and since the other conditions in the theorem are satisfied, the derivative $f'(z)$ exists at each point where $f(z)$ is defined. The theorem tells us, moreover, that

$$f'(z) = e^{-i\theta}\left[\frac{1}{3(\sqrt[3]{r})^2}\cos\frac{\theta}{3} + i\frac{1}{3(\sqrt[3]{r})^2}\sin\frac{\theta}{3}\right],$$

or

$$f'(z) = \frac{e^{-i\theta}}{3(\sqrt[3]{r})^2}e^{i\theta/3} = \frac{1}{3(\sqrt[3]{r}e^{i\theta/3})^2} = \frac{1}{3[f(z)]^2}.$$

Note that when a specific point z is taken in the domain of definition of f, the value $f(z)$ is one value of $z^{1/3}$ (see Sec. 9). Hence this last expression for $f'(z)$ can be put in the form

$$\frac{d}{dz}z^{1/3} = \frac{1}{3(z^{1/3})^2}$$

when that value is taken. Derivatives of such power functions will be elaborated on in Chap. 3 (Sec. 33).

EXERCISES

1. Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if

(a) $f(z) = \overline{z}$; *(b)* $f(z) = z - \overline{z}$;
(c) $f(z) = 2x + ixy^2$; *(d)* $f(z) = e^x e^{-iy}$.

2. Use the theorem in Sec. 22 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when

(a) $f(z) = iz + 2$; *(b)* $f(z) = e^{-x}e^{-iy}$;
(c) $f(z) = z^3$; *(d)* $f(z) = \cos x \cosh y - i \sin x \sinh y$.

Ans. *(b)* $f''(z) = f(z)$; *(d)* $f''(z) = -f(z)$.

3. From results obtained in Secs. 21 and 22, determine where $f'(z)$ exists and find its value when

(a) $f(z) = 1/z$; *(b)* $f(z) = x^2 + iy^2$; *(c)* $f(z) = z\,\text{Im}\,z$.

Ans. *(a)* $f'(z) = -1/z^2$ $(z \neq 0)$; *(b)* $f'(x + ix) = 2x$; *(c)* $f'(0) = 0$.

4. Use the theorem in Sec. 23 to show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$:

(a) $f(z) = 1/z^4$ $\quad(z \neq 0)$;

(b) $f(z) = \sqrt{r}e^{i\theta/2}$ $\quad(r > 0, \alpha < \theta < \alpha + 2\pi)$;

(c) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$ $\quad(r > 0, 0 < \theta < 2\pi)$.

Ans. *(b)* $f'(z) = \dfrac{1}{2f(z)}$; *(c)* $f'(z) = i\dfrac{f(z)}{z}$.

5. Show that when $f(z) = x^3 + i(1 - y)^3$, it is legitimate to write

$$f'(z) = u_x + iv_x = 3x^2$$

only when $z = i$.

6. Let u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Verify that the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at the origin $z = (0, 0)$. [Compare with Exercise 9, Sec. 20, where it is shown that $f'(0)$ nevertheless fails to exist.]

7. Solve equations (2), Sec. 23 for u_x and u_y to show that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, \quad u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

Then use these equations and similar ones for v_x and v_y to show that in Sec. 23 equations (4) are satisfied at a point z_0 if equations (6) are satisfied there. Thus complete the verification that equations (6), Sec. 23, are the Cauchy–Riemann equations in polar form.

8. Let a function $f(z) = u + iv$ be differentiable at a nonzero point $z_0 = r_0 \exp(i\theta_0)$. Use the expressions for u_x and v_x found in Exercise 7, together with the polar form (6), Sec. 23, of the Cauchy–Riemann equations, to rewrite the expression

$$f'(z_0) = u_x + iv_x$$

in Sec. 22 as

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where u_r and v_r are to be evaluated at (r_0, θ_0).

9. *(a)* With the aid of the polar form (6), Sec. 23, of the Cauchy–Riemann equations, derive the alternative form

$$f'(z_0) = \frac{-i}{z_0}(u_\theta + iv_\theta)$$

of the expression for $f'(z_0)$ found in Exercise 8.

(b) Use the expression for $f'(z_0)$ in part *(a)* to show that the derivative of the function $f(z) = 1/z \ (z \neq 0)$ in Example 1, Sec. 23, is $f'(z) = -1/z^2$.

10. *(a)* Recall (Sec. 5) that if $z = x + iy$, then

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.$$