When n is a positive integer greater than 2, various mapping properties of the transformation $w = z^n$, or $w = r^n e^{in\theta}$, are similar to those of $w = z^2$. Such a transformation maps the entire z plane onto the entire w plane, where each nonzero point in the w plane is the image of n distinct points in the z plane. The circle $r = r_0$ is mapped onto the circle $\rho = r_0^n$; and the sector $r \le r_0, 0 \le \theta \le 2\pi/n$ is mapped onto the disk $\rho \le r_0^n$, but not in a one to one manner.

Other, but somewhat more involved, mappings by $w = z^2$ appear in Example 1, Sec. 97, and Exercises 1 through 4 of that section.

14. MAPPINGS BY THE EXPONENTIAL FUNCTION

In Chap. 3 we shall introduce and develop properties of a number of elementary functions which do not involve polynomials. That chapter will start with the exponential function

$$e^z = e^x e^{iy} \qquad (z = x + iy), \tag{1}$$

the two factors e^x and e^{iy} being well defined at this time (see Sec. 6). Note that definition (1), which can also be written

$$e^{x+iy} = e^x e^{iy},$$

is suggested by the familiar additive property

$$e^{x_1+x_2} = e^{x_1} e^{x_2}$$

of the exponential function in calculus.

The object of this section is to use the function e^z to provide the reader with additional examples of mappings that continue to be reasonably simple. We begin by examining the images of vertical and horizontal lines.

EXAMPLE 1. The transformation

$$w = e^z \tag{2}$$

can be written $w = e^x e^{iy}$, where $z = x + iy$, according to equation (1). Thus, if $w = \rho e^{i\phi}$, transformation (2) can be expressed in the form

$$\rho = e^x, \quad \phi = y. \tag{3}$$

The image of a typical point $z = (c_1, y)$ on a vertical line $x = c_1$ has polar coordinates $\rho = \exp c_1$ and $\phi = y$ in the w plane. That image moves counterclockwise around the circle shown in Fig. 20 as z moves up the line. The image of the line is evidently the entire circle; and each point on the circle is the image of an infinite number of points, spaced 2π units apart, along the line.

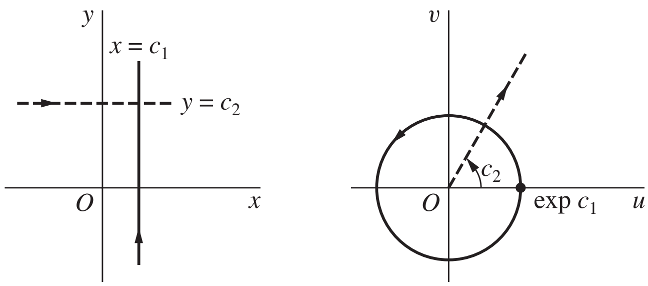

FIGURE 20
$w = \exp z$.

A horizontal line $y = c_2$ is mapped in a one to one manner onto the ray $\phi = c_2$. To see that this is so, we note that the image of a point $z = (x, c_2)$ has polar coordinates $\rho = e^x$ and $\phi = c_2$. Consequently, as that point z moves along the entire line from left to right, its image moves outward along the entire ray $\phi = c_2$, as indicated in Fig. 20.

Vertical and horizontal line *segments* are mapped onto portions of circles and rays, respectively, and images of various regions are readily obtained from observations made in Example 1. This is illustrated in the following example.

EXAMPLE 2. Let us show that the transformation $w = e^z$ maps the rectangular region $a \le x \le b, c \le y \le d$ onto the region $e^a \le \rho \le e^b, c \le \phi \le d$. The two regions and corresponding parts of their boundaries are indicated in Fig. 21. The vertical line segment AD is mapped onto the arc $\rho = e^a, c \le \phi \le d$, which is labeled $A'D'$. The images of vertical line segments to the right of AD and joining the horizontal parts of the boundary are larger arcs; eventually, the image of the line segment BC is the arc $\rho = e^b, c \le \phi \le d$, labeled $B'C'$. The mapping is one to one if $d - c < 2\pi$. In particular, if $c = 0$ and $d = \pi$, then $0 \le \phi \le \pi$; and the rectangular region is mapped onto half of a circular ring, as shown in Fig. 8, Appendix 2.

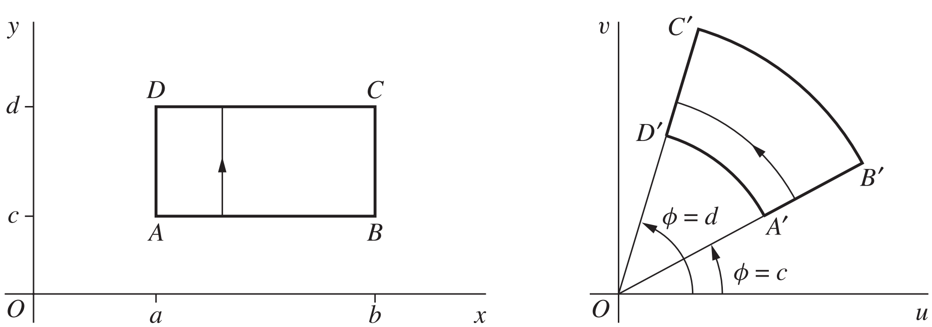

FIGURE 21
$w = \exp z$.

