

Construction of suitable weak solutions of the Navier-Stokes equations

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ABSTRACT. Weak solutions of the Navier – Stokes system of equations are *suitable* if they satisfy a localized version of the energy inequality. The interest in this notion is that the partial regularity theorems of Scheffer [10, 11] and Caffarelli, Kohn and Nirenberg [2] apply to suitable weak solutions of the Navier-Stokes equations in three spatial dimensions, limiting the parabolic Hausdorff dimension of their singular set. We show that the weak solutions obtained by the approximation method of Leray [8] are suitable, as are weak solutions obtained by the super-viscosity approximation. However it is not known whether the weak solutions obtained by Hopf’s method of Galerkin approximation [7] are suitable. For the problem on the torus \mathbb{T}^d we give a new estimate of weak solutions which has some bearing on this question.

1. Introduction

It is an important problem to show that there exist weak solutions of the Navier-Stokes equations whose space-time singular set S is empty. Such solutions would be automatically strong ones, and the Serrin weak-strong uniqueness theorem [14] would imply the uniqueness of weak solutions. Moreover, Serrin’s regularity theorem would imply that this solution is a smooth classical solution. While these considerations are out of reach at present, the well-known partial regularity results of Scheffer [10, 11] and Caffarelli, Kohn and Nirenberg [2] provide limits on the size of the singular set S in the case of weak solutions which satisfy the additional condition of a local energy inequality. Such solutions are called *suitable*. The result of [2] is that for these weak solutions the singular set is small, in the sense that the one-dimensional parabolic Hausdorff measure of S is zero.

Strong solutions of the Navier – Stokes equations satisfy a global energy identity, and it is generally agreed that weak solutions are physically meaningful only if they satisfy the closely related global energy inequality. Most methods of construction of weak solutions provide ones which satisfy this property. In order to be

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a suitable weak solution, a localized version of the energy inequality must also be satisfied. Roughly speaking, every method of constructing weak solutions gives its own class of solutions, and because there lacks a uniqueness theorem, these may well be different classes. Because of the importance of the properties of partial regularity, it is a question as to which of these weak solutions are suitable. The method of Leray and the super-viscosity approximation method provide suitable weak solutions, as does the modification of Leray's method presented in [2]. It is a bit surprising however, that for the famous method of Hopf of Galerkin approximation it is not known whether the resulting weak solutions are suitable.

In this article we give elementary proofs that Leray's construction method [8] of weak solutions and the method of super-viscosity give rise to suitable weak solutions. We also discuss the construction method of Hopf [7]. With regard to the latter method on the torus $\mathbb{T}^d = \mathbb{R}^d / (\ell\mathbb{Z})^d$, we give a new global bound on the Fourier coefficients of weak solutions of the Navier – Stokes equations, which may be useful in future work on the subject.

1.1. The local energy inequality. The Navier – Stokes system of equations is

$$(1.1) \quad \begin{aligned} \partial_t u + \nabla \cdot (u \otimes u) - \nu \Delta u + \nabla p &= 0, \\ \nabla \cdot u &= 0. \end{aligned}$$

DEFINITION 1.1. Let $Q \subset \mathbb{T}^3 \times \mathbb{R}$ be an open set. We say that a pair (u, p) , where $u \in (L_{\text{loc}}^2(Q))^3$ and $p \in \mathcal{D}'(Q)$ is a *weak solution* for the Navier – Stokes system in the domain Q if equations (1.1) are satisfied in the distributional sense.

Using common abuse of definitions we equivalently allow to say that a function $u \in (L_{\text{loc}}^2(Q))^3$ is a weak solution of the Navier-Stokes system in Q if there is $p \in \mathcal{D}'(Q)$, such that (u, p) is a weak solution in the sense of the above definition. In other words we say that a function $u \in (L_{\text{loc}}^2(Q))^3$ is a weak solution for the Navier-Stokes system on an open set Q , if it is weakly divergence free, i.e., for any C^∞ -smooth real valued function $\varphi(t, x)$ with $\text{supp } \varphi \subset\subset Q$ we have

$$(1.2) \quad \iint u \cdot \nabla \varphi \, dx dt = 0,$$

and the following integral identity holds

$$(1.3) \quad \iint u \cdot \psi_t + \nu u \cdot \Delta \psi + u \otimes u : \nabla \otimes \psi \, dx dt = 0.$$

for any divergence free, C^∞ -smooth vector-valued function $\psi(t, x)$ with $\text{supp } \psi \subset \subset Q$. The double dot symbol above stands for summation in both indexes. From now on we will use the common abuse of notation in this topic, such as writing $u \in L_{\text{loc}}^2(Q)$ instead of $u \in (L_{\text{loc}}^2(Q))^3$; writing ∇u instead of $\nabla \otimes u$, etc.

DEFINITION 1.2. A pair (u, p) is called a *suitable weak solution* of (1.1) in an open set $Q \subset \mathbb{T}^3 \times \mathbb{R}$ if the pair of functions (u, p) satisfies the following conditions;

- (1) (Equations) The pair (u, p) is a weak solution of the Navier-Stokes system.
- (2) (Integrability hypothesis) The functions u, p are measurable on Q , and moreover

$$(1.4) \quad \sup_t \int_{Q \cap \{\mathbb{T}^3 \times \{t\}\}} |u(t, x)|^2 \, dx \leq E_0 < \infty, \quad \nabla u \in L_{\text{loc}}^2(Q), \quad p \in L_{\text{loc}}^{3/2}(Q).$$

- (3) (The local energy inequality) For any non-negative C^∞ -smooth function $\varphi(t, x)$ with $\text{supp } \varphi \subset\subset Q$ we have

$$(1.5) \quad 2\nu \iint |\nabla u|^2 \varphi \, dx dt \leq \iint |u|^2 (\varphi_t + \nu \Delta \varphi) + (|u|^2 + 2p) u \cdot \nabla \varphi \, dx dt.$$

We remark that while definition 1.1 remains the same in any space dimension d , the definition 1.2 is only “suitable” for spatial dimension $d = 3$. Clearly, there is a monotonicity property of notions of weak solution and suitable weak solution with respect to the domain, i.e., these notions are preserved when we restrict consideration to a subdomain. It is therefore reasonable to look for suitable solutions on a “largest” domain. In relation to the Cauchy problem for the Navier – Stokes system we take $Q = \mathbb{T}^3 \times (0, T)$, where $T \in (0, +\infty]$.

DEFINITION 1.3. Let $T \in (0, +\infty]$. We say that a vector field $u \in L^2(\mathbb{T}^3 \times (0, T))$ is a solution of the Cauchy problem for the Navier – Stokes system (1.1) with an initial condition $u_0 \in L^2(\mathbb{T}^3)$ if

- (1) the $L^2(\mathbb{T}^3)$ -weak essential limit of $\lim_{t \rightarrow 0} u(\cdot, t)$ exists and equal to u_0 ,
- (2) there exists $p \in \mathcal{D}'(\mathbb{T}^3 \times (0, T))$, such that the pair (u, p) is a weak solution of the Navier-Stokes system in $\mathbb{T}^3 \times (0, T)$ in the sense of the Definition 1.1.

