

Bounds on Kolmogorov spectra for the Navier – Stokes equations

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Abstract

Let $u(x, t)$ be a (possibly weak) solution of the Navier - Stokes equations on all of \mathbb{R}^3 , or on the torus $\mathbb{R}^3/\mathbb{Z}^3$. Denoting the Fourier transform by $\hat{u} = \mathcal{F}u$, the *energy spectrum* of $u(\cdot, t)$ is the spherical integral

$$E(\kappa, t) = \int_{|k|=\kappa} |\hat{u}(k, t)|^2 dS(k), \quad 0 \leq \kappa < \infty,$$

or alternatively, a suitable approximate sum. An argument involving scale invariance and dimensional analysis given by Kolmogorov [7, 9] and Obukhov [14] predicts that in three dimensions large Reynolds number solutions of the Navier - Stokes equations should obey

$$E(\kappa, t) \sim C_0 \varepsilon^{2/3} \kappa^{-5/3}$$

over an *inertial range* $\kappa_1 \leq \kappa \leq \kappa_2$, at least in an average sense. We derive a global estimate on weak solutions in the norm $\|\mathcal{F}\partial_x u(\cdot, t)\|_\infty$ which gives bounds on a solution's ability to satisfy the Kolmogorov law. A subsequent result gives rigorous upper and lower bounds on the inertial range, and an upper bound on the time of validity of the Kolmogorov spectral regime.

Keywords: Navier – Stokes equations, energy spectrum, inertial range

1. Introduction

An important issue in the study of solutions of the Navier – Stokes equations in the large is the principle governing the distribution of energy in Fourier space. The theory of Kolmogorov [7, 8, 9] and Obukhov [14] plays a central rôle, predicting power law decay behavior of the Fourier space energy density for solutions which exhibit fully developed turbulence. In outline, a basic prediction is that energy spectral functions $E(\kappa, t)$, or possibly its average over a statistical ensemble, is expected to satisfy

$$E(\kappa, t) \simeq C_0 \varepsilon^{2/3} \kappa^{-5/3} \tag{1}$$

over an inertial range of wavenumbers $\kappa \in [\kappa_1, \kappa_2]$, where C_0 is a dimensionless constant, ε is a parameter interpreted physically as the energy transfer rate per unit volume, and the exponents are determined by dimensional analysis [14][6]. This famous statement has been very influential in the field, and considerable experimental and numerical evidence has been gathered to support it. Despite its success, there have been

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relatively few rigorous mathematical results on the analysis of solutions of the Navier – Stoke equations, with or without bulk *inhomogeneous* forces, which have addressed the question as to whether solutions exhibit spectral behavior as described by (1). Among those papers which do address certain aspects of these questions, we cite in particular two sources. Firstly, the book by C. Doering & J. Gibbon [5] reviews the Kolmogorov – Obukhov theory, and discusses the compatibility of spectral aspects of solutions with the L^2 regularity theory for the Navier – Stokes equations. Secondly, S. Kuksin [11] proves that solutions to the nonlinear Schrödinger equation with added dissipation and with stochastic forces exhibit spectral behavior over an inertial range, with some positive exponent (which is not known explicitly). This latter work serves as an important mathematical model of generation of spectral behavior of solutions under stochastic forcing, despite the basic difference in the equations that are addressed.

In this paper we give a new global estimate in the norm $\|\mathcal{F} \partial_x u(\cdot, t)\|_{L^\infty}$ on weak solutions of the Navier – Stokes equations which have reasonably smooth initial data, where it is also possible that the solution is subject to reasonably smooth inhomogeneous forces. This estimate has implications on the energy spectral function for such solutions, and in particular in the case that there is no inhomogeneous force, we show that weak solutions of the initial value problem have spectral energy function which for all $\kappa \in \mathbb{R}^+$ satisfies

$$E(\kappa, t) \leq 4\pi R_1^2, \quad (2)$$

and time averages which satisfy

$$\frac{1}{T} \int_0^T E(\kappa, t) dt \leq \frac{4\pi R_2^2}{\nu \kappa^2 T}, \quad (3)$$

again for all κ , where ν is the coefficient of viscosity. In the case that a bounded inhomogeneous force is applied to the solution of the initial value problem, we find similarly that

$$E(\kappa, t) \leq 4\pi R_1^2(t), \quad (4)$$

and furthermore

$$\frac{1}{T} \int_0^T E(\kappa, t) dt \leq \frac{4\pi R_2^2(T)}{\nu \kappa^2 T}, \quad (5)$$

for constants $R_1(t)$ and $R_2(t)$ which may be time dependent. In the situation of forcing being given by a stationary process, it is to be expected that for typical solutions the quantity $R_2^2(T)/T$ has a limit $\overline{R_2^2}$ for large T , giving a constant upper bound for the time average of $E(\kappa, T)$. Since these estimates give rigorous bounds on $E(\kappa, t)$ with a faster rate of decay in wavenumber κ than that stated in (1), this result presents a conundrum. Either it is the case that solutions which exhibit large scale spectral behavior as in (1) are not smooth, and in particular do not arise from the initial value problem with reasonably smooth initial data. Or else the bounds (2)(3) (and (4)(5) respectively, in the case with inhomogeneous forces) give restrictions on the spectral behavior of solutions, and in particular an upper bound on the value of the parameter ε , a restriction on the extent of the inertial range $[\kappa_1, \kappa_2]$, and in the case of (2)(3), an upper bound on the time interval $[0, T_0]$ over which spectral behavior may occur for a solution.

