)\), respectively. Find \(P(X_1 > X_2)\).

*Hint:* Write \(P(X_1 > X_2) = P(X_1 - X_2 > 0)\) and determine the distribution of \(X_1 - X_2\).

Answer: Let \(Y = X_1 - X_2\), since \(X_1\) and \(X_2\) are independent with normal distributions, we have that

\[
M_{X_1}(t) = Ee^{tX_1} = e^{(6t + \frac{1}{2}t^2)}, \quad M_{(-X_2)}(t) = Ee^{tX_2} = e^{(-7t + \frac{1}{2}t^2)},
\]

and

\[
M_Y(t) = Ee^{tY} = Ee^{tX_1}e^{-tX_2} = e^{(-t + t^2)}.
\]

Therefore, we obtain that \(Y\) has a normal distribution \(N(-1, 2)\). Thus,

\[
P(X_1 - X_2 > 0) = P(Y > 0) = 1 - \Phi(1/\sqrt{2}).
\]

7. Exercise 3.6.10 on Page 196:

Let \(T = W/\sqrt{V/r}\), where the independent variables \(W\) and \(V\) are, respectively, normal with mean zero and variance 1 and chi-square with \(r\) degrees of freedom. Show that \(T^2\) has an \(F\)-distribution with parameters \(r_1 = 1\) and \(r_2 = r\).

*Hint:* What is the distribution of the numerator of \(T^2\)?

Answer: Since \(T = W/\sqrt{V/r}\), we have that \(T^2 = W^2/(V/r) = (W^2/1)/(V/r)\). Moreover, \(W\) is \(N(0, 1)\), then we have that \(W^2\) is chi-square with 1 degrees of freedom. The variables \(W\) and \(V\) are independent, so \(W^2\) and \(V\) are independent. Thus, \(T^2\) is \(F\)-distribution with 1 and \(r\) degrees of freedom.

8. Exercise 3.6.13 on Page 196:

Let \(X_1, X_2\) be iid with common distribution having the pdf \(f(x) = e^{-x}, 0 < x < \infty\), zero elsewhere. Show that \(Z = X_1/X_2\) has an \(F\)-distribution.

Answer: This question I explained wrongly in the class, as you can not multiply the m.g.f. directly
as $X_1$ and $X_1$ itself are dependent. Therefore, we need to use the Jacobian:

$$\left| \frac{dx}{dy} \right| = \frac{1}{2}$$

Then, the p.d.f. is

$$g(y) = f\left(\frac{1}{2}y\right) \times \frac{1}{2} = \frac{1}{2}e^{-\frac{y}{2}}$$

Then correspondingly we can have the m.g.f.

$$M_{Y_i}(t) = (1 - 2t)^{-2}, \ (t < 1/2).$$

Or otherwise you do not use the m.g.f, just directly observe the p.d.f and recognise that $Y_i$ is chi-square with 2 degrees of freedom. Moreover, $X_1$ and $X_2$ are iid, so we have that

$$Z = \frac{X_1}{X_2} = \frac{2X_1/2}{2X_2/2} = \frac{Y_1}{Y_2}$$

Thus $Z$ has an $F$-distribution.