Chapter 3
Compact and Connected Sets

In this chapter, we study two of the most important and useful kinds of sets in metric spaces and especially in $\mathbb{R}^n$. Intuitively, we want to say that a set in $\mathbb{R}^n$ is compact when it is closed and is contained in a bounded region, and that a set is connected when it is "in one piece." Figure 3-1 gives some examples. As usual, it is necessary to turn these ideas into rigorous definitions. In each case the most useful technical definition appears to be a little removed from our intuition, but in the end we will see that it is in good accord with it. The fruitfulness of these notions will be revealed in Chapter 4, where they will be applied to the study of continuous functions.

§3.1 Compactness

In this section we give the general definition and properties of compact sets in metric spaces. A criterion for recognizing compact sets, called the ***Heine-Borel theorem,*** states that a set in $\mathbb{R}^n$ is compact iff it is closed and bounded. This result, special to the metric space $\mathbb{R}^n$, is discussed in §3.2.

Recall from our discussion of completeness of $\mathbb{R}^n$ in Chapter 1 that every bounded sequence has a convergent subsequence. This can be rephrased: If $A \subset \mathbb{R}^n$ is a closed and bounded set, then every sequence in A has a subsequence converging to a point of A. Historically, this was recognized to be an important property of sets, and so was elevated to a definition. This property plays a crucial role in many basic theorems such as the existence of maxima and minima of continuous functions on closed intervals, as we shall see in Chapter 4.

3.1.1 Definition *Let M be a metric space. A subset $A \subset M$ is called* ***sequentially compact*** *if every sequence in A has a subsequence that converges to a point in A.*

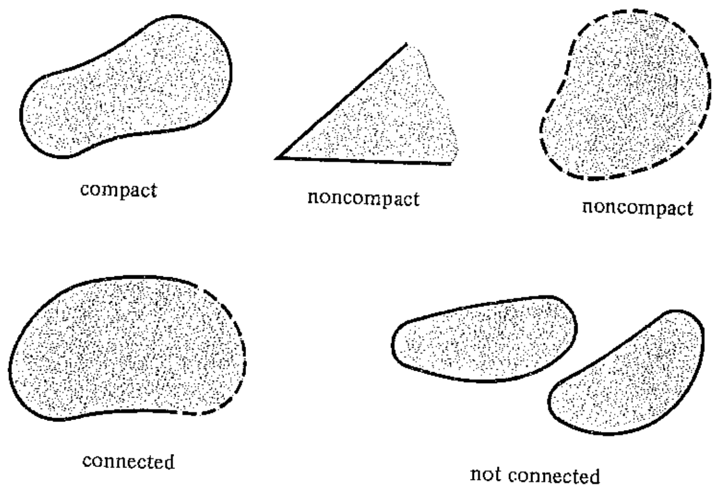

FIGURE 3-1 Compact and connected sets in $\mathbb{R}^2$

This property is equivalent to another property, called *compactness*, that we shall now develop. This property is less obvious, and its equivalence to sequential compactness is far from clear, at least at first.

Here is some terminology we need for our formal definition. Let M be a metric space and $A \subset M$ a subset. ***A cover*** of A is a collection $\{U_i\}$ of sets whose union contains A; it is an ***open cover*** if each U_i is open. A ***subcover*** of a given cover is a subcollection of $\{U_i\}$ whose union also contains A or, as we say, ***covers*** A; it is a ***finite subcover*** if the subcollection contains only a finite number of sets.

Open covers are not necessarily countable collections of open sets. For example, the uncountable set of disks $\{D((x, 0), 1) \mid x \in \mathbb{R}\}$ in $\mathbb{R}^2$ covers the real axis, and the subcollection of all disks $D((n, 0), 1)$ centered at integer points on the real line forms a countable subcover. Note that the set of disks $D((2n, 0), 1)$ centered at even integer points on the real line does not form a subcovering (why?).

3.1.2 Definition *A subset A of a metric space M is called* ***compact*** *if every open cover of A has a finite subcover.*

Here is the first major result, which links compactness and sequential compactness.

3.1.3 Bolzano-Weierstrass Theorem *A subset of a metric space is compact iff it is sequentially compact.*

Some simple observations will help give a feel for compactness and for this theorem. First, *a sequentially compact set must be closed.* Indeed, if $x_n \in A$ converges to $x \in M$, then by assumption there is a subsequence converging to a point $x_0 \in A$; by uniqueness of limits, $x = x_0$, and so A is closed. Second, a sequentially compact set A *must be bounded*, for if not, there is a point $x_0 \in A$ and a sequence $x_n \in A$ with $d(x_n, x_0) \geq n$. Then x_n cannot have any convergent subsequence. To show directly that a compact set is bounded, use the fact that for any $x_0 \in A$, the open balls $D(x_0, n)$, $n = 1, 2, \ldots$, cover A, so there is a finite subcover.