There is a well developed literature on the energy transfer rate ε , and other aspects of the Kolmogorov – Obukhov theory, based on physical assumptions on the character of the fluid motion. These assumptions are for flows exhibiting fully developed turbulence, and are described in Obukhov [14], for example. They include the hypothesis that the flow is in a stochastically steady state, the energy transfer rate is of a certain form and exhibits a particular scale invariance, and that the energy spectral function is negligible for wave numbers higher than a cutoff κ_ν . Under these assumptions, the cutoff κ_ν is determined (it is known as the Kolmogorov scale) and the energy transfer rate ε is identified with the energy dissipation rate

$$\varepsilon_1 := \frac{\nu}{(2\pi)^3} \int_0^{+\infty} \kappa^2 E(\kappa, t) d\kappa. \quad (6)$$

Using the latter statement, that $\varepsilon = \varepsilon_1$, it is possible to find a better estimate of the behavior of ε with respect to Reynolds' number than ours in this paper, as for example in C. Doering & C. Foias [4]. The difference between this body of work and our analysis is that we make no physical assumptions on solutions of the Navier – Stokes equations, deducing our conclusions purely from known results about such flows. It is worthwhile to point out as well that the upper and lower bounds to the inertial range in our work are conclusions, as compared to prior work in which the upper bound κ_ν on the inertial range is an assumption of the theory, and no explicit lower bound is given.

In Section 2 we give a statement and the proofs of our estimates on the Fourier transform of weak solutions of the Navier – Stokes equations, posed either on all of $x \in \mathbb{R}^3$ or else for $x \in \mathbb{T}^3$. Since there is no known uniqueness result, one cannot speak of the solution map for Navier – Stokes flow, and we emphasize that this estimate is valid for any weak solution that satisfies the energy inequality. In section 3 we interpret these estimates in the context of the spectral energy function, and we analyse the constraints on spectral behavior of solutions mentioned above, giving specific and dimensionally appropriate estimates for the endpoints of the inertial range κ_1, κ_2 . In the case of no inhomogeneous forces, we give an upper bound T_0 on the time of validity of the spectral regime. The bounds on κ_1 and κ_2 are also valid in the probabilistic setting, for statistical ensembles of solutions. That is, suppose that one is given an ergodic probability measure $(\mathbb{P}, \mathcal{M})$ on the space of divergence free vector fields which is invariant under some choice of definition of Navier – Stokes flow. As long as the inhomogeneous force and the support of the invariant probability measure are contained in the closure of the set of reasonably smooth divergence free vector fields, then our constraints on the spectral behavior of solutions apply. The final section gives a comparison of our constraints on $(\kappa_1, \kappa_2, T_0)$ to the Kolmogorov length and time-scales of the classical theory, and a discussion of the dimensionless parameter $r_\nu := \kappa_2/\kappa_1$ as an indicator of spectral behavior of solutions.

2. Estimates on the Fourier transform in L^∞

The incompressible Navier – Stokes equations in their usual form are written for the velocity field $u(x, t)$ of a fluid, its pressure $p(x, t)$, and a divergence-free force $f(x, t)$,

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u &= -\nabla p + \nu \Delta u + f \\ \nabla \cdot u &= 0, \\ u(\cdot, 0) &= u_0(\cdot), \end{aligned} \tag{7}$$

where we consider spatial domains either all of Euclidian space $x \in \mathbb{R}^3$, or else the compact and boundaryless torus $x \in \mathbb{T}^3 := \mathbb{R}^3/\Gamma$, where $\Gamma \subseteq \mathbb{R}^3$ is a lattice of full rank. Denote by D either of the above spatial domains. The time domain is $0 < t < +\infty$, and the inhomogeneous force function f is assumed to be divergence-free and to satisfy $f \in L_{loc}^\infty([0, +\infty); H^{-1}(D) \cap L^2(D))$. A ‘*Leray*’ weak solution to (7) on $D \times [0, +\infty)$ satisfies the three conditions.