Equivalently, we require that u be divergence free in the sense of (1.2) and for any smooth divergence free (vector function) ψ , with bounded support we have

$$(1.6) \quad \int_{\mathbb{T}^3} u \cdot \psi|_{t=0} \, dx + \int_0^\infty \int_{\mathbb{T}^3} u \cdot \psi_t + \nu u \cdot \Delta \psi + u \otimes u : \nabla \otimes \psi \, dx dt = 0.$$

DEFINITION 1.4. Let $I = (t_0, t_1)$, or $I = [t_0, t_1)$, or $I = (t_0, t_1]$, or $I = [t_0, t_1]$, where $t_0 < t_1 \leq \infty$. Let $\overset{\circ}{I}$ denote the interior of the interval I . Let u be a weak solution of the Navier – Stokes system on $\mathbb{T}^3 \times \overset{\circ}{I}$. We say that u satisfies the *energy inequality*¹ on the time interval I , if

- (1) $u \in L^\infty(I, L^2(\mathbb{T}^3)) \cap L^2(I, H^1(\mathbb{T}^3))$.
- (2) in the case $t_i \in I$ ($i = 0, 1$) we require that there exist $L^2(\mathbb{T}^3)$ -weak essential limit $\lim_{t \rightarrow t_i} u(\cdot, t)$ and define $u(t_i) \in L^2(\mathbb{T}^3)$ as the value of this limit.
- (3) for every C^∞ -smooth function $h(t)$ with $\text{supp } h \subset\subset I$ we have

$$(1.7) \quad \int_{\mathbb{T}^3} |u(x, t_1)|^2 h(t_1) \, dx + 2\nu \iint |\nabla u|^2 h \, dx dt \leq \iint |u|^2 h' \, dx dt + \int_{\mathbb{T}^3} |u(x, t_0)|^2 h(t_0) \, dx.$$

Normally our domain of suitability Q will be everywhere relevant to the problem, namely $Q = \mathbb{T}^3 \times (0, +\infty)$. The global energy inequality is that

$$(1.8) \quad \int_{\mathbb{T}^3} |u(x, T)|^2 \, dx + 2\nu \iint_{\mathbb{T}^3 \times [0, T]} |\nabla u(x, t)|^2 \, dx dt \leq \int_{\mathbb{T}^3} |u_0(x)|^2 \, dx.$$

It does not follow precisely from the local one with $Q = \mathbb{T}^3 \times (0, T)$, but we must also allow for the support of φ to intersect $\mathbb{T}^3 \times \{t = 0\}$ and $\mathbb{T}^3 \times \{t = T\}$.

Define the singular set $S \in Q$ of a weak solution (u, p) of (1.1) to consist of those points (x_0, t_0) such that the vector field $u(x, t)$ is not in $L^\infty(B)$ for any

¹In the literature this is sometimes referred to as a Leray solution. We however reserve this notion for use later, for weak solutions generated by the approximation scheme of Leray.

space-time neighborhood $B \subseteq Q$ containing (x_0, t_0) . By definition it is a closed set. It makes sense to speak of the singular set in terms of only the L^∞ norm of u , as the classical *a priori* regularity result of Serrin [15] shows that a local L^∞ estimate implies that the solution u is in fact locally C^∞ in spatial variables (however not necessarily in t), and Hölder continuous in the t variable. Thus we can equivalently define S as the set of points of discontinuity of u . Furthermore the famous partial regularity results of Scheffer [10, 11] and Caffarelli, Kohn and Nirenberg [2] apply to a suitable weak solution, implying that its singular set has zero parabolic Hausdorff one-dimensional measure, $\mathcal{P}^1(S) = 0$.

The paper [2] includes an appendix in which a class of suitable weak solutions using an adaptation of the method of Leray [8] is constructed. The paper [1] gives a proof that weak solutions constructed by the super-viscosity method for the a power $\alpha = 2$ are in fact suitable. The papers [4, 5] discuss various approximation methods for the Navier – Stokes system. Among other things, they give a proof that the Galerkin – super-viscosity method for $\alpha > 5/4$ and some particular choice of the sequence of scales ε_n gives a suitable solution. As remarked above, it is not currently known whether the weak solutions produced by the Galerkin truncation method of Hopf [7] are suitable. However very recently Guermond [6] has given a proof that Galerkin approximations with respect to certain classes of finite-element bases do give suitable weak solutions. The question of the suitability of solutions stemming from the classical Galerkin approximation using Fourier series, or more generally a Laplacian eigenfunction expansion, is still open.

In this paper we give an uniform approach to Galerkin and super-viscosity method, treating both as extreme points of a two-dimensional ($n \rightarrow \infty, \varepsilon \rightarrow 0$) approximation with a parameter $\alpha \geq 1$. We obtain suitable solutions under the condition $\alpha > 1$, just short of the Galerkin method which corresponds to $\alpha = 1$. We also give a wider range of admissible scales than in [4, 5].

Since the approaches for proving the suitability of solutions essentially follow the same steps, we first consider Leray’s method, followed by the Galerkin approximation where we localize the problem. Then we consider a Galerkin – super-viscosity method where we give a complete proof that this procedure leads to suitable weak solutions.

Finally, we give a conditional theorem, that Galerkin solution are suitable provided that energy does not concentrate within certain annuli. In addition, we derive a new estimate of solutions of the Navier – Stokes equations on the torus \mathbb{T}^d , which has some bearing on the question of suitability of weak solutions constructed by the method of Hopf. But we do not have a proof either way as to their actual suitability.

2. The Leray approximation to the Navier-Stokes system.

We say that a pair (u, p) is a *Leray² weak solution* of (1.1) if it is obtained by the Leray approximation scheme, which is that there exists a sequence $\{(u^n, p^n)\}_{n=1}^\infty$ of smooth functions and a sequence of positive numbers $\varepsilon_n \rightarrow 0$ such that

$$(2.1) \quad u^n \rightarrow u \text{ in } L^2(L^2) \quad \text{strongly}$$

$$(2.2) \quad u^n \rightharpoonup u \text{ in } L^2(H^1) \quad \text{weakly ,}$$

²This is somewhat nonstandard terminology. Usually the term “Leray weak solution” stands for any weak solution that satisfies the energy inequality “from $t = 0$ ”.

with (u^n, p^n) satisfying the following approximation scheme due to Leray

$$(2.3) \quad \begin{cases} \partial_t u^n - \nu \Delta u^n + ((\eta^n * u^n) \cdot \nabla) u^n + \nabla p^n = 0 \\ \nabla \cdot u^n = 0 \\ u^n|_{t=0} = \eta^n * u_0 . \end{cases}$$

Here $\eta^n(x) = \frac{1}{\varepsilon_n^3} \eta(\frac{x}{\varepsilon_n})$, where $\eta(x)$ is a standard mollifier in \mathbb{R}^3 . The existence of Leray solutions is well known, as is the fact that they satisfy certain additional conditions required of weak solutions (which are by now standard), and in particular they satisfy the global energy inequality; indeed this is one of the founding results of the subject [8]. The fact that they are also suitable weak solutions, satisfying (1.5) is the result of the following theorem.

THEOREM 2.1. *Let (u, p) be a Leray solution of the Navier-Stokes system on the torus \mathbb{T}^3 , then (u, p) is a suitable weak solution in $\mathbb{T}^3 \times \mathbb{R}_+$.*