Note that in the definitions, one can take $A = M$, in which case one just speaks of a ***compact metric space.*** We shall develop examples of compact spaces in due course.

Another characterization of compactness relates to completeness. It is a useful technical tool used in the proof of the Bolzano-Weierstrass theorem.

3.1.4 Definition *A set $A \subset M$ is called **totally bounded** if for each $\varepsilon > 0$ there is a finite set $\{x_1, \ldots, x_N\}$ in M such that $A \subset \cup_{i=1}^{N} D(x_i, \varepsilon)$.*

3.1.5 Theorem *A metric space is compact iff it is complete and totally bounded.*

Let $A \subset M$, and assume that M is complete. If we apply this theorem to the metric space A, we conclude that *A is compact iff it is closed and totally bounded.*

In Theorem **3.1.5**, a few things are obvious, others less obvious. First, note that $D(x_i, \varepsilon) \subset D(x_1, \varepsilon + d(x_i, x_1))$, so that if

$$R = \varepsilon + \max\{d(x_2, x_1), \ldots, d(x_N, x_1)\},$$

then $A \subset D(x_1, R)$ and so *a totally bounded set is bounded.* This is consistent with our earlier remark that compact sets are bounded.

At this stage we do not have effective methods for telling when a given set is compact. We will remedy this in the next section.

3.1.6 Example The entire real line $\mathbb{R}$ is not compact, for it is unbounded. Another reason is that

$$\{D(n, 1) =]n - 1, n + 1[\mid n = 0, \pm 1, \pm 2, \ldots\}$$

is an open cover of $\mathbb{R}$ but does not have a finite subcover (why?). ♦

3.1.7 Example *Let $A =]0, 1]$. Find an open cover with no finite subcover.*

Solution Consider the open cover $\{]1/n, 2[\mid n = 1, 2, 3, \ldots\}$. (Why does the union contain all of A?) It clearly cannot have a finite subcover. This time, compactness fails because A is not closed; the point 0 is "missing" from A. This collection is not a cover for $[0, 1]$; in fact any open cover for $[0, 1]$ must have a finite subcover, because, as we prove in the next section, $[0, 1]$ is compact. ♦

3.1.8 Example *Give an example of a bounded and closed set that is not compact.*

Solution Let M be any infinite set with the discrete metric: $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. Clearly, $M \subset D(x_0, 2)$ for any $x_0 \in M$, and so M is bounded. Since it is already the entire metric space, it is closed. However, it is not compact. Indeed, $\{D(x, 1/2) \mid x \in M\}$ is an open cover with no finite subcover. ♦

3.1.9 Example *A collection of closed sets $\{K_\alpha\}$ in a metric space M is said to have the* ***finite intersection property*** *for A if the intersection of any finite number of the K_α with A is nonempty. Show that $A \subset M$ is compact iff every collection of closed sets with the finite intersection property for A has nonempty intersection with A.*

Solution First, assume A is compact. Let $\{F_i\}$ be a collection of closed sets and let $U_i = M \backslash F_i$, so that U_i is open. Suppose that $A \cap (\cap_{i=1}^{\infty} F_i) = \varnothing$. Taking complements, this means that the U_i cover A. Since the covering is open, there is a finite subcovering, say, $A \subset U_1 \cup \cdots \cup U_N$. Then $A \cap (F_1 \cap \cdots \cap F_N) = \varnothing$, and so $\{F_i\}$ does not have the finite intersection property. Thus, if $\{F_i\}$ is a collection of closed sets with the finite intersection property, then $A \cap \{F_i\} \neq \varnothing$.

Conversely, let $\{U_i\}$ be an open covering of A and let $F_i = M \backslash U_i$. Then $A \cap (\cap_{i=1}^{\infty} F_i) = \varnothing$, and so, by assumption, $\{F_i\}$ cannot have the finite intersection property for A. Thus, $A \cap (F_1 \cap \cdots \cap F_N) = \varnothing$ for some members $F_1, \ldots, F_N$ of the collection. Hence, $U_1, \ldots, U_N$ is the required finite subcover and thus A is compact. ♦

Exercises for §3.1

1. Show that $A \subset M$ is sequentially compact iff every infinite subset of A has an accumulation point in A.

2. Prove that $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x < 1, 0 \leq y \leq 1\}$ is not compact.

3. Let M be complete and $A \subset M$ be totally bounded. Show that cl(A) is compact.

4. Let $x_k \to x$ be a convergent sequence in a metric space and let $A = \{x_1, x_2, \ldots\} \cup \{x\}$.

   a. Show that A is compact.

   b. Verify that every open cover of A has a finite subcover.