1. *Integrability conditions:* For any $T > 0$ the vector function (u, p) lies in the following function spaces

$$u \in L^\infty([0, T]; L^2(D)) \cap L^2([0, T]; \dot{H}^1(D)), \tag{8}$$

$$p \in L^{5/3}(D \times [0, T]), \tag{9}$$

2. *Weak solution of the equation:* the pair (u, p) is a distributional solution of (7), and furthermore $\lim_{t \rightarrow 0^+} u(\cdot, t) = u_0(\cdot)$ exists in the strong L^2 sense,
3. *Energy inequality:* the energy inequality is satisfied

$$\begin{aligned} \frac{1}{2} \int_D |u(x, t)|^2 dx + \nu \int_0^t \int_D |\nabla u(x, s)|^2 dx ds \\ - \int_0^t \int_D u(x, s) \cdot f(x, s) dx ds \leq \frac{1}{2} \int_D |u_0(x)|^2 dx \end{aligned} \tag{10}$$

for all $0 < t < +\infty$. The inequality (10) is an identity for solutions which are regular. It is well known that weak solutions exist globally in time, either when $f = 0$, a result due to Leray [12, 13], or when f is nonzero. The question of their uniqueness and regularity remains open.

Many facts are known about weak solutions, including that for any $T > 0$ the interpolation inequalities hold; $u \in L^s([0, T]; L^p(D))$ for all $3/p + 2/s = 3/2$, for $2 \leq p \leq 6$. That the $L^{5/3}$ estimate for the pressure in (9) is sufficient is due to [15]. Considering a weak solution as a curve in $L^2(D)$ defined over $t \in \mathbb{R}^+$, the following proposition holds.

Proposition 2.1. *A weak solution is a mapping $[0, T] \mapsto L^2(D)$ satisfying the continuity properties*

$$u(\cdot) \in L^\infty([0, T]; L^2(D)) \cap C([0, T]; L^2(D)_{\text{weak topology}}) \cap C([0, T]; H^{-\delta}(D)) \quad (11)$$

for any $\delta > 0$. Furthermore, as a curve in Sobolev space,

$$\frac{du}{dt} \in L^2([0, T]; H^{-3/2}(D)) .$$

A clear exposition which includes these basic facts is the lecture notes of J.-Y. Chemin [2].

Being a curve in $L^2(D)$, the Fourier transform of a weak solution makes sense, and $\hat{u}(\cdot, t) = \mathcal{F}u(\cdot, t)$ is again a curve in $L^\infty([0, T]; L_k^2) \cap C([0, T]; L_{k; \text{weak topology}}^2)$. We will make use of a dimensionally adapted Fourier transform \mathcal{F} , namely

$$\hat{u}(k) = (\mathcal{F}u)(k) := \frac{1}{V^{1/2}} \int_D e^{-ik \cdot x} u(x) dx , \quad (12)$$

where $k \in \mathbb{R}^3$ when the spatial domain is $D = \mathbb{R}^3$, and we set $V = (2\pi)^3$ in standard units of volume in \mathbb{R}^3 . In this setting, the norm of the Fourier transform is given by $\|\hat{u}\|^2 = \int_{\mathbb{R}^3} |\hat{u}(\xi)|^2 d\xi$. When the domain is $D = \mathbb{T}^3 = \mathbb{R}^3/\Gamma$, we take $k \in \Gamma'$ the lattice dual to Γ , we set $V = |\Gamma| := \text{vol}(\mathbb{R}^3/\Gamma)$, and we define $\|\hat{u}\|^2 := \sum_{k \in \Gamma'} |\hat{u}(k)|^2 |\Gamma'|$. With this choice, the Plancherel identity reads

$$\|u\|^2 = \frac{V}{(2\pi)^3} \|\hat{u}\|^2 . \quad (13)$$

With respect to the normalization, the function $u(x, t)$ has units of velocity L/T , and its Fourier transform is such that $|\hat{u}(k, t)|^2$ has units of Fourier space energy density $(L/T)^2 L^3$.

2.1. An estimate on $\mathcal{F}\partial_x u(\cdot, t)$ on the torus

Focus the discussion on the case of the spatial domain $D = \mathbb{T}^3$. Then any choice of initial data $u_0(x) \in L^2(\mathbb{T}^3)$ has uniformly bounded Fourier coefficients, indeed $|\hat{u}_0(k)| \leq \|u_0\|$. Furthermore, since the complex exponential $e^{ik \cdot x}$ is a perfectly good element of $(H^{-3/2})^*$ which, being tested against $u(\cdot, t)$ gives the Fourier coefficients, we also have the result

Proposition 2.2. *For each $k \in \Gamma'$ the Fourier coefficient $\hat{u}(k, t) \in \mathbb{C}^3$ is a Lipschitz function of $t \in \mathbb{R}^+$.*

It is again made clear by this that the problem of singularity formation is not that $\hat{u}(k, t)$ becomes unbounded, but rather that H^1 mass, including possibly L^2 mass, is propagated to infinity in k -space in finite time.