PROOF. Denote by $\{(u^n, p^n)\}_{n=1}^\infty$ a sequence of solutions of (2.3) which converges in the sense of (2.1)–(2.2) to (u, p) , a Leray solution of (1.1). Using Sobolev embedding $H^1(\mathbb{T}^3) \rightarrow L^6(\mathbb{T}^3)$ and (2.2) we deduce that $\{u^n\}_{n=1}^\infty$ is a bounded sequence in $L^2(L^6)$ (where we are using the norm convention that $v \in L^q(L^p)$ means that $\int (\int |v(x, t)|^p dx)^{q/p} dt < \infty$). Interpolating $L^\infty(L^2)$ with $L^2(L^6)$, we conclude that $\{u^n\}_{n=1}^\infty$ is bounded in $L^{\frac{10}{3}}(L^{\frac{10}{3}})$. Interpolating once more $L^{\frac{10}{3}}(L^{\frac{10}{3}})$ with (2.1) we obtain

$$(2.4) \quad u^n \rightarrow u \text{ in } L^3_{t,x} \text{ strongly .}$$

The goal is to prove that the limit (u, p) satisfies the local energy inequality (1.5) on $Q := \mathbb{T}^3 \times \mathbb{R}_+$. Let φ be a non-negative C^∞ -smooth function with $\text{supp } \varphi \subset\subset Q$ (in particular φ vanishes near $t = 0$). Multiplying (2.3) by $2\varphi u^n$ and integrating by parts, and using the key identity

$$(2.5) \quad 2 \int_{\mathbb{T}^3} \eta^n * u_j(t, x) (\partial_j u_k^n) u_k^n(t, x) \varphi(t, x) dx = - \int_{\mathbb{T}^3} \eta^n * u_j^n |u_k^n|^2 \partial_j \varphi(t, x) dx ,$$

this yields

$$(2.6) \quad \begin{aligned} 2\nu \iint_Q |\nabla u^n|^2 \varphi dxdt &= \iint_Q |u^n|^2 (\varphi_t + \nu \Delta \varphi) dxdt \\ &+ \iint_Q (|u^n|^2 (\eta^n * u^n) \cdot \nabla \varphi + 2p^n u^n \cdot \nabla \varphi) dxdt . \end{aligned}$$

The Hölder inequality implies

$$| \iint_Q |u^n - u|^2 (\eta^n * u^n) \cdot \nabla \varphi dxdt | \leq |\eta^n * u^n|_{L^3(L^3)} |u^n - u|_{L^3(L^3)}^2 |\nabla \varphi|_{L^\infty(L^\infty)} .$$

Similarly we estimate

$$| \iint_Q (u \cdot u^n) (\eta^n * (u^n - u)) \cdot \nabla \varphi dxdt | \leq |\eta^n * (u^n - u)|_{L^3(L^3)} |u^n|_{L^3(L^3)} |u|_{L^3(L^3)} |\nabla \varphi|_{L^\infty} .$$

This proves that $\iint |u^n|^2 (\eta^n * u^n) \cdot \nabla \varphi dxdt$ converges to $\iint |u|^2 (u \cdot \nabla \varphi) dxdt$. It remains to prove the convergence of the pressure term. Taking the divergence of equation (2.3) we obtain

$$\Delta p^n = \sum_{k,j} \partial_j ((\eta^n * u_k^n) \partial_k u_j^n) = \sum_{k,j} \partial_j \partial_k ((\eta^n * u_k^n) u_j^n),$$

which is the analog of the classical Biot-Savart law. Therefore

$$p^n(t, \cdot) = R \left(\sum_{k,j} ((\eta^n * u_k^n) u_j^n)(t, \cdot) \right),$$

where the linear map R is defined by

$$\begin{aligned} R : (L^r(\mathbb{T}^3))^9 &\longrightarrow L^r(\mathbb{T}^3) \\ (g^{j,k})_{j,k=1,2,3} &\mapsto (-\Delta)^{-1} \left(\sum_{j,k} \partial_j \partial_k g^{j,k} \right). \end{aligned}$$

By the theory of Riesz transforms ([16], chapter III), R is a continuous map for any $r \in (1, +\infty)$.

Since $\{(\eta^n * u_k^n) u_j^n\}_{n=1}^\infty$ converges strongly to $(u_k u_j)$ in $L^{3/2}(L^{3/2})$, it is in particular bounded in $L^{3/2}(L^{3/2})$ and, up to a subsequence, we may assume that for almost all $t > 0$, $\{((\eta^n * u_k^n) u_j^n)(t, \cdot)\}_{n=1}^\infty$ converges to $(u_k u_j)(t, \cdot)$ in $L^{3/2}(\mathbb{T}^3)$. Using the dominate convergence theorem (for time integration) and the continuity of the operator R , we conclude that p^n converges strongly to p in $L^{\frac{3}{2}}(L^{\frac{3}{2}})$ and therefore (1.5) holds. \square

3. Galerkin approximation of the Navier – Stokes system

This section is devoted to a discussion of the suitability of weak solutions of (1.1) which are obtained through Hopf's method of Galerkin approximation [7]. More precisely, we derive a condition which is sufficient to deduce suitability of these solutions. Unfortunately, in the general case we have not succeeded to prove or to disprove that such a condition holds for solutions obtained by the Galerkin method.

At this point we state the Galerkin approximation of Navier – Stokes system, which is in one view surprisingly close to the approximation scheme of Leray (2.3). However we will show that this procedure presents certain difficulties when one seeks to show that the resulting weak solutions are suitable. Let Π denote the Leray projector onto divergence free vector fields in $(L^2(\mathbb{T}^3))^3$, given in the basis of Fourier coefficients by the expression

$$\begin{aligned} \Pi & : (L^2(\mathbb{T}^3))^3 \longrightarrow (L^2(\mathbb{T}^3))^3 \\ g(x) = \sum_{k \in \mathbb{Z}^d} g_k \exp\left(\frac{2\pi i}{\ell} k \cdot x\right) &\mapsto \Pi g(x) = \sum_{k \in \mathbb{Z}^d} \left(g_k - (g_k \cdot k) \frac{k}{|k|^2}\right) \exp\left(\frac{2\pi i}{\ell} k \cdot x\right). \end{aligned}$$

For any integer n , we denote by Π_n the projector onto the divergence free vector fields whose Fourier coefficients u_k vanish for all $|k| > n$. Namely,

$$\begin{aligned} \Pi_n & : (L^2(\mathbb{T}^3))^3 \longrightarrow (L^2(\mathbb{T}^3))^3 \\ g(x) = \sum_{k \in \mathbb{Z}^d} g_k \exp\left(\frac{2\pi i}{\ell} k \cdot x\right) &\mapsto \Pi_n g(x) = \sum_{|k| \leq n} \left(g_k - (g_k \cdot k) \frac{k}{|k|^2}\right) \exp\left(\frac{2\pi i}{\ell} k \cdot x\right). \end{aligned}$$

The Galerkin approximation of Navier-Stokes, denoted by u^n , is the sequence satisfying the following system of equations

$$(3.1) \quad \begin{cases} \partial_t u^n - \nu \Delta u^n + \Pi_n((u^n \cdot \nabla) u^n) = 0 \\ \Pi_n u^n = u^n, \quad u^n|_{t=0} = \Pi_n u_0. \end{cases}$$

This statement of the system of equations for u^n does not contain a pressure term. However, an approximation p^n to the pressure can be recovered by the analog of the Biot-Savart law, in particular it satisfies

$$\Delta p^n = - \sum_{k,j} \partial_k \partial_j (u_k^n u_j^n) .$$

Define a second projection Q_n by $\Pi = \Pi_n + Q_n$, then there is an identity

$$(u^n \cdot \nabla) u^n = \Pi_n((u^n \cdot \nabla) u^n) + Q_n((u^n \cdot \nabla) u^n) - \nabla p^n ,$$

from which the system (3.1) can be rewritten as

$$(3.2) \quad \partial_t u^n - \nu \Delta u^n + (u^n \cdot \nabla) u^n - Q_n((u^n \cdot \nabla) u^n) + \nabla p^n = 0 .$$

Limits (in the sense of (2.1)–(2.2)) of solutions u^n of (3.1) are energy inequality satisfying weak solutions of the Navier-Stokes system. We would like to show that any such limit is in fact a suitable weak solution. Let φ be a positive smooth function as in (1.5). Multiplying (3.2) by $2\varphi u^n$ and integrating by parts we obtain the following identity.