5. Let M be a set with the discrete metric. Show that any infinite subset of M is noncompact. Why does this not contradict the statement in Exercise 4?

§3.2 The Heine-Borel Theorem

In Euclidean space we can easily tell if a set is compact from the following theorem:

3.2.1 Heine-Borel Theorem *A set $A \subset \mathbb{R}^n$ is compact iff it is closed and bounded.*

One half of this was already indicated in §**3.1**. In fact, a compact set is closed and bounded in *any* metric space. The converse must be special in view of Example **3.1.8.** Indeed, it is not even obvious that the closed interval $[0, 1]$ in $\mathbb{R}$ is compact. In fact, $[0, 1]$ is compact, and one of the proofs of the Heine-Borel theorem begins by treating this case.

3.2.2 Example *Determine which of the following are compact:*

a. $\{x \in \mathbb{R} \mid x \geq 0\} \subset \mathbb{R}$

b. $[0, 1] \cup [2, 3] \subset \mathbb{R}$

c. $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \subset \mathbb{R}^2$

Solution

a. Noncompact, because it is unbounded.

b. Compact, because it is closed and bounded.

c. Noncompact, because it is not closed. ♦

3.2.3 Example *Let x_k be a sequence of points in $\mathbb{R}^n$ with $||x_k|| \leq 3$ for all k. Show that x_k has a convergent subsequence.*

Solution The set $A = \{x \in \mathbb{R}^n \mid ||x|| \leq 3\}$ is closed and bounded, and hence compact. Since $x_k \in A$, we can apply the Bolzano-Weierstrass theorem to obtain the conclusion. ♦

3.2.4 Example *In the definition of a compact set, can "every" be replaced by "some"?*

Solution No. Let $A = \mathbb{R}$, and let the open cover consist of the single open set $\mathbb{R}$. This has a finite subcover, namely, itself, but being unbounded, $\mathbb{R}$ is not compact. ♦

3.2.5 Example *Let $A = \{0\} \cup \{1, 1/2, \ldots, 1/n, \ldots\}$. Show directly that A satisfies the definition of compactness.*

Solution Let $\{U_i\}$ be an arbitrary open cover of A. We must show that there is a finite subcover. The point 0 lies in one of the open sets—relabeling if needed, we can suppose that $0 \in U_1$. Since U_1 is open and $1/n \to 0$, there is an N such that $1/N, 1/(N+1), \ldots$ lie in U_1. Relabeling again if needed, suppose that $1 \in U_2, \ldots, 1/(1-N) \in U_N$. Then $U_1, \ldots, U_N$ is a finite subcover, since it is a finite subcollection of the $\{U_i\}$ and it includes all of the points of A. Notice that if A were the set $\{1, 1/2, \ldots\}$, then the argument would not work. In fact, this set is not closed, and so it is not compact. ♦

Exercises for §3.2

1. Which of the following sets are compact?

a. $\{x \in \mathbb{R} \mid 0 \leq x \leq 1 \text{ and } x \text{ is irrational}\}$

b. $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$

c. $\{(x, y) \in \mathbb{R}^2 \mid xy \geq 1\} \cap \{(x, y) \mid x^2 + y^2 < 5\}$

2. Let $r_1, r_2, r_3, \ldots$ be an enumeration of the rational numbers in [0, 1]. Show that there is a convergent subsequence.

3. Let $M = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ with the standard metric. Show that $A \subset M$ is compact iff A is closed.

4. Let A be a bounded set in $\mathbb{R}^n$. Prove that $\text{cl}(A)$ is compact.

5. Let A be an infinite set in $\mathbb{R}$ with a single accumulation point in A. Must A be compact?

§3.3 Nested Set Property

The next theorem is an important consequence of the Bolzano-Weierstrass theorem.

3.3.1 Nested Set Property *Let F_k be a sequence of compact nonempty sets in a metric space M such that $F_{k+1} \subset F_k$ for all $k = 1, 2, \ldots$. Then there is at least one point in $\cap_{k=1}^{\infty} F_k$.*

Intuitively, the sets F_k are nonempty and decreasing, and so it seems reasonable that there should be a point in all of them. However, if the F_k are not compact, then the intersection can be empty (see Example **3.3.4**). Thus, the actual proof requires more care.