A (future) invariant set A is one such that $u_0 \in A$ implies for all $t > 0$, $u(t) \in A$ as well. When $f = 0$ the energy inequality (10) can be viewed as implying that the ball $B_R(0) \subseteq L^2(D)$ is an invariant set for weak solutions satisfying $\|u_0\|_{L^2(D)} \leq R$. In similar terms, we give another invariant set for weak solutions. Define the set

$$A_{R_1} := \{u : \forall k \in \Gamma', |k| |\hat{u}(k)| \leq R_1\} , \quad (14)$$

and as above let $B_R(0)$ denote the ball of radius R in $L^2(\mathbb{T}^3)$.

Theorem 2.3. *In the case that $f = 0$, whenever*

$$\frac{R^2}{\sqrt{V}} \leq \nu R_1 \quad (15)$$

then the set $A_{R_1} \cap B_R(0)$ is invariant for weak solutions of the Navier – Stokes equations (7). Thus, if the initial data satisfies $u_0 \in A_{R_1} \cap B_R(0)$, for all $0 < t < +\infty$ the Fourier coefficients of any Leray weak solution emanating from this data satisfy

$$\sup_{0 < t < +\infty} |\hat{u}(k, t)| \leq \frac{R_1}{|k|}, \quad \forall k \in \Gamma'. \quad (16)$$

This result appears in the paper [1] in a slightly different form. For nonzero f the ball $B_R(0) \subseteq L^2(D)$ is not necessarily invariant. However given $u_0 \in B_R(0)$ and our hypothesis that $f \in L_{loc}^\infty([0, +\infty); H^{-1}(D) \cap L^2(D))$, there is always a nondecreasing function $R(T) \geq R$ such that for all $T > 0$, $u(\cdot, T) \in B_{R(T)}(0)$. Indeed, suppose that a Galilean frame is chosen and the pressure p is suitably normalized so that $\int_D u(x, T) dx = 0 = \int_D f(x, T) dx$. Let $F^2(T) := \int_0^T \|f\|_{H^{-1}}^2 dt$. Then by standard interpolation one has that

$$\|u(\cdot, T)\|_{L^2}^2 + \nu \int_0^T \|\nabla u(\cdot, s)\|_{L^2}^2 ds \leq R^2(T). \quad (17)$$

The function $R^2(T)$ is an upper bound for the LHS of the energy inequality, for which there is an estimate $R^2(T) \leq R^2 + \frac{1}{\nu} F^2(T)$. In case of a bounded inhomogeneous force $f(\cdot, t) \in L^\infty(\mathbb{R}^+; H^{-1} \cap L^2)$, there is an upper estimate $F^2(T) \leq CT$, so that $R^2(T)$ exhibits (not more than) linear growth in T .

Theorem 2.4. *In the case of nonzero $f(x, t)$, let $R(t)$ be an a priori upper bound for $\|u(\cdot, t)\|_{L^2}$, for example as in (17). Suppose that $R_1(t)$ is a nondecreasing function such that for all (k, t) we have*

$$\frac{R^2(t)}{\sqrt{V}} + \frac{|\hat{f}(k, t)|}{|k|} < \nu R_1(t), \quad (18)$$

then the set $\{(u, t) : 0 < t, u(\cdot, t) \in A_{R_1(t)} \cap B_{R(t)}(0)\}$ is invariant for weak solutions of the equations (7). That is, if the initial data satisfies $u_0 \in A_{R_1(0)} \cap B_{R(0)}(0)$ then for all $0 < t < +\infty$ the Fourier coefficients of any weak solution emanating from u_0 and being subject to the force f will obey the estimate

$$|\hat{u}(k, t)| \leq \frac{R_1(t)}{|k|}. \quad (19)$$

It is a common situation for the inhomogeneous force to have properties of recurrence, such as if it were time-periodic, or if given by a statistical process which is stationary with respect to time. For bounded f as above, the estimate $F^2(T) \leq CT$ holds. Furthermore, one is interested in those solutions which are themselves statistically stationary. For these solutions it will be the case that the force adds to the total energy at the same rate as the dissipation depletes it. Indeed, one assumes that there is a constant \bar{R} such that $\|u(\cdot, T)\|_{L^2}^2 \leq \bar{R}^2$ and $\nu \int_0^T \|\nabla u(\cdot, s)\|_{L^2}^2 ds \leq \bar{R}^2 T$. Therefore, in the statistically stationary case we expect that for typical solutions the upper bound $R(t)$ in the hypotheses of Theorem 2.4 is given by the constant \bar{R} , which in particular gives a uniform bound in time. In this situation, the constant R_1 is also time independent.

