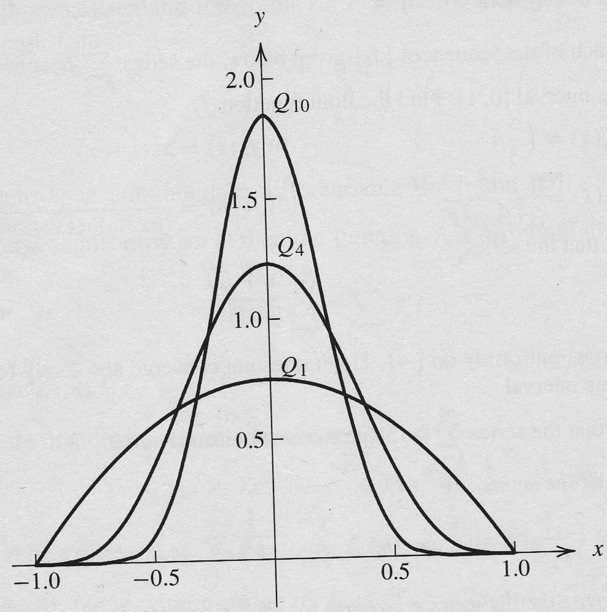

Figure 8.3 The graphs of Q_1, Q_4, and Q_{10}

counter to intuition and the proofs are a bit involved. When this happens, it is a good idea to slow down, read carefully, and think hard.

The first result gives an interesting relationship between continuous functions and polynomials, a particularly simple class of continuous functions. Let f be a continuous function on $[a, b]$ and let $\epsilon > 0$. Then there exists a polynomial P such that $|P(x) - f(x)| < \epsilon$ for all $x \in [a, b]$. There are no restrictions on f other than that of continuity. Consequently, every continuous function on a closed and bounded interval is the uniform limit of a sequence of polynomials. There are several ways to prove this result. One method is presented below. A few of the computations in the proof will be left to the reader.

THEOREM 8.11 Weierstrass Approximation Theorem If f is a continuous function on $[a, b]$, then there exists a sequence $\{P_n\}$ of polynomials such that $\{P_n\}$ converges uniformly to f on $[a, b]$.

Proof. We will prove the theorem for the case in which f is a continuous function on $[0, 1]$ with $f(0) = 0 = f(1)$. The proof that the general case follows from this result will be left as an exercise.

Extend the function f to $\mathbb{R}$ by letting $f(x) = 0$ for $x \in \mathbb{R} \setminus [0, 1]$. The function f is then continuous for all real numbers. For each positive integer n, let Q_n be the polynomial defined by $Q_n(x) = c_n(1 - x^2)^n$ where c_n is a constant chosen so that