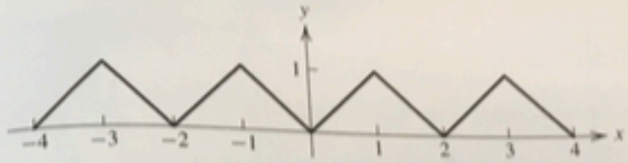

Figure 8.4 The graph of the function g in the proof of Theorem 8.12

$|g(y) - g(x)| \le |y - x|$ for all x and y. The function

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k g(4^k x)$$

is defined for all x and the series converges uniformly by the Weierstrass M-test. Since g is continuous on $\mathbb{R}$, the function f is continuous on $\mathbb{R}$ by Theorem 8.10. We will show that for each $x \in \mathbb{R}$, there exists a sequence $\{x_n\} \subseteq \mathbb{R} \setminus \{x\}$ such that $\{x_n\}$ converges to x, but the sequence

$$\left\{ \frac{f(x_n) - f(x)}{x_n - x} \right\}$$

does not converge. This will show that f is not differentiable at x.

Fix $x \in \mathbb{R}$ and let n be a positive integer. Choose $\delta_n = \pm 4^{-n}/2$ so that there are no integers between $4^n x$ and $4^n(x + \delta_n)$. Now

$$|g(4^k x + 4^k \delta_n) - g(4^k x)| \begin{cases} = 0, & \text{if } k > n; \\ = 1/2, & \text{if } k = n; \\ \le |4^k \delta_n|, & \text{if } k < n. \end{cases}$$

This assertion requires some justification. For $k > n$, the result is 0 since $4^k \delta_n$ is a multiple of 2 and g is a periodic function with period 2. For $k = n$, the result is 1/2 since $|4^n \delta_n| = 1/2$ and the function g is linear on this interval with a slope of 1 or −1. (This is where the choice of δ_n is used.) For $k < n$, the inequality follows from the fact that $|g(y) - g(x)| \le |y - x|$. The above results and the Reverse Triangle Inequality yield

$$\begin{aligned} \left| \frac{f(x + \delta_n) - f(x)}{\delta_n} \right| &= \left| \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k \frac{g(4^k x + 4^k \delta_n) - g(4^k x)}{\delta_n} \right| \\ &\ge \left(\frac{3}{4}\right)^n 4^n - \sum_{k=0}^{n-1} \left(\frac{3}{4}\right)^k \left| \frac{g(4^k x + 4^k \delta_n) - g(4^k x)}{\delta_n} \right| \\ &\ge 3^n - \sum_{k=0}^{n-1} 3^k \\ &= 3^n - \frac{3^n - 1}{3 - 1} > 3^n/2. \end{aligned}$$

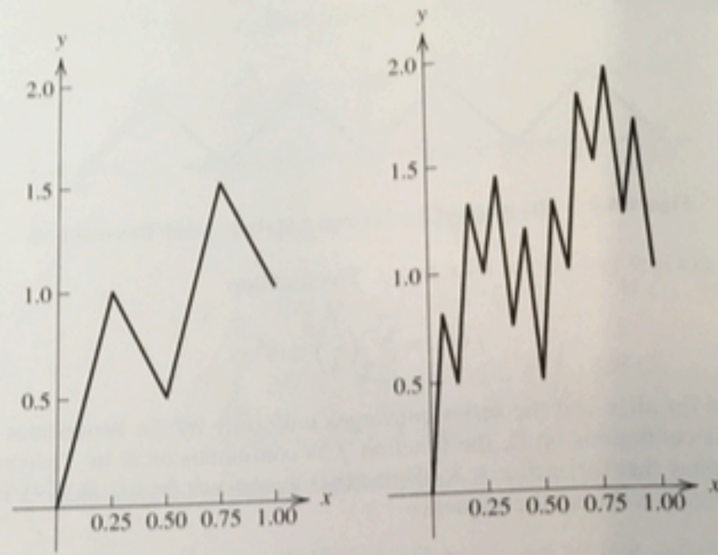

Figure 8.5 The graphs of $\sum_{k=0}^{n} (\frac{3}{4})^k g(4^k x)$ for $n = 1$ and $n = 2$

Let $x_n = x + \delta_n$ for each n. The sequence $\{x_n\}$ converges to x while the sequence

$$\left\{ \frac{f(x_n) - f(x)}{x_n - x} \right\}$$

is unbounded and hence does not converge. It follows that f is not differentiable at x. This completes the proof. ∎

Each of the continuous functions $g(4^k x)$ has points of nondifferentiability. As k increases, the number of points of nondifferentiability increases. By summing all of these functions, we end up with a continuous function with no points of differentiability. However, a nowhere differentiable function can be the limit of differentiable functions. By Theorem 8.11, the function f constructed above is the uniform limit of a sequence of polynomials on the interval $[0, 1]$. The actual construction of these polynomials is very difficult, but such a sequence does exist. Consequently, uniform convergence does not preserve differentiability at all.

The graph of a nowhere differentiable function is impossible to sketch because it is so jagged. Two of the partial sums of the series defining f are graphed in Figure 8.5. The erratic behavior of this function is already evident. The best image for the graph of a nowhere differentiable function is the view of the edge of a table under a high-powered microscope.