Proof of Theorems 2.3 and 2.4. For each $k \in \Gamma'$ the vector $\hat{u}(k) \in \mathbb{C}_k^2 \subseteq \mathbb{C}^3$, where $\mathbb{C}_k^2 = \{w \in \mathbb{C}^3 : w \perp k = 0\}$ is specified by the divergence-free condition. Because (u, p) is a distributional solution, the Fourier

coefficients satisfy

$$\begin{aligned}\partial_t \hat{u}(k, t) &= -\nu |k|^2 \hat{u}(k, t) - i \Pi_k k \cdot \frac{1}{\sqrt{V}} \sum_{k_1} \hat{u}(k - k_1) \otimes \hat{u}(k_1) + \hat{f}(k, t), \\ &:= X(u)_k,\end{aligned}\tag{20}$$

at least in the weak sense, after testing with a smooth cutoff function $\varphi(t) \in C^\infty$. We use the notation that $X(u)_k$ is the k^{th} component of the vector field represented by the RHS. The convolution has introduced the factor $1/\sqrt{V}$. The operator $\Pi_k : \mathbb{C}^3 \rightarrow \mathbb{C}_k^2$ is given by

$$\Pi_k(z) = z - (z \cdot k) \frac{k}{|k|^2}.$$

The Leray projector onto the divergence-free distributional vector fields, considered in Fourier space coordinates, is the direct sum of the Π_k .

The radial component of the vector field $X(u)_k$ in $\mathbb{C}_k^2 \subseteq \mathbb{C}^3$ is expressed by $\text{re}(\overline{\hat{u}(k)} \cdot X(u)_k)/|\hat{u}(k)|$. Consider first Theorem 2.3, which is the case that $f = 0$. Suppose that $|k||\hat{u}(k)| = R_1$ for some k , that is, the solution is on the boundary of the region A_{R_1} . Since $|\Pi_k k \cdot \sum_{k_1} \hat{u}(k - k_1) \otimes \hat{u}(k_1)| \leq |k| \|u\|_{L^2}^2$, an estimate of the radial component of $X(u)_k$ is that

$$\begin{aligned}\text{re}(\overline{\hat{u}(k)} \cdot X(u)_k) &= -\nu |k|^2 |\hat{u}(k)|^2 + \frac{1}{\sqrt{V}} \text{im}(\overline{\hat{u}(k)}) k \Pi_k \sum_{k_1} \hat{u}(k - k_1) \otimes \hat{u}(k_1) \\ &\leq -\nu R_1^2 + \frac{1}{\sqrt{V}} \|u(\cdot, t)\|_{L^2}^2 R_1.\end{aligned}$$

When $u(\cdot, t) \in B_R(0)$ and $R^2 < \nu R_1 \sqrt{V}$ the RHS is negative, implying that integral curves $\hat{u}(k, t)$ cannot exit the region. Thus the ball of radius $R_1/|k|$ in \mathbb{C}_k^2 is a trapping set, or a future invariant set, for the vector field $X(u)_k$.

In case of the presence of a force f , suppose again that $|k||\hat{u}(k, t)| = R_1(t)$ for some (k, t) . The radial component of $X(u)_k$ satisfies

$$\begin{aligned}\text{re}(\overline{\hat{u}(k)} \cdot X(u)_k) &= -\nu |k|^2 |\hat{u}(k)|^2 + \frac{1}{\sqrt{V}} \text{im}(\overline{\hat{u}(k)}) k \Pi_k \sum_{k_1} \hat{u}(k - k_1) \otimes \hat{u}(k_1) \\ &\quad + \text{re}(\overline{\hat{u}(k)} \cdot \hat{f}(k, t)) \\ &\leq -\nu R_1^2 + \frac{1}{\sqrt{V}} \|u(\cdot, t)\|_{L^2}^2 R_1 + |\hat{f}(k, t)| \frac{R_1}{|k|}.\end{aligned}\tag{21}$$

Furthermore, the energy $\|u(\cdot, t)\|_{L^2}^2$ is bounded by $R^2(t)$. As long as the radial component of $X(u)_k$ (namely the quantity in (21) normalized by the length $|\hat{u}(k)| = R_1(t)/|k|$) is bounded above by the growth rate of the ball itself, namely by $\dot{R}_1/|k|$, then solution curves $(u(\cdot, t), t)$ do not exit the set $\{(u(\cdot), t) : 0 < t, u(\cdot, t) \in A_{R_1(t)} \cap B_{R(t)}(0)\}$. In particular this happens for nondecreasing $R_1(t)$ whenever $(\sqrt{V})^{-1} R^2(t) + |\hat{f}(k, t)|/|k| < \nu R_1(t)$.

Therefore when the initial data $u_0 \in A_{R_1(0)}$ and the force f satisfies (18), then the solution satisfies $u(x, t) \in A_{R_1(t)}$ for $R_1(t)$ finite, for all positive times t . \square

It is natural to ask what constraints are imposed on the data u_0 by the condition (15). Given smooth initial data u_0 and a force f satisfying $|\hat{f}(k, t)| < F_2 |k|$, the constant νR_1 can always be chosen so as to satisfy (15) (the case $f = 0$), or if the force $f \neq 0$, the function $R_1(t)$ can be chosen to be nondecreasing and

to satisfy (18). Thus the hypotheses to this theorem encompass any reasonable smooth class of initial data and inhomogeneous forcing terms. We note that the constant R_1 scales dimensionally in terms of the units $L^{3/2}/T$.