$$(3.3) \quad 2\nu \iint |\nabla u^n|^2 \varphi \, dxdt = \iint |u^n|^2 (\varphi_t + \nu \Delta \varphi) \, dxdt \\ + \iint (|u^n|^2 + 2p^n) u^n \cdot \nabla \varphi \, dxdt - 2 \iint Q_n((u^n \cdot \nabla) u^n) \cdot u^n \varphi \, dxdt .$$

It is well known that the sequence $\{u^n\}_{n=1}^\infty$ satisfies (2.1) and (2.2). Hence, using the same ideas as we used for Leray solutions, we can prove that $\iint |u^n|^2 (\varphi_t + \nu \Delta \varphi)$ and $\iint (|u^n|^2 + 2p^n) u^n \cdot \nabla \varphi$ converge to $\iint |u|^2 (\varphi_t + \nu \Delta \varphi)$ and $\iint (|u|^2 + 2p) u \cdot \nabla \varphi$ respectively. The issue is to show that

$$(3.4) \quad \nu \iint |\nabla u|^2 \varphi \, dxdt \leq \liminf_{n \rightarrow \infty} \left\{ \nu \iint |\nabla u^n|^2 \varphi \, dxdt \right. \\ \left. + \iint Q_n((u^n \cdot \nabla) u^n) \cdot u^n \varphi \, dxdt \right\} .$$

The presence of the term containing the projection Q_n is the difference between the error estimates in the Hopf Galerkin approximation scheme and the Leray scheme. Since the inequality

$$(3.5) \quad \liminf_{n \rightarrow \infty} \nu \iint |\nabla u^n|^2 \varphi \, dxdt \geq \nu \iint |\nabla u|^2 \varphi \, dxdt$$

is known (by Fatou's lemma), then a credible approach to (3.4) is to prove that for the sequence of numbers

$$(3.6) \quad I_n := \iint Q_n((u^n \cdot \nabla) u^n) \cdot u^n \varphi \, dxdt$$

we have $\liminf_{n \rightarrow \infty} I_n \geq 0$ or even that $\lim_{n \rightarrow \infty} I_n = 0$.

We postpone the analysis of the suitability of solutions obtained through the Galerkin scheme to section 5 where a conditional theorem is stated. The condition concerns the assumption of H^1 -mass in annuli $A_{C,n} = \{k \in \mathbb{Z}^3 : n - C < |k| \leq n\}$ for any fixed C .

4. The Galerkin approximation for the super-viscosity model

This section studies the Galerkin approximation for the super-viscosity model of the Navier – Stokes system. More precisely, we will show that adding a super-viscosity term of the type $\varepsilon(-\Delta)^\alpha u$ to equations (1.1), the Galerkin approximation of the new system will lead to suitable weak solutions of the Navier – Stokes system. The sequence ε has to be suitably chosen with respect to the cutoff wave number n in the Galerkin approximation. We restrict our considerations to the case of three space dimensions, and to the spatial domain \mathbb{T}^3 . Using the notation of the previous section and referring to equation (3.1), the Galerkin approximation for the super-viscosity model of the Navier-Stokes system is defined as

$$(4.1) \quad \begin{cases} \partial_t u^n - \nu \Delta u^n + \Pi_n((u^n \cdot \nabla)u^n) + \varepsilon_n(-\Delta)^\alpha u^n = 0 \\ u^n|_{t=0} = \Pi_n u(0), \quad \operatorname{div} u^n = 0, \quad Q_n u^n = 0. \end{cases}$$

Here, $\alpha > 1$ and ε_n is a sequence of positive numbers representing the scale of the approximation scheme at index n . The case $n = \infty$ and $\varepsilon_\infty = 0$ corresponds to the Navier-Stokes system. The case $\varepsilon_n = 0$ (or $\alpha = 1$, $\varepsilon_n \rightarrow 0$) corresponds to the usual Galerkin scheme. The case $n = \infty$ but “ $\varepsilon_n \rightarrow 0$ ” is the standard super-viscosity method, which will be considered later.

First, we recall that for an initial state $u(0) \in L^2(\mathbb{T}^3)$, we have existence, uniqueness and smoothness of u^n , since (4.1) is a smooth ODE in a finite dimensional space. Moreover, u^n satisfies the following energy inequality

PROPOSITION 4.1. *Let $u(0) \in L^2(\mathbb{T}^3)$. For any finite n , the unique solution u^n of problem (4.1) satisfies (for any $T > 0$) the following a priori estimate,*

$$(4.2) \quad |u^n(T, \cdot)|_{L^2}^2 + 2\nu \int_0^T |\nabla u^n(t, \cdot)|_{L^2}^2 dt + 2\varepsilon_n \int_0^T |\nabla^\alpha u^n(t, \cdot)|_{L^2}^2 dt \leq |u_0|_{L^2}^2.$$

The proof of the proposition is straightforward. Multiplying the momentum equation (4.1) by u^n and integrating by parts, we obtain (4.2); the further details are omitted.

The following result generalizes the results of [1], [4] and [5] by extending the range of admissible sequences ε_n and the possible values of α .

THEOREM 4.2. *Let $\alpha > 1$ and $u_0 \in L^2$. Then the following holds.*

- (1) *If the sequence ε_n is non-negative, then up to a subsequence, there exists a limit u such that*

$$(4.3) \quad u^n \rightarrow u \text{ in } L^2(L^2) \quad \text{strongly,}$$

$$(4.4) \quad u^n \rightharpoonup u \text{ in } L^2(H^1) \quad \text{weakly.}$$

- (2) *If in addition we assume that*

$$\lim_{n \rightarrow +\infty} \varepsilon_n = 0,$$

then any of the above limits is a weak solution of (NS) satisfying the global energy inequality.

- (3) *Assume moreover that the sequence of scales ε_n satisfies $\varepsilon_n \gg \frac{1}{n^{2\alpha-2}}$ as $n \rightarrow +\infty$. More precisely, we assume*

$$(4.5) \quad \frac{1}{\varepsilon_n} = o(n^{2\alpha-2}) \quad \text{as } n \rightarrow +\infty,$$

then any of the above limits is in fact suitable.

PROOF. (of Theorem 4.2) To prove the first statement of Theorem 4.2, one can use a standard compactness argument based on the energy estimate (4.2). Since $\varepsilon_n \geq 0$, then we deduce that the sequence u^n is bounded in $L^2(H^1)$. Hence, we can extract a weakly convergent subsequence in $L^2(H^1)$. Moreover, this sequence is bounded in $L^\infty(L^2)$ and (using (4.1)) $\frac{d}{dt}u^n$ is bounded in a sufficiently negative Sobolev space $L^2(H^{-M})$, where one can take, for example, $M = \max(\frac{3}{2}, 2\alpha - 1)$. By Theorem 2.1 from [17], up to a subsequence, we also have strong convergence in $L^2(L^2)$. This completes the proof of the first statement.

