Practice Problems for the space C(X) and Weierstrass Approx Thm

From the text:

pp. 168-169, # 7, 10, 14, 15 (Hint: $|x - \frac{1}{2}|$ is continuous but not differentiable at $x = \frac{1}{2}$); 20, fill in the details of Application 11.6.

Other Problems:

1. Assume $(x_n(t))_{n\in\mathbb{N}}$ is a sequence of continuous functions on $[0,T]$, $x_n \to x$ uniformly on $t \in [0,T]$, and $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Show that $f(x_n) \to f(x)$ uniformly on $t \in [0,T]$.

2. Show that there cannot be a sequence of polynomials P_n for which $P_n \to \sin x$ uniformly on $\mathbb{R}$.

3. Suppose f is continuous on $[0,1]$. Find the limit, and prove your answer:

$$\lim_{n\to\infty} \int_0^1 (n+1)\, x^n\, f(x)\, dx$$

Hints: use $\int_0^1 (n+1)x^n\, dx = 1$ and the trick from the proof of Weierstrass to formulate the limit statement as an estimate of an integral. Divide the interval of integration into two pieces, $[0, 1-\delta] \cup (1-\delta, 1]$, and treat each one separately.

4. Define $P_t(x) = \dfrac{1}{\pi}\dfrac{t}{x^2+t^2}$.

(a) verify the following properties of $P_t(x)$:

1. $P_t(x) > 0$ for all $x \in \mathbb{R}$, $t > 0$;
2. $\int_{-\infty}^{\infty} P_t(x)\, dx = 1$ for all $t > 0$;
3. For all $\delta > 0$,

$$\lim_{t\to 0^+} \left[\int_{-\infty}^{-\delta} P_t(x)\, dx + \int_{\delta}^{\infty} P_t(x)\, dx \right] = 0.$$

(b) Let f be bounded ($|f(x)| \le M\ \forall x \in \mathbb{R},$) and continuous on $\mathbb{R}$. Prove that

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} f(x+y) P_t(x)\, dx = f(y)$$

for all $y \in \mathbb{R}$.