Under changes of scale, the quantity $\sup_t \sup_k |k| |\hat{u}(k, t)|$ transforms like the *BV-norm* $\sup_t \|\partial_x u(\cdot, t)\|_{L^1}$, and indeed the latter being finite implies the former. However as far as we know there is no known global bound on the *BV-norm* of weak solutions to (7). A related inequality appears in [3], which is a global upper bound on the L^1 -norm of the vorticity $\omega := \nabla \times u$, again uniformly in time.

A corollary to this result gives a stronger estimate for time integrals of the Fourier coefficients of weak solutions.

Theorem 2.5. *Suppose that a weak solution $u(x, t)$ satisfies (17), with its initial conditions satisfying $u_0 \in A_{R_1}$. Then time integrals of the Fourier coefficients obey the stronger estimate*

$$\int_0^T |\hat{u}(k, t)|^2 dt \leq \frac{R_2^2(T)}{\nu |k|^4} . \quad (22)$$

The constant is given by

$$R_2(T) = \frac{1}{2} \left(R_4(T) + \sqrt{2R_1^2(0) + R_4^2(T)} \right) ,$$

where

$$R_4(T) = \frac{R^2(T)}{\nu \sqrt{V}} + \frac{F_1(k, T)}{\sqrt{\nu}} , \quad F_1(k, T) = \left(\int_0^T |\hat{f}(k, t)|^2 dt \right)^{1/2} .$$

Assuming that $\sup_{k \in \mathbb{Z}^3} F_1(k, T) := F_\infty(T) < +\infty$, the constant $R_2(T)$ will be independent of k .

When the force $f = 0$, the constants R and R_1 can be taken independent of T , implying that R_2 is also constant in time, and the estimate (22) holds uniformly over $0 < T < +\infty$. For nonzero forces which are L^∞ with respect to time, there is an upper bound $F_1(k, T) \sim \sqrt{T}$, and $R_2(T)$ will grow at most linearly in time for large T . In the situation in which the forcing is given by a stationary process, it is expected (but not proven at this point in time) that for typical solutions, the quantity $R_2^2(T)/T$ will have a limit \bar{R}_2^2 for large time T , representing a balance between energy input and dissipation. Notice that R_2 scales dimensionally in terms of $L^{3/2}/T$.

Proof. Because the field $u(\cdot, t)$ is divergence-free, $k \cdot \hat{u}(k, \cdot) = 0$, implying the vector identity $k \cdot \hat{u}(k - k_1) \otimes \hat{u}(k_1) = \hat{u}(k - k_1) \cdot k_1 \otimes \hat{u}(k_1)$. The absolute value of \hat{u} can be estimated from (20),

$$\begin{aligned} \frac{1}{2} \partial_t |\hat{u}(k, t)|^2 + \nu |k|^2 |\hat{u}(k, t)|^2 &= \text{im} \frac{1}{\sqrt{V}} \left(\overline{\hat{u}(k, t)} \cdot \Pi_k \sum_{k_1} \hat{u}(k - k_1, t) \cdot k_1 \otimes \hat{u}(k_1, t) \right) \\ &\quad + \overline{\hat{u}(k, t)} \cdot \hat{f}(k, t) , \end{aligned}$$

which is valid for each k in the sense of weak solutions in time. When integrated over the time interval $[0, T]$ it gives

$$\begin{aligned} \nu |k|^2 \int_0^T |\hat{u}(k, t)|^2 dt &= \frac{1}{2} |\hat{u}_0(k)|^2 - \frac{1}{2} |\hat{u}(k, T)|^2 \\ &\quad + \text{im} \frac{1}{\sqrt{V}} \int_0^T \left(\overline{\hat{u}(k, t)} \cdot \Pi_k \sum_{k_1} \hat{u}(k - k_1, t) \cdot k_1 \otimes \hat{u}(k_1, t) \right) dt \\ &\quad + \int_0^T \overline{\hat{u}(k, t)} \cdot \hat{f}(k, t) dt . \end{aligned} \quad (23)$$

