is uniformly continuous on W. Thus, for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that, if $(x,y) \in W$ and $(\bar{x},\bar{y}) \in W$ with $|x-\bar{x}| < \delta$ and $|y-\bar{y}| < \delta$, then

$$|f(\bar{x},\bar{y}) - f(x,y)| < \varepsilon.$$

Choose points $x_1, x_2, \ldots, x_n$ such that

$$x_0 < x_1 < x_2 < \cdots < x_n = b \quad \text{and} \quad |x_i - x_{i-1}| < \delta/M$$

for all $i = 1, \ldots, n$. Define a function k_ε on $[x_0, b]$ as follows: $k_\varepsilon(x_0) = y_0$ and, on $[x_0, x_1]$, k_ε is linear with slope $f(x_0, y_0)$; on $[x_1, x_2]$, take k_ε to be linear with slope $f(x_1, k_\varepsilon(x_1))$; continuing in this way, we extend the definition of k_ε to all of $[x_0, b]$.

We have arrived at a function k_ε defined on $[x_0, b]$ whose graph is a polygonal arc through the point (x_0, y_0) and is contained in W. Since the slopes of the line segments composing the graph of k_ε are determined by values of the function f in W, we see that

$$|k_\varepsilon(x) - k_\varepsilon(\bar{x})| \leq M|x - \bar{x}| \tag{34}$$

for all $x, \bar{x} \in [x_0, b]$. Now let $x \in [x_0, b]$, $x \neq x_i$, $i = 0, 1, \ldots, n$. Then there exists $j \in \{1, 2, \ldots, n\}$ such that $x_{j-1} < x < x_j$. Noting that

$$|x_j - x_{j-1}| < \delta/M$$

and using (34), we see that

$$|k_\varepsilon(x) - k_\varepsilon(x_{j-1})| \leq M|x - x_{j-1}| < \delta.$$

This implies that

$$|f(x_{j-1}, k_\varepsilon(x_{j-1})) - f(x, k_\varepsilon(x))| < \varepsilon.$$

But

$$k_\varepsilon'(x) = f(x_{j-1}, k_\varepsilon(x_{j-1})),$$

so

$$|k_\varepsilon'(x) - f(x, k_\varepsilon(x))| < \varepsilon. \tag{35}$$

The inequality (35) is valid for all $x \in [x_0, b]$ except at points x in the finite set $\{x_0, \ldots, x_n\}$, at which k_ε need not be differentiable. By (35), we see that the functions k_ε are approximate solutions to (33).

We have constructed a family K of functions, one function corresponding to every $\varepsilon > 0$. The family K is uniformly bounded on $[x_0, b]$, since the graph of each of the functions k_ε is contained in W. It follows from (34) that K is an equicontinuous family. The Arzelà-Ascoli theorem now implies that $\overline{K}$ is compact in $C[x_0, b]$.

We can now complete the proof of the theorem. For all $x \in [x_0, b]$, we have

$$\begin{aligned} k_\varepsilon(x) &= y_0 + \int_{x_0}^{x} k'_\varepsilon(t)\,dt \\ &= y_0 + \int_{x_0}^{x} (f(t, k_\varepsilon(t)) + (k'_\varepsilon(t) - f(t, k_\varepsilon(t)))\,dt. \end{aligned} \tag{36}$$

The fact that k'_ε may fail to exist on the set $\{x_0, x_1, \ldots, x_n\}$ does not affect the integral.

Since $K \subset \overline{K}$ and $\overline{K}$ is compact the sequence $\{k_{(1/n)}\}$ contains a subsequence $\{k_{1/n_i)}\}$ that converges uniformly to some function $k \in \overline{K}$. Note that k must be continuous on $[x_0, b]$. Since f is uniformly continuous on W, the functions $f(t, k_{(1/n_i)}(t))$ converge uniformly to the function $f(t, k(t))$ on $[x_0, b]$. Noting (35), we now infer from (36) that

$$k(x) = y_0 + \int_{x_0}^{x} f(t, k(t))\,dt$$

for all $x \in [x_0, b]$. It follows that k is a solution to (33) on $[x_0, b]$.

In a similar manner, we obtain a solution $\overline{k}$ to (33) on $[a, x_0]$. The function y given by

$$y(x) = \begin{cases} k(x) & \text{for } x \in [x_0, b]; \\ \overline{k}(x) & \text{for } x \in [a, x_0], \end{cases}$$

satisfies (33) on all of $[a, b]$, as required. ■

Exercises

13.12.59 **(Peano's Example)** Show that the hypotheses given in Theorem 13.103 are not sufficient to guarantee uniqueness of solutions to the equation $y' = f(x, y)$ by taking, for example, the equation $y' = 3y^{2/3}$, $y(0) = 0$. Does this example conflict with the uniqueness assertion of Theorem 13.86?