Our final example here uses the images of *horizontal* lines to find the image of a horizontal strip.

EXAMPLE 3. When $w = e^z$, the image of the infinite strip $0 \le y \le \pi$ is the upper half $v \ge 0$ of the w plane (Fig. 22). This is seen by recalling from Example 1 how a horizontal line $y = c$ is transformed into a ray $\phi = c$ from the origin. As the real number c increases from $c = 0$ to $c = \pi$, the y intercepts of the lines increase from 0 to π and the angles of inclination of the rays increase from $\phi = 0$ to $\phi = \pi$. This mapping is also shown in Fig. 6 of Appendix 2, where corresponding points on the boundaries of the two regions are indicated.

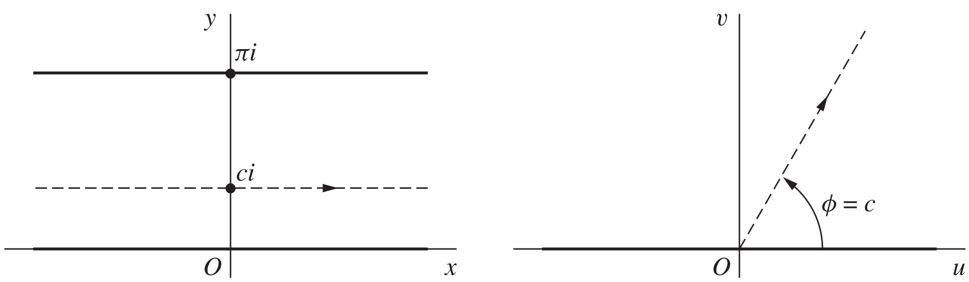

FIGURE 22
$w = \exp z$.

EXERCISES

1. By referring to Example 1 in Sec. 13, find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1, u = 2, v = 1$, and $v = 2$. (See Fig. 2, Appendix 2.)
2. Find and sketch, showing corresponding orientations, the images of the hyperbolas
$$x^2 - y^2 = c_1 \ (c_1 < 0) \quad \text{and} \quad 2xy = c_2 \ (c_2 < 0)$$
under the transformation $w = z^2$.
3. Sketch the region onto which the sector $r \le 1, 0 \le \theta \le \pi/4$ is mapped by the transformation *(a)* $w = z^2$; *(b)* $w = z^3$; *(c)* $w = z^4$.
4. Show that the lines $ay = x$ $(a \ne 0)$ are mapped onto the spirals $\rho = \exp(a\phi)$ under the transformation $w = \exp z$, where $w = \rho \exp(i\phi)$.
5. By considering the images of *horizontal* line segments, verify that the image of the rectangular region $a \le x \le b, c \le y \le d$ under the transformation $w = \exp z$ is the region $e^a \le \rho \le e^b, c \le \phi \le d$, as shown in Fig. 21 (Sec. 14).
6. Verify the mapping of the region and boundary shown in Fig. 7 of Appendix 2, where the transformation is $w = \exp z$.
7. Find the image of the semi-infinite strip $x \ge 0, 0 \le y \le \pi$ under the transformation $w = \exp z$, and label corresponding portions of the boundaries.

8. One interpretation of a function $w = f(z) = u(x, y) + iv(x, y)$ is that of a *vector field* in the domain of definition of f. The function assigns a vector w, with components $u(x, y)$ and $v(x, y)$, to each point z at which it is defined. Indicate graphically the vector fields represented by *(a)* $w = iz$; *(b)* $w = z/|z|$.

15. LIMITS

Let a function f be defined at all points z in some deleted neighborhood (Sec. 11) of z_0. The statement that the *limit* of $f(z)$ as z approaches z_0 is a number w_0, or that

$$\lim_{z \to z_0} f(z) = w_0, \tag{1}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. We now express the definition of limit in a precise and usable form.

Statement (1) means that for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{2}$$

Geometrically, this definition says that for each ε neighborhood $|w - w_0| < \varepsilon$ of w_0, there is a deleted δ neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε neighborhood (Fig. 23). Note that even though all points in the deleted neighborhood $0 < |z - z_0| < \delta$ are to be considered, their images need not fill up the entire neighborhood $|w - w_0| < \varepsilon$. If f has the constant value w_0, for instance, the image of z is always the center of that neighborhood. Note, too, that once a δ has been found, it can be replaced by any smaller positive number, such as $\delta/2$.

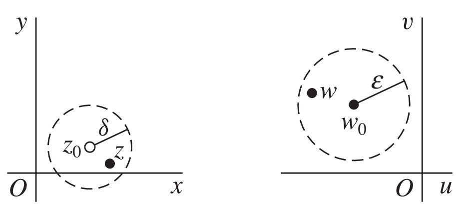

FIGURE 23

It is easy to show that *when a limit of a function $f(z)$ exists at a point z_0, it is unique*. To do this, we suppose that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_1.$$

Then, for each positive number ε, there are positive numbers δ_0 and δ_1 such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_0$$