Multiplying this identity by $|k|^2$, the terms of the RHS are then bounded as follows:

$$\begin{aligned} \frac{1}{2}|k|^2|\hat{u}_0(k)|^2 &\leq \frac{1}{2}R_1^2(0), \\ |k|^2\left|\int_0^T \overline{\hat{u}(k,t)} \cdot \hat{f}(k,t) dt\right| &\leq \frac{1}{\sqrt{\nu}}(\nu|k|^4 \int_0^T |\hat{u}(k,t)|^2 dt)^{1/2} \left(\int_0^T |\hat{f}(k,t)|^2 dt\right)^{1/2}, \end{aligned}$$

and finally

$$\begin{aligned} &|k|^2 \frac{1}{\sqrt{V}} \left| \text{im} \int_0^T \overline{\hat{u}(k,t)} \cdot \Pi_k \sum_{k_1} \hat{u}(k-k_1,t) \cdot k_1 \otimes \hat{u}(k_1,t) dt \right| \\ &\leq \frac{1}{\nu\sqrt{V}} (\nu|k|^4 \int_0^T |\hat{u}(k,t)|^2 dt)^{1/2} \sup_{0 < t < T} \|u(\cdot,t)\|_{L^2} \left(\nu \int_0^T |\nabla u(\cdot,s)|^2 ds\right)^{1/2}. \end{aligned}$$

Define $I^2(k,T) = \nu|k|^4 \int_0^T |\hat{u}(k,t)|^2 dt$, then the identity (23) implies the inequality for $I(k,T)$

$$I^2(k,T) \leq \frac{R_1^2(0)}{2} + \left(\frac{R^2(T)}{\nu\sqrt{V}} + \frac{F_1(k,T)}{\sqrt{\nu}} \right) I(k,T). \quad (24)$$

where we have used that $R(t)$ is nondecreasing. The quantity $I(k,T)$, being nonnegative, cannot exceed the largest positive root of the quadratic equation where equality is attained, giving the estimate (22). This estimate is uniform in $k \in \mathbb{Z}^3$ as long as $\sup_{k \in \mathbb{Z}^3} F_1(k,T) = F_\infty(T) < +\infty$. We note that

$$F^2(T) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{F_1^2(k,T)}{|k|^2},$$

which appears in the discussion of energy estimate bounds (17). \square

2.2. The analogous estimate on \mathbb{R}^3

Suppose that $\|u(\cdot,t)\|_{L^2} \leq R(t)$ (if there is no force, then $R(t) = R(0)$ suffices). The main difference in the case of $D = \mathbb{R}^3$ is that the functions $\hat{u}(k,t)$ are elements of a Hilbert space, whose values at a particular Fourier space-time point (k,t) are not well defined. We work instead with filtered values of the vector field $u(x,t)$. Let $0 \neq k \in \mathbb{R}^3$, and for $\delta < |k|/(2\sqrt{3})$ define $\hat{\chi}_k(\xi)$ a smooth cutoff function of the cube Q_k about $k \in \mathbb{R}^3$ of side length 2δ ($\delta \leq 1$ is acceptable for large $|k|$) which takes value $\hat{\chi}_k = 1$ on a cube of half the sidelength. The point is that for $\xi \in \text{supp}(\hat{\chi}_k)$ then $|k|/2 \leq |\xi| \leq 3|k|/2$. Now define $(\hat{\chi}_k(D)u)(x,t) = \mathcal{F}^{-1}\hat{\chi}_k(\xi)\hat{u}(\xi,t) = (\chi_k * u)(x,t)$. Since $\chi_k \in H^m$ for all m , it and its translations are admissible test functions, the statement (11) implies that $(\chi_k * u)(x,t)$ is a Lipschitz function of t for each x . Define $e_p(k,t) := (\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p d\xi)^{1/p}$ for $2 \leq p < +\infty$, the conclusion is the following.

Proposition 2.6. *The function $e_p^p(k,t)$ is a Lipschitz function of $t \in \mathbb{R}^+$.*

We quantify the Fourier behavior of the force f in similar terms. Consider the function $\hat{\chi}_k(D)f(x,t) = \mathcal{F}^{-1}\hat{\chi}_k(\xi)\hat{f}(\xi,t)$ and let $f_p(k,t) := \sup_{0 \leq s \leq t} (\int (|\hat{\chi}_k(\xi)\hat{f}(\xi,t)|^p/|\xi|^p) d\xi)^{1/p}$. Recalling that $f \in L_{loc}^\infty([0, +\infty); \dot{H}^{-1} \cap L^2(D))$ we have $f_2(k,t) \leq \sup_{0 \leq s \leq t} \|f(k,s)\|_{\dot{H}^{-1}}$. However the fact that f_p is finite is in general additional information about the regularity of the forcing.