To prove the second statement, we only need to show that for any smooth divergence free function φ , we have

$$\varepsilon_n \iint (-\Delta)^\alpha u^n \cdot \varphi \, dx \, dt \rightarrow 0, \quad \text{as } n \text{ tends to } \infty.$$

This follows by noticing that

$$\varepsilon_n \iint (-\Delta)^\alpha u^n \cdot \varphi \, dx \, dt = \varepsilon_n \iint u^n \cdot (-\Delta)^\alpha \varphi \, dx \, dt.$$

It remains to prove the last (and main) statement of Theorem 4.2. Similar to (3.2) we rewrite (4.1) as

$$(4.6) \quad \partial_t u^n - \nu \Delta u^n + (u^n \cdot \nabla) u^n - Q_n((u^n \cdot \nabla) u^n) - \varepsilon_n (-\Delta)^\alpha u^n + \nabla p^n = 0.$$

We multiply (4.6) by $2\varphi u^n$, where φ is nonnegative, smooth, compactly supported function, and integrate by parts to obtain

$$\begin{aligned} 2\nu \iint |\nabla u^n|^2 \varphi \, dx \, dt &= \iint |u^n|^2 (\varphi_t + \nu \Delta \varphi) \, dx \, dt + 2\varepsilon_n \iint (-\Delta)^\alpha u^n \cdot u^n \varphi \, dx \, dt \\ &+ \iint (|u^n|^2 + 2p^n) u^n \cdot \nabla \varphi \, dx \, dt - 2 \iint Q_n((u^n \cdot \nabla) u^n) \cdot u^n \varphi \, dx \, dt. \end{aligned}$$

Due to arguments in the previous section, it is sufficient to prove that the super-viscous term behaves properly;

$$\liminf_{n \rightarrow +\infty} \varepsilon_n \iint (-\Delta)^\alpha u^n \cdot u^n \varphi \, dx \, dt \geq 0,$$

and that the Q_n -term vanishes. That is, defining I_n as (3.6), we need to show that

$$\lim_{n \rightarrow +\infty} I_n = 0.$$

This is done in Lemmas 4.3 and 4.4 respectively, from which the theorem follows. \square

The following lemma deals with the super-viscosity term.

LEMMA 4.3. *For any sequence of non-negative numbers $\varepsilon_n \rightarrow 0$, and for any non-negative test function $\varphi \in C^\infty$ on $\mathbb{T}^3 \times (0, \infty)$, with $\text{supp } \varphi \subset \mathbb{T}^3 \times (0, T)$, we have*

$$(4.7) \quad \liminf_{n \rightarrow +\infty} \varepsilon_n \int_0^T \int_{\mathbb{T}^3} (-\Delta)^\alpha u^n \cdot u^n \varphi \, dx \, dt \geq 0.$$

PROOF. (of Lemma 4.3). Let $d\mu = \frac{1}{\ell^3} dx$ be a unit measure on \mathbb{T}^3 .

$$\varepsilon_n \int_0^T \int_{\mathbb{T}^3} (-\Delta)^\alpha u^n \cdot u^n \varphi \, d\mu \, dt = \left(\frac{2\pi}{\ell}\right)^{2\alpha} \varepsilon_n \int_0^T \sum_{k+j+m=0} |k|^{2\alpha} u_k^n u_j^n \varphi_m \, dt = A_n + B_n,$$

where we set

$$A_n = \left(\frac{2\pi}{\ell}\right)^{2\alpha} \varepsilon_n \int_0^T \sum_{k+j+m=0} |k|^\alpha u_k^n |j|^\alpha u_j^n \varphi_m dt$$

and

$$B_n = \left(\frac{2\pi}{\ell}\right)^{2\alpha} \varepsilon_n \int_0^T \sum_{k+j+m=0} |k|^\alpha u_k^n (|k|^\alpha - |j|^\alpha) u_j^n \varphi_m dt.$$

We have

$$A_n = \varepsilon_n \int_0^T \int_{\mathbb{T}^3} ((-\Delta)^{\alpha/2} u^n)^2 \varphi d\mu dt \geq 0.$$

Here we have used the fact that the pseudo-differential operator $(-\Delta)^{\alpha/2}$ preserves the reality of functions. To estimate B_n we use the inequality

$$(4.8) \quad \left| |k|^\alpha - |j|^\alpha \right| \leq C_\alpha (|k+j| |j|^{\alpha-1} + |k+j|^\alpha),$$

where one can take, for example, $C_\alpha = \alpha 2^{\alpha-1}$. Indeed, using the 1-dimensional inequality $|f(a) - f(b)| \leq |a-b| \sup_{[a,b]} |f'|$ we have for $\alpha - 1 \geq 0$ the following:

$$\left| |b+x|^\alpha - |b|^\alpha \right| \leq \alpha |x| (|b| + |x|)^{\alpha-1} \leq \alpha 2^{\alpha-1} |x| (|b|^{\alpha-1} + |x|^{\alpha-1}).$$

Set $b = |j|$ and $b+x = |k|$. Noticing that $|k+j| \geq ||k| - |j|| = |x|$, we arrive at (4.8).

Since $|k+j| = |m|$, then inequality (4.8) implies

$$\begin{aligned} B_n &\leq C \varepsilon_n \int_0^T \sum_{k+j+m=0} |k|^\alpha |u_k^n| |j|^{\alpha-1} |u_j^n| |m| |\varphi_m| dt \\ &\quad + C \varepsilon_n \int_0^T \sum_{k+j+m=0} |k|^\alpha |u_k^n| |u_j^n| |m|^\alpha |\varphi_m| dt = B'_n + B''_n. \end{aligned}$$

We first estimate B'_n as follows

$$B'_n \leq C \varepsilon_n \int_0^T \left(\sum_k |k|^{2\alpha} |u_k^n|^2 \right)^{1/2} \left(\sum_j |j|^{2\alpha-2} |u_j^n|^2 \right)^{1/2} \left(\sum_m |m| |\varphi_m| \right) dt.$$

Since φ is C^∞ -smooth we have that $\sup_{[0,T]} \sum_m |m| |\varphi_m|$ is bounded. Thus

$$\begin{aligned} B'_n &\leq C \varepsilon_n \int_0^T \|u^n\|_{H^\alpha} \|u^n\|_{H^{\alpha-1}} dt \leq C \varepsilon_n \int_0^T \|u^n\|_{H^\alpha}^{2-\frac{1}{\alpha}} \|u^n\|_{L^2}^{\frac{1}{\alpha}} dt = \\ &\quad C (\varepsilon_n)^{\frac{1}{2\alpha}} \int_0^T (\varepsilon_n \|u^n\|_{H^\alpha}^2)^{1-\frac{1}{2\alpha}} (\|u^n\|_{L^2}^2)^{\frac{1}{2\alpha}} dt. \end{aligned}$$

Using Hölder inequality

$$\int_0^T (\varepsilon_n \|u^n\|_{H^\alpha}^2)^{1-\frac{1}{2\alpha}} (\|u^n\|_{L^2}^2)^{\frac{1}{2\alpha}} dt \leq \left(\int_0^T \varepsilon_n \|u^n\|_{H^\alpha}^2 dt \right)^{\frac{2\alpha-1}{2\alpha}} \left(\int_0^T \|u^n\|_{L^2}^2 dt \right)^{\frac{1}{2\alpha}},$$

and taking into account the energy estimate (4.2) we have $B'_n \rightarrow 0$ as $n \rightarrow \infty$, since we have assumed that $\varepsilon_n \rightarrow 0$.

To estimate term B''_n we use similar arguments. Since φ is C^∞ -smooth we deduce that $\sup_{[0,T]} \sum_m |m|^\alpha |\varphi_m|$ is bounded. Thus

$$B''_n \leq C \sqrt{\varepsilon_n} \int_0^T \sqrt{\varepsilon_n} \|u^n\|_{H^\alpha} \|u^n\|_{L^2} dt \leq C \sqrt{\varepsilon_n} \left(\int_0^T \varepsilon_n \|u^n\|_{H^\alpha}^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \|u^n\|_{L^2}^2 dt \right)^{\frac{1}{2}}.$$

Again, using the energy estimate (4.2) we deduce that $B_n'' \rightarrow 0$ as $n \rightarrow \infty$. Taking into account the sign of term A_n , the proof of Lemma 4.3 is complete. \square

LEMMA 4.4. *Suppose that ε_n is a nonnegative sequence satisfying (4.5), then we have $\lim_{n \rightarrow +\infty} I_n = 0$.*

