$\int_{-1}^{1} Q_n = 1$. To estimate the magnitude of c_n, compute

$$\begin{aligned}\int_{-1}^{1} (1-x^2)^n\,dx &= 2\int_0^1 (1-x^2)^n\,dx \\ &\geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n\,dx \\ &\geq 2\int_0^{1/\sqrt{n}} (1-nx^2)\,dx \\ &= \frac{4}{3\sqrt{n}} \\ &> \frac{1}{\sqrt{n}}.\end{aligned}$$

This shows that $c_n < \sqrt{n}$ for all n. The graphs of several Q_n's are shown in Figure 8.3. Define a function $P_n\colon [0,1] \to \mathbb{R}$ by

$$P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)\,dt.$$

Noting that $f(t) = 0$ for all t outside of the interval $[0, 1]$ and making a substitution of variables in the integral yields

$$P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)\,dt = \int_0^1 f(t)Q_n(t-x)\,dt.$$

The expression $f(t)Q_n(t-x)$ can be written as a polynomial in x with coefficients that are functions of t. Integrating with respect to t yields a polynomial in x. Since the functions of t are independent of x, it follows that P_n is a polynomial in x. We will prove that the sequence $\{P_n\}$ converges uniformly to f on $[0, 1]$.

Let M be a bound for f and let $\epsilon > 0$. Since f is uniformly continuous on $\mathbb{R}$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon/2$ whenever $|y - x| \leq \delta$. For $x \in [\delta, 1]$,

$$Q_n(x) = c_n(1-x^2)^n < \sqrt{n}(1-\delta^2)^n.$$

It is not difficult to prove that the sequence $\{\sqrt{n}(1-\delta^2)^n\}$ converges to 0. By Theorem 8.3, the sequence $\{Q_n\}$ converges uniformly to 0 on $[\delta, 1]$. It then follows from Theorem 8.6 that $\lim_{n\to\infty} \int_\delta^1 Q_n = 0$. Consequently, there exists a positive integer N such that $\int_\delta^1 Q_n < \epsilon/(8M)$ for all $n \geq N$. If $x \in [0, 1]$ and $n \geq N$, then

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} f(x+t)Q_n(t)\,dt - \int_{-1}^{1} f(x)Q_n(t)\,dt \right| \\
&= \left| \int_{-1}^{1} \big(f(x+t) - f(x)\big) Q_n(t)\,dt \right| \\
&\leq \int_{-1}^{1} |f(x+t) - f(x)| Q_n(t)\,dt \\
&\leq \int_{-1}^{-\delta} 2M\, Q_n + \int_{-\delta}^{\delta} (\epsilon/2) Q_n + \int_{\delta}^{1} 2M\, Q_n \\
&\leq \int_{\delta}^{1} 2M\, Q_n + \int_{-1}^{1} (\epsilon/2) Q_n + \int_{\delta}^{1} 2M\, Q_n \\
&< \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon.
\end{aligned}$$

Hence, the sequence $\{P_n\}$ converges uniformly to f on $[0, 1]$. This completes the proof. ■

The above proof, although not terribly difficult, is not all that enlightening. It is not easy to actually find the polynomials that approximate the function, and the convergence of the polynomials is difficult to visualize. Nevertheless, a sequence of polynomials that converges uniformly to f on $[a, b]$ does exist.

The second result is even more surprising than the first. It states that there exists a continuous, nowhere differentiable function. An example of such a function was first published by Weierstrass in 1872, and it created quite a stir among mathematicians. It had been taken for granted that continuous functions were differentiable at most points; think about the type of graph you normally draw to represent a continuous function. After rigorous definitions for continuity of functions and convergence of series were given, it was possible to see where these definitions led. It is imperative that a simple mental picture of a continuous function be set aside; a continuous function is a function that satisfies the definition of continuity. Results contrary to intuition sometimes appear. When this occurs, either the definition has been poorly formulated (and thus needs to be altered) or intuition needs to be expanded to include new possibilities. In this case, since the definitions of continuity and convergence are well established, it is the intuition that must adapt.

A construction of a continuous, nowhere differentiable function is given below. (This example is different than the one published by Weierstrass.) This construction uses a common device for creating continuous functions. Any function that is the uniform limit of a series of continuous functions is itself continuous.

THEOREM 8.12 There exists a continuous function that is not differentiable at any point.

Proof. Define a function $g: \mathbb{R} \to \mathbb{R}$ by letting $g(x) = |x|$ for $-1 \leq x \leq 1$ and $g(x+2) = g(x)$ for all other values of x. (The graph of g can be found in Figure 8.4.) By definition, the function g is continuous on $\mathbb{R}$, $0 \leq g(x) \leq 1$ for all x, and