3. Suppose that v is a harmonic conjugate of u in a domain D and also that u is a harmonic conjugate of v in D. Show how it follows that both $u(x, y)$ and $v(x, y)$ must be constant throughout D.

4. Use Theorem 2 in Sec. 26 to show that v is a harmonic conjugate of u in a domain D if and only if $-u$ is a harmonic conjugate of v in D. (Compare with the result obtained in Exercise 3.)

Suggestion: Observe that the function $f(z) = u(x, y) + iv(x, y)$ is analytic in D if and only if $-if(z)$ is analytic there.

5. Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be analytic in a domain D that does not include the origin. Using the Cauchy–Riemann equations in polar coordinates (Sec. 23) and assuming continuity of partial derivatives, show that throughout D the function $u(r, \theta)$ satisfies the partial differential equation

$$r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0,$$

which is the *polar form of Laplace's equation*. Show that the same is true of the function $v(r, \theta)$.

6. Verify that the function $u(r, \theta) = \ln r$ is harmonic in the domain $r > 0, 0 < \theta < 2\pi$ by showing that it satisfies the polar form of Laplace's equation, obtained in Exercise 5. Then use the technique in Example 5, Sec. 26, but involving the Cauchy–Riemann equations in polar form (Sec. 23), to derive the harmonic conjugate $v(r, \theta) = \theta$. (Compare with Exercise 6, Sec. 25.)