PROOF. By (4.2) we have $\varepsilon_n \int \sum |k|^{2\alpha} |u_k^n| dt \leq C$, therefore

$$\int_0^T \sum |k|^{2\alpha} |u_k^n| dt = o(n^{2\alpha-2}).$$

Thus

$$(4.9) \quad n^2 \sum_{|k| \geq \frac{n}{2}} \int_0^T |u_k^n|^2 dt \leq n^2 \left(\frac{n}{2}\right)^{-2\alpha} \sum_{|k| \geq \frac{n}{2}} \int_0^T |k|^{2\alpha} |u_k^n|^2 dt = o(1).$$

Applying the Hölder inequality to definition (3.6) of I_n , we obtain

$$|I_n| \leq C |u^n|_{L^\infty(\mathbb{R}, L^2(\mathbb{T}^3))} |\nabla u^n|_{L^2(\mathbb{R}, L^2(\mathbb{T}^3))} |Q_n(\varphi u^n)|_{L^2(\mathbb{R}, L^\infty(\mathbb{T}^3))}.$$

Both $|u^n|_{L^\infty(\mathbb{R}, L^2(\mathbb{T}^3))}$ and $|\nabla u^n|_{L^2(\mathbb{R}, L^2(\mathbb{T}^3))}$ are uniformly bounded (in n) by the L^2 norm of the initial data u_0 . Hence we only need to prove that

$$\lim_{n \rightarrow \infty} |Q_n(\varphi u^n)|_{L^2(\mathbb{R}, L^\infty(\mathbb{T}^3))} = 0.$$

Defining $g(t) = |Q_n(\varphi u^n)(t)|_{L^\infty(\mathbb{T}^3)}$ (which, of course, also depends on n), our goal is to estimate $\int_0^T g^2(t) dt$. Let

$$\Omega_{q,n} = \{l \in \mathbb{Z}^3 : |l| \leq n \text{ and } |q+l| > n\}.$$

Since the L^∞ norm of a function (including vector valued) is no grater than the l^1 norm of its Fourier coefficients and the operator Q_n can not increase absolute value of individual coefficients, we have

$$g(t) \leq \sum_{|k| > n} \sum_{|l| \leq n} |\varphi_{k-l}(t)| |u_l^n(t)| = \sum_{q \in \mathbb{Z}^3} |\varphi_q(t)| \sum_{l \in \Omega_{q,n}} |u_l^n(t)|.$$

Writing $|\varphi_q(t)| \sum_{l \in \Omega_{q,n}} |u_l^n(t)| = \sqrt{|\varphi_q(t)|} \left(\sqrt{|\varphi_q(t)|} \sum_{l \in \Omega_{q,n}} |u_l^n(t)| \right)$ and applying the Cauchy – Schwarz inequality, we have

$$g^2(t) \leq \left(\sum_{q \in \mathbb{Z}^3} |\varphi_q(t)| \right) \left(\sum_{q \in \mathbb{Z}^3} |\varphi_q(t)| \left(\sum_{l \in \Omega_{q,n}} |u_l^n(t)| \right)^2 \right).$$

Applying the Cauchy – Schwarz inequality in the following form

$$\left(\sum_{l \in \Omega_{q,n}} 1 \cdot |u_l^n(t)| \right)^2 \leq \sum_{l \in \Omega_{q,n}} 1^2 \sum_{l \in \Omega_{q,n}} |u_l^n(t)|^2 = |\Omega_{q,n}| \sum_{l \in \Omega_{q,n}} |u_l^n(t)|^2,$$

and defining $C = \sup_{t \in [0, T]} \sum_{q \in \mathbb{Z}^3} |\varphi_q(t)|$, we have

$$(4.10) \quad g^2(t) \leq C \sum_{q \in \mathbb{Z}^3} \left(|\varphi_q(t)| |\Omega_{q,n}| \sum_{l \in \Omega_{q,n}} |u_l^n(t)|^2 \right).$$

Since $|\Omega_{0,n}| = 0$ we can decompose the outer sum as follows

$$\sum_{q \in \mathbb{Z}^3} (\dots) = \sum_{0 < |q| \leq n/2} (\dots) + \sum_{|q| > n/2} (\dots).$$

Using the fact that φ is C^∞ and therefore $|\varphi_q(t)| \leq C_M/|q|^M$ for any M , and the inequality $|\Omega_{q,n}| \leq C' \min\{|q|n^2, n^3\}$ for a second constant C' , we have with $C'' = CC'$ the following

$$g^2(t) \leq C'' \left(\sum_{|q| \leq n/2} \frac{C_M n^2}{|q|^{M-1}} \sum_{l \in \Omega_{q,n}} |u_l^n(t)|^2 \right) + \frac{C''}{n} \left(\sum_{|q| > n/2} \frac{8C_M}{|q|^{M-4}} \sum_{l \in \Omega_{q,n}} |u_l^n(t)|^2 \right).$$

(For the first sum we assume $q \neq 0$). Finally, taking M large enough, so that both $\sum_{q \in \mathbb{Z}^3 \setminus 0} \frac{1}{|q|^{M-1}}$ and $\sum_{q \in \mathbb{Z}^3 \setminus 0} \frac{1}{|q|^{M-4}}$ converge, we get

$$g^2(t) \leq C''' n^2 \sum_{|k| \geq n/2} |u_k^n(t)|^2 + \frac{C'''}{n} \sum_{k \in \mathbb{Z}^3} |u_k^n(t)|^2.$$

Integrating from $t = 0$ to $t = T$ and using (4.9) we arrive at the statement of the lemma. \square

Remark. The result of the theorem 4.2 can be interpreted as follows. Since the super-viscous term is meant to be a perturbation of the Navier-Stokes system, one wants it to be as small as possible. The proof shows that in order to guarantee that the limit solution of the Navier – Stokes equations is suitable, a certain proportion of super-viscosity is needed. Below that amount, the regularizing term cannot play the rôle it is assigned.

5. Conditional suitability of the Galerkin scheme

THEOREM 5.1. *Assume that the initial state u_0 is in L^2 . Consider a (sub)sequence u^n of solutions for (3.1), that converges in the sense (2.1), (2.2). Assume further, that for some $T > 0$ this sequence satisfies the following property*

$$(5.1) \quad \text{for any } c > 0 \text{ we have } \lim_{n \rightarrow \infty} \int_0^T \sum_{n-c < |k| \leq n} |k|^2 |u_k^n(t)|^2 dt = 0.$$

Then the limit u is a suitable weak solution for the Navier-Stokes system.

PROOF. It is only remains to prove that I_n which defined in (3.6) tends to zero as $n \rightarrow \infty$. Following the steps of the proof of the Lemma 4.4 we introduce $g(t) = |Q_n(\varphi u^n)(t)|_{L^\infty(\mathbb{T}^3)}$, which also depends on n , but for brevity we do not indicate this in our notation. As in (4.10), setting $C = \sup_{t \in [0, T]} \sum_{q \in \mathbb{Z}^3} |\varphi_q(t)|$, we have

$$g^2(t) \leq C \sum_{q \in \mathbb{Z}^3} \left(|\varphi_q(t)| |\Omega_{q,n}| \sum_{l \in \Omega_{q,n}} |u_l^n(t)|^2 \right) = C \sum_{q \in \mathbb{Z}^3} g_q(t),$$

where $g_q(t) = |\varphi_q(t)| |\Omega_{q,n}| \sum_{l \in \Omega_{q,n}} |u_l^n(t)|^2$. Defining $G_q^n = \int_0^T g_q(t) dt$ we have

$$\int_0^T g^2(t) dt \leq \sum_{0 < |q| \leq n/2} G_q^n + \sum_{|q| > n/2} G_q^n.$$