To prove the nested set property using the Bolzano-Weierstrass theorem, pick $x_k \in F_k$ for each k. The sequence x_k has a convergent subsequence, since it lies in the compact set F_1. The limit point lies in all of the sets F_k because they are closed (see Figure 3.3-1). An alternative proof is given at the end of the chapter.

One can rephrase the nested set property in terms of "growing sets" this way. Let $U_k = M \backslash F_k$, so that the U_k are open and $U_{k+1} \supset U_k$. Then $\cup_{k=1}^{\infty} U_k \neq M$ is equivalent to $\cap_{k=1}^{\infty} F_k \neq \varnothing$. Thus, *if M is a metric space and the open sets U_k are increasing—i.e., $U_{k+1} \supset U_k$—and have compact complements, then the union of the U_k is not all of M.*

Theorem Proofs for Chapter 3

3.1.3 Bolzano-Weierstrass Theorem *A subset of a metric space is compact iff it is sequentially compact.*

Proof We begin with two lemmas.

Lemma 1 *A compact set $A \subset M$ is closed.*

Proof We will show that $M \backslash A$ is open. Let $x \in M \backslash A$ and consider the following collection of open sets: $U_n = \{y \mid d(y, x) > 1/n\}$. Since every $y \in M$ with $y \neq x$ has $d(y, x) > 0$, y lies in some U_n. Thus, the U_n cover A, and so there must be a finite subcover. One of these has a largest index, say, U_N. If $\varepsilon = 1/N$, then, by construction, $D(x, 1/N) \subset M \backslash A$, and so $M \backslash A$ is open. ▼

Lemma 2 *If M is a compact metric space and $B \subset M$ is closed, then B is compact.*

Proof Let $\{U_i\}$ be an open covering of B and let $V = M \backslash B$, so that V is open. Thus $\{U_i, V\}$ is an open cover of M. Therefore, M has a finite cover, say, $\{U_1, \ldots, U_N, V\}$. Then $\{U_1, \ldots, U_N\}$ is a finite open cover of B. ▼

Proof of 3.1.3 Let A be compact. Assume there exists a sequence $x_k \in A$ that has *no* convergent subsequences. In particular, this means that x_k has infinitely many distinct points, say, $y_1, y_2, \ldots$. Since there are no convergent subsequences, there is some neighborhood U_k of y_k containing no other y_i. This is because if every neighborhood of y_k contained another y_j, we could, by choosing the neighborhoods $D(y_k, 1/m)$, $m = 1, 2, \ldots$, select a subsequence converging to y_k. We claim that the set $\{y_1, y_2, \ldots\}$ is closed. Indeed, it has no accumulation points, by the assumption that there are no convergent subsequences. Applying **Lemma 2** to $\{y_1, y_2, \ldots\}$ as a subset of A, we find that $\{y_1, y_2, \ldots\}$ is compact. But $\{U_k\}$ is an open cover that has no finite subcover, a contradiction. Thus x_k has a convergent subsequence. The limit lies in A, since A is closed, by **Lemma 1**.

Conversely, assume that A is sequentially compact. To prove that A is compact, let $\{U_i\}$ be an open cover of A. We need to prove that this has a finite subcover. To show this, we proceed in several steps.

Lemma 3 *There is an $r > 0$ such that for each $y \in A$, $D(y, r) \subset U_i$ for some U_i.*

Proof If not, then for every integer n, there is some y_n such that $D(y_n, 1/n)$ is not contained in any U_i. By hypothesis, y_n has a convergent subsequence, say, $z_n \to z \in A$. Since the U_i cover A, $z \in U_{i_0}$ for some U_{i_0}. Choose $\varepsilon > 0$ such that $D(z, \varepsilon) \subset U_{i_0}$, which is possible since U_{i_0} is open. Choose N large enough so that $d(z_N, z) < \varepsilon/2$ and $1/N < \varepsilon/2$. Then $D(z_N, 1/N) \subset U_{i_0}$, a contradiction. ▼

Lemma 4 *A is totally bounded (see Definition* **3.1.4**).

