13.12.6 Peano's Theorem

Enrichment

In Section 13.11.4 we saw how the contraction mapping principle can be used to prove an existence and uniqueness theorem for solutions to the differential equation $y' = f(x, y)$. The requirement that was needed in order to apply the contraction mapping principle involved a Lipschitz condition on the function f.

In order to obtain an existence theorem (this time without uniqueness) under weaker hypotheses we need a different approach. For this we go back to a simple idea of Euler from 1768. To solve numerically an initial value problem

$$y' = f(x, y) \qquad y(x_0) = y_0$$

on an interval $[x_0, b]$, divide the interval into n equally spaced points (equal subdivision is not an essential feature, but is traditional in numerical methods)

$$x_0 < x_1 < x_2 < \cdots < x_n = b$$

and approximate a solution by a piecewise linear function. Write

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0),$$

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1),$$

and so on through to

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})(x_n - x_{n-1}).$$

We can let $k_n(x)$ denote the function on $[x_0, b]$ that is continuous, passes through the points (x_0, y_0), (x_1, y_1), ... ,(x_n, y_n), and is linear in between. Note that $k_n'(x) = f(x_i, k(x_i))$ if x is in the ith interval (x_i, x_{i+1}) and that there is no derivative at the corner points.

In practice this method gives a reasonable approximation to solutions. We could make this the basis of an existence proof if we could show that the sequence $\{k_n\}$ converges uniformly to a function k and that the function k solves the initial value problem. Unfortunately, the hypothesis that we wish to use, the continuity of the function f, is too weak to allow a proof that that the sequence $\{k_n\}$ converges uniformly. But we can use a compactness argument to obtain a uniformly convergent subsequence. The key is to use the continuity of f to design an interval $[x_0, b]$ on which such a sequence

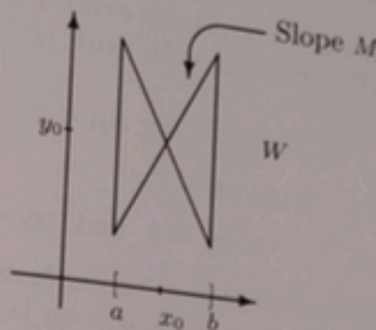

Figure 13.8. The set W and its projection to $[a, b]$ in the proof of Theorem 13.103.

of functions $\{k_n\}$ is bounded and equicontinuous. Then the Arzelà-Ascoli theorem supplies the subsequence.

This improvement of the classical existence theorems of Cauchy and Lipschitz was obtained by Peano in 1886. In 1890 he showed that this weakening of hypotheses occurs at the expense of uniqueness by supplying the example that we consider in Exercise 13.12.59.

Theorem 13.103 (Peano) *Let f be continuous on an open subset D of $\mathbb{R}^2$, and let $(x_0, y_0) \in D$. Then the differential equation*

$$y' = f(x, y)$$

has a local solution passing through the point (x_0, y_0).

Proof We are seeking an exact solution that is valid in some interval containing the point x_0. Thus we wish to find an interval $[a, b]$ containing x_0 and a differentiable function k defined on $[a, b]$ such that

$$k(x_0) = y_0 \text{ and } k'(x) = f(x, k(x)) \text{ for all } x \in [a, b]. \tag{33}$$

Our strategy is essentially to construct a family K of approximate solutions through (x_0, y_0) on $[a, b]$. We then show that the set $\overline{K}$ is compact in $\mathcal{C}([a, b])$ and use compactness to show the existence of the function k as a point in $\overline{K}$.

We first select the interval $[a, b]$. Let R be a closed rectangle contained in D having sides parallel to the coordinate axes and having (x_0, y_0) as center. Let $M \geq 1$ be an upper bound for $|f|$ on R. Let

$$W = \{(x, y) \in R : |y - y_0| \leq M|x - x_0|\},$$

nd let $[a, b]$ be the projection of W onto the x-axis, as in Figure 13.8.

We next obtain a family K of functions defined on $[x_0, b]$ that can be onsidered as approximate solutions to (33). Since W is compact in $\mathbb{R}^2$, f