The second sum clearly tends to zero as $n \rightarrow \infty$ due to rapid decay of φ_q . For the first sum we also use the rapid decay of φ_q , the inequality $|\Omega_{q,n}| < C|q|N^2$ and the fact that $|k| > N - |q|$ if $k \in \Omega_{q,n}$ to get

$$G_q(t) \leq C \frac{1}{|q|^M} (|q|n^2) \frac{1}{(n - |q|)^2} \sum_{n - |q| < |k| \leq n} |k|^2 \int_0^T |u_k^n(t)|^2 dt$$

with sufficiently large M . Since $|q| \leq n/2$ we have

$$G_q(t) \leq \frac{C}{|q|^{M-1}} \int_0^T \sum_{n-|q| < |k| \leq n} |k|^2 |u_k^n(t)|^2 dt.$$

We see that by (5.1) each fixed q the term G_q^n tends to zero as $n \rightarrow \infty$ while at the same times the terms are bounded by an $\ell^1(\mathbb{Z}^3)$ sequence. By the Lebesgue dominate convergence theorem we done. \square

As a corollary we have the following theorem

THEOREM 5.2. *If a Galerkin weak solution u of the Navier – Stokes system satisfies the global energy equality³, then it is suitable, that is, it satisfies the local energy inequality. However it moreover satisfies the local energy equality⁴ as well.*

PROOF. Indeed, the global energy equality implies that for the sequence of approximations we have convergence of the sequence of their norms $|u^n|_{L^2(H^1)} \rightarrow |u|_{L^2(H^1)}$. Together with weak convergence we obtain the strong convergence in $L^2(H^1)$ space. But this implies (5.1) and so we can use theorem 5.1 to concluded suitability. Since we also have an equality in (3.5) in this case, then finally we arrive to local energy equality, i.e., at (1.5) with equality there. \square

5.1. A new estimate. In this section we will give a new estimate on the Fourier coefficients of solutions of the Navier – Stokes equations, and of the coefficients in Galerkin approximations of these equations. Our main estimate in this section is independent of the dimension, and we will work on the d -dimensional torus. We emphasize that the estimates are also independent of the Galerkin truncation, and are satisfied uniformly in n , including $n = \infty$. In this estimate, we wish to keep track of its dependence on the size of the physical domain, as well as the coefficient of viscosity ν . Let $\mathbb{T}^d = \mathbb{R}^d / (\ell\mathbb{Z})^d$ be the fundamental domain for functions which are ℓ -periodic, however for convenience we will normalize the measure on \mathbb{T}^3 to be 1. This sets the L^p norm of any (scalar or vector valued) function g to be

$$\|g\|_{L^p} = \left(\frac{1}{\ell^d} \int_{\mathbb{T}^d} |g(x)| dx \right)^{1/p}.$$

For any $k \in \mathbb{Z}^d$ the k -th Fourier coefficient g_k of a function or a distribution g is defined by the Fourier series expansion

$$(5.2) \quad g(x) = \sum_{k \in \mathbb{Z}^d} g_k \exp\left(\frac{2\pi i}{\ell} k \cdot x\right);$$

equivalently, we can define them by $g_k = \ell^{-d} \int_{\mathbb{T}^d} g(x) \exp(-\frac{2\pi i}{\ell} k \cdot x) dx$. For each $s \geq 0$, define the Sobolev space H^s as a subset of L^2 which consists of functions with finite homogeneous H^s -Sobolev (semi) norm:

$$\|g\|_{H^s}^2 := \|(-\Delta)^{s/2} g\|_{L^2}^2 = \sum_{k \in \mathbb{Z}^d} \left| \frac{2\pi}{\ell} k \right|^{2s} |g_k|^2.$$

³i.e. (1.8) with equality

⁴i.e. (1.5) with equality

For $s < 0$ we define the Sobolev space H^s as completion of L^2 with respect to the semi-norm

$$\|g\|_{H^s}^2 := \sum_{k \in \mathbb{Z}^d \setminus 0} \left| \frac{2\pi}{\ell} k \right|^{2s} |g_k|^2 .$$

For any two vectors $x, k \in \mathbb{R}^d$ with $k \neq 0$, we denote by $\pi_k(x)$ the orthogonal projection of x onto the hyperplane orthogonal to k , namely

$$(5.3) \quad \pi_k(x) = x - (x \cdot k) \frac{k}{|k|^2}, \quad k \neq 0 \quad \text{for which} \quad |\pi_k(x)| \leq |x| .$$

The latter statement is the remark that the euclidean length of an orthogonal projection is non-increasing. If $u(t, x)$ is a weak solution of Navier-Stokes system, satisfying (1.4) and (1.8), then its k -th Fourier mode $u_k(t)$ satisfies

$$(5.4) \quad \begin{aligned} \frac{d}{dt} u_k + \nu \left(\frac{2\pi}{\ell} |k| \right)^2 u_k + \frac{2\pi i}{\ell} \pi_k \left(\sum_{m \in \mathbb{Z}^d} (u_{k-m} \cdot m) u_m \right) &= 0 \quad \text{if } k \neq 0 \\ u_k \cdot k &= 0, \quad \frac{d}{dt} u_0 = 0 . \end{aligned}$$

Similarly, considering the Galerkin approximation (3.1), we have (5.4) restricted to wavenumbers $|k| \leq n$. The following theorem treats both solutions of the Navier – Stokes equations, and solutions of the Galerkin approximation, for which the constants are uniform in $n \in \mathbb{N} \cup \{+\infty\}$ and the dimension d of space. In our presentation we omit reference to the index n for brevity.

THEOREM 5.3. *Assume that the initial data is such that $u(0) \in L^2(\mathbb{T}^d)$, and that its Fourier coefficients $u_k(0)$ satisfy*

$$(5.5) \quad |u_k(0)| \leq \frac{C}{|k|}$$

with some positive constant C . Then there exists C_1 and C_2 such that for any later time $T > 0$ each of the Fourier coefficients $u_k(t)$ satisfies

$$(5.6) \quad |u_k(T)| \leq \frac{C_1}{|k|},$$

$$(5.7) \quad \int_0^T |u_k(t)|^2 dt \leq \frac{C_2}{|k|^4},$$

where one can take, for example, $C_1 = \max\{C, \frac{\ell \|u(0)\|_{L^2}}{2\pi\nu}\}$ and $C_2 = \frac{\ell^2 \|u(0)\|_{L^2}^4}{8\pi^2\nu^3} + \frac{\ell^2 C^2}{4\pi^2\nu^2}$. If (5.5) holds only for some k 's then (5.6) and (5.7) hold for these k 's. Regardless of (5.5) we have

$$(5.8) \quad \int_0^T |u_k(t)| dt \leq \frac{C_3}{|k|^2},$$

where $C_3 = \frac{\ell^2 |u_k(0)|}{4\pi^2\nu} + \sqrt{2T\nu} \frac{\ell}{4\pi} \|u(0)\|_{L^2}^2$.

The proof of Theorem 5.3 is based on the following lemma, which has to do with elementary behavior of vector fields.