Proof If A is not totally bounded, then for some $\varepsilon > 0$ we cannot cover A with finitely many disks. Choose $y_1 \in A$ and $y_2 \in A \backslash D(y_1, \varepsilon)$. By assumption, we can repeat; choose $y_n \in A \backslash [D(y_1, \varepsilon) \cup \cdots \cup D(y_{n-1}, \varepsilon)]$. This is a sequence with $d(y_n, y_m) \geq \varepsilon$ for all n and m, and so y_n has no convergent subsequence, a contradiction to the assumption that A is sequentially compact. ▼

To complete our proof, let r be as in Lemma **3**. By Lemma **4** we can write $A \subset D(y_1, r) \cup \cdots \cup D(y_n, r)$ for finitely many y_j. By Lemma **3**, $D(y_j, r) \subset U_{i_j}$, $j = 1, \ldots, n$, for some index i_j. Then $U_{i_1}, \ldots, U_{i_n}$ cover A. ■

3.1.5 Theorem *A metric space is compact iff it is complete and totally bounded.*

Proof First assume that M is compact. By **3.1.3**, it is sequentially compact. Thus, if x_k is a Cauchy sequence, it has a convergent subsequence, and so, as in **1.4.7**, the whole sequence converges. Thus M is complete. It is also totally bounded, by Lemma **4**.

Conversely, assume that M is complete and totally bounded. By **3.1.3**, it is enough to show that M is sequentially compact. Let y_k be a sequence in M. We can assume that the y_k are all distinct, for if y_k has infinitely many repetitions, there is a trivially convergent subsequence, and if there are finite repetitions we may delete them. Given an integer N, cover M with finitely many balls, $D(x_{L_1}, 1/N), \ldots, D(x_{L_N}, 1/N)$. An infinite number of the y_k lie in one of these balls. Start with $N = 1$. Write $M = D(x_{L_1}, 1) \cup \cdots \cup D(x_{L_N}, 1)$, and so we can select a subsequence of y_k lying entirely in one of these balls. Repeat for $N = 2$, getting a further subsequence lying in a fixed ball of radius $1/2$, and so on. Now choose the "diagonal" subsequence, the first member from the first sequence, the second from the second, and so on. This sequence is Cauchy and since M is complete, it converges. ■

3.2.1 Heine-Borel Theorem *A set $A \subset \mathbb{R}^n$ is compact iff it is closed and bounded.*

Proof We have already proved that compact sets are closed and bounded. We must now show that a set $A \subset \mathbb{R}^n$ is compact if it is closed and bounded. We will give two proofs of this.

First Proof This proof is based on the Bolzano-Weierstrass theorem and the fact that any bounded sequence in $\mathbb{R}$ has a convergent subsequence, proved in **1.4.3.** In fact, we shall prove that a closed and bounded set A is sequentially compact. Let $x_k = (x_k^1, x_k^2, \ldots, x_k^n) \in \mathbb{R}^n$ be a sequence. Since A is bounded, x_k^1 has a convergent subsequence, say, ${x^1}_{f_1(k)}$. Then ${x^2}_{f_1(k)}$ has a convergent subsequence, say ${x^2}_{f_2(k)}$. Continuing, we get a further subsequence $x_{f_n(k)} = ({x^1}_{f_n(k)}, \ldots, {x^n}_{f_n(k)})$, all of whose components converge. Thus $x_{f_n(k)}$ converges in $\mathbb{R}^n$. The limit lies in A since A is closed. Thus A is sequentially compact, and so is compact. ■

Second Proof This proof uses the definition of compactness in terms of open covers directly. We begin with a special case:

Lemma 1 *Closed intervals $[a,b]$ in $\mathbb{R}$ are compact.*

Proof Let $\mathcal{U} = \{U_i\}$ be an open covering of $[a,b]$. Define

$$C = \{x \in [a,b] \mid \text{the set } [a,x] \text{ can be covered by a finite collection of the } U_i\}.$$

We want to show that $C = [a,b]$. To this end, let $c = \sup(C)$. The sup exists because $C \neq \varnothing$ (since $a \in C$) and C is bounded above by b. Since $a \in C$ and b is an upper bound for C, $c \in [a,b]$, by definition of $\sup(C)$. Suppose $c \in U_{i_0}$; such a U_{i_0} exists, since the U_i's cover $[a,b]$. Since U_{i_0} is open, there is an $\varepsilon > 0$ such that $]c-\varepsilon, c+\varepsilon[\subset U_{i_0}$. Since $c = \sup(C)$, there exists an $x \in C$ such that $c - \varepsilon < x \leq c$ (see Proposition **1.3.2**). Because $x \in C$, $[a,x]$ has a finite subcover, say, $U_1, \ldots, U_N$; then $[a, c+\varepsilon/2]$ also has the finite subcover $U_1, \ldots, U_N, U_{i_0}$. Thus we conclude that $c \in C$ and moreover that $c = b$. Indeed, if $c < b$, we would get a member of C larger than c, since $[a, c+\varepsilon/2]$ has a finite subcover. The latter cannot happen, since $c = \sup(C)$. ▼