LEMMA 5.4. *Let $X(t)$ and $f(t)$ be two smooth curves in \mathbb{C}^d ($d \geq 1$) satisfying*

$$(5.9) \quad \frac{d}{dt} X(t) = -aX(t) + f(t) ,$$

where a is a positive real number. Then for any $T > 0$ we have

$$(5.10) \quad \sup_{t \in [0, T]} |X(t)| \leq \max \left\{ |X(0)|, \frac{1}{a} \sup_{t \in [0, T]} |f(t)| \right\};$$

$$(5.10') \quad |X(t)| \leq \left(|X(0)| - \frac{\sup_{\tau \in [0, T]} |f(\tau)|}{a} \right) e^{-at} + \frac{\sup_{\tau \in [0, T]} |f(\tau)|}{a} \quad \forall t \in [0, T];$$

$$(5.11) \quad \int_0^T |X(t)|^2 dt \leq \frac{1}{a^2} \int_0^T |f(t)|^2 dt + \frac{1}{a} |X(0)|^2;$$

$$(5.12) \quad \int_0^T |X(t)| dt \leq \frac{|X(0)| + \int_0^T |f(t)| dt}{a}.$$

PROOF. (of Theorem 5.3 modulo Lemma 5.4.) Set $X(t) = u_k(t)$. Then according to (5.4), X satisfies (5.9) with $a = \nu \left(\frac{2\pi}{\ell} |k| \right)^2$ and $f(t) = -\frac{2\pi i}{\ell} \pi_k \left(\sum_{m \in \mathbb{Z}^d} (u_{k-m} \cdot m) u_m \right)(t)$. The condition of incompressibility implies that $(u_{k-m} \cdot m) = (u_{k-m} \cdot k)$; then using (5.3) and Cauchy-Schwartz inequality, we deduce that

$$\sup_{t \geq 0} |f(t)| \leq \frac{2\pi |k|}{\ell} \|u(0)\|_{L^2}^2.$$

Estimate (5.6) is then straightforward. Furthermore,

$$(5.13) \quad \begin{aligned} |f(t)| &\leq \frac{2\pi}{\ell} \sum_{m \in \mathbb{Z}^d} |u_{k-m}(t)| |m| |u_m(t)| \\ &\leq \frac{2\pi}{\ell} \left(\sum_{m \in \mathbb{Z}^d} |u_{k-m}(t)|^2 \right)^{\frac{1}{2}} \left(\sum_{m \in \mathbb{Z}^d} |m|^2 |u_m(t)|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{2\pi}{\ell} \|u(t)\|_{L^2} \|\nabla u(t)\|_{L^2}. \end{aligned}$$

Hence

$$\int_0^T |f(t)|^2 dt \leq \left(\frac{2\pi}{\ell} \right)^2 \|u\|_{L^\infty((0, T), L^2)}^2 \|\nabla u(t)\|_{L^2((0, T), L^2)}^2 \leq \frac{1}{2\nu} \left(\frac{2\pi}{\ell} \right)^2 \|u(0)\|_{L^2}^4.$$

In the last estimate we used the energy inequality

$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \leq \|u(0)\|_{L^2}^2.$$

The proof of (5.7) is then achieved using (5.11). The estimate (5.8) follows from (5.12) and the standard inequality $\int_0^T |f(t)| dt \leq \sqrt{T} \left(\int_0^T |f(t)|^2 dt \right)^{1/2}$. \square

PROOF. (of Lemma 5.4.) We start from the inequality

$$(5.14) \quad \frac{d}{dt} |X(t)| \leq -a |X(t)| + |f(t)|$$

from which one deduces that $|X(t)|$ is decreasing whenever $|X(t)| > \frac{|f(t)|}{a}$; the estimate (5.10) follows.

The explicit solution of (5.9) is given by

$$X(t) = e^{-at} \left(X_0 + \int_0^t e^{as} f(s) ds \right).$$

Hence $|X(t)| \leq e^{-at}(|X(0)| + \frac{e^{at}-1}{a} \sup_{\tau \in [0,t]} |f(\tau)|)$ and therefore (5.10') follows.

To prove (5.11), multiply (5.9) by $\overline{X}(t)$, integrate in time and use the Cauchy-Schwartz inequality to obtain

$$\frac{1}{2}|X(T)|^2 - \frac{1}{2}|X(0)|^2 \leq -a \int_0^T |X(t)|^2 dt + \sqrt{\int_0^T |X(t)|^2 dt} \sqrt{\int_0^T |f(t)|^2 dt}.$$

Using the standard inequality $xy \leq \frac{a}{2}x^2 + \frac{y^2}{2a}$ we arrive at (5.11). To prove estimate (5.12) we integrate (5.14) from 0 to T and note that $|X(T)| \geq 0$. \square

Estimate (5.7) implies that the crucial term in (5.1) is $\mathcal{O}(1)$ but not $o(1)$, hence this strategy has not shown that Galerkin solutions are suitable. In fact another ingredient is necessary for this result to hold as is shown by the following construction.

5.2. A counterexample to the method. In fact, the phenomenon exhibited by the sequence of steps in the above estimate is not completely an artifice of the proof of the inequality. It is uncomfortably close to the frontier between weak and strong solutions, and the behavior of the error term I_n at a hypothetical singular time $T \in \mathbb{R}^+$ could conceivably be that nontrivial H^1 mass of a solution is propagated to infinity in Fourier space as $N \rightarrow \infty$. To exhibit this, we construct an example to show that the input (or ‘black-box data’) for the method cannot preclude that $I_n = \mathcal{O}(1)$ and not $o(1)$ as $n \mapsto \infty$. Let $u^n(x, t)$ be a solution for n -th Galerkin approximation of Navier-Stokes equation (for which in particular $u_k^n \equiv 0$ for $|k| \geq n$) and consider $u_k^n(t) = \int u^n(t, x) \exp(-ik \cdot x) dx$. Then functions $u_k^n(t)$ satisfy

$$(5.15) \quad \partial_t |u_k^n| \leq -k^2 |u_k^n| + f_k^n(t) \quad \text{where} \quad f_k^n(t) \leq \sum_{l \in \mathbb{Z}^3} |u_l \cdot k| |u_{k-l}|,$$

and

$$(5.16) \quad (u_k, k) = 0.$$

The basic energy inequality implies that we have

$$(5.17) \quad \int_0^T \sum_{|k| \leq n} |k|^2 |u_k^n(t)|^2 dt \leq C, \quad \sup_{t \in [0, T]} \sum_{|k| \leq n} |u_k^n(t)|^2 \leq C,$$

while Theorem 5.3 gives that there exists an ε such that for each k and n ,

$$(5.18) \quad \sup_{t \in [0, T]} |u_k^n(t)| \leq \frac{c}{|k|}, \quad \int_0^T |u_k^n(t)|^2 dt \leq \frac{c}{|k|^4}, \quad \int_0^T |u_k^n(t)| dt \leq \frac{c}{|k|^2}.$$

The goal is to establish whether or not we have

$$(5.19) \quad \lim_{n \rightarrow \infty} \int_0^T \sum_{n-7 < |k| \leq n} |k|^2 |u_k^n|^2 dt = 0?$$

The ‘counterexample’ is a sequence of collections of functions $\{\{u_k^n(t)\}_{|k| \leq n}\}_{n=100}^\infty$, which are not solutions to the Galerkin approximation to the Navier-Stokes, however do satisfy (5.15) for $|k| > \frac{2}{3}n$ (and hence for $|k| \geq n-7$), as well as all of the

estimates (5.16)(5.17)(5.18). However (5.19) is violated. Essentially set

$$|u_k^n(t)| = \begin{cases} \frac{1}{n^{3/2}}, & \text{for } \frac{n}{3} < |k| \leq \frac{2n}{3} \text{ and } 1 - \frac{1}{n^2} < t < 1; \\ 0, & \text{otherwise for } |k| < n - 7; \\ \text{“maximal” solution of (5.15),} & \text{for } |k| \geq n - 7. \end{cases}$$

While considering the maximal solution above it is sufficient to consider

$$f_k^n(t) = \sum_{\max\{|l|, |k-l|\} < n-7} |u_l \cdot k| |u_{k-l}|.$$

One can furthermore implement that $k \cdot u_k(t) = 0$, so that (5.16) is satisfied. However all this can be done so that u_k^n satisfies $\sum_l |u_l \cdot k| |u_{k-l}| \sim cn$. Such sequences concentrate all of their H^1 mass in an annulus of radius n , but of small cross section, essentially transporting H^1 mass to infinity for fixed t as $n \mapsto \infty$.

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