Theorem 2.7. *Let initial conditions $u_0(x)$ give rise to a weak solution $u(x,t)$ which satisfies $\|u(\cdot,t)\|_{L^2} \leq R(t)$. Suppose that there exists a nondecreasing function $R_1(t)$ such that for all $2 \leq p < +\infty$ and $t \in \mathbb{R}^+$*

$$(2\delta)^{3/p} \frac{R^2(t)}{\sqrt{V}} + f_p(k,t) < \frac{\nu}{6} R_1(t), \quad (25)$$

where $\delta < |k|/2\sqrt{3}$. Consider a solution to (7) that initially satisfies $\sup_{2 \leq p < +\infty} e_p(k, 0) < R_1(0)/|k|$. Then for all $t \in \mathbb{R}^+$

$$|\hat{\chi}_k(\xi)\hat{u}(\xi, t)|_{L^\infty} < \frac{R_1(t)}{|k|}. \quad (26)$$

Theorem 2.8. Suppose a weak solution $u(x, t)$ satisfies (17), and furthermore ask that $|\hat{\chi}_k(\xi)\hat{u}_0(\xi)|_{L^\infty} \leq R_1(0)/|k|$. Then for all $T \in \mathbb{R}^+$,

$$\int_0^T |\hat{\chi}_k\hat{u}(\cdot, t)|_{L^\infty}^2 dt \leq \frac{R_2^2(T)}{\nu|k|^4}, \quad (27)$$

where the constant $R_2(T)$ is given by

$$R_2(T) = \frac{1}{2} \left(R_5(T) + \sqrt{4R_1^2(0) + R_5^2(T)} \right), \quad (28)$$

where

$$R_5(T) = \frac{2R^2(T)}{\nu\sqrt{V}} + \frac{2F_\infty(T)}{\sqrt{\nu}}, \quad F_\infty(T) = \sup_{k \in \mathbb{R}^3 \setminus \{0\}} \left(\int_0^T |\hat{\chi}_k(\xi)\hat{f}(\xi, t)|_{L^\infty}^2 dt \right)^{1/2}. \quad (29)$$

The strategy of the proof of these two results is to give an analysis similar to that of Section 2.1 for a uniform bound on $e_p(k, t)$ with the correct behavior in the parameter k . The first lemma controls the behavior of $e_2(k, t)$, pointwise in t .

Lemma 2.9. Suppose that $R_1(t)$ is nondecreasing, and is such that for all $t \in \mathbb{R}^+$

$$(2\delta)^{3/2} \frac{R^2(t)}{\sqrt{V}} + f_2(k, t) < \frac{\nu}{6} R_1(t), \quad (30)$$

where $\delta < |k|/2\sqrt{3}$. If $e_2(k, 0) < R_1(0)/|k|$, then for all $0 < t < +\infty$

$$e_2(k, t) \leq \frac{R_1(t)}{|k|}. \quad (31)$$

Proof. The quantity $e_2^2(k, t)$ satisfies the identity

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} e_2^2(k, t) &= \frac{1}{2} \partial_t \int |\hat{\chi}_k \hat{u}|^2 d\xi \\ &= \operatorname{re} \int \overline{(\hat{\chi}_k(\xi)\hat{u}(\xi, t))} \left(-\nu |\xi|^2 (\hat{\chi}_k(\xi)\hat{u}(\xi, t)) \right. \\ &\quad \left. - i \hat{\chi}_k(\xi) \frac{\sqrt{V}}{(2\pi)^3} \Pi_\xi \int \hat{u}(\xi - \xi_1, t) \cdot \xi_1 \otimes \hat{u}(\xi_1, t) d\xi_1 \right) d\xi \\ &\quad + \operatorname{re} \int \overline{(\hat{\chi}_k(\xi)\hat{u}(\xi, t))} \hat{\chi}_k(\xi) \hat{f}(\xi, t) d\xi. \end{aligned} \quad (32)$$

The first term of the RHS is negative, bounded above by

$$\begin{aligned} -\operatorname{re} \nu \int \overline{(\hat{\chi}_k(\xi)\hat{u}(\xi, t))} |\xi|^2 (\hat{\chi}_k(\xi)\hat{u}(\xi, t)) d\xi &= -\nu \int |\xi|^2 |\hat{\chi}_k(\xi)\hat{u}(\xi, t)|^2 d\xi \\ &\leq -\nu \frac{|k|^2}{4} \int |\hat{\chi}_k(\xi)\hat{u}(\xi, t)|^2 d\xi, \end{aligned}$$

where we recall that $|\xi| > (|k| - \sqrt{3}\delta) > |k|/2$ holds for $\xi \in \text{supp}(\hat{\chi}_k)$. The second term of the RHS of (32) is bounded with two applications of the Cauchy – Schwartz inequality;

$$\begin{aligned} & \left| \text{im} \int \overline{(\hat{\chi}_k(\xi)\hat{u}(\xi, t))} \left(\hat{\chi}_k(\xi) \frac{\sqrt{V}}{(2\pi)^3} \Pi_\xi \int \hat{u}(\xi - \xi_1, t) \cdot \xi_1 \otimes \hat{u}(\xi_1, t) d\xi_1 \right) d\xi \right| \\ & \leq \frac{\sqrt{V}}{(2\pi)^3} \|\hat{\chi}_k \hat{u}\|_{L^2} \|\hat{\chi}_k \Pi_\xi \xi \cdot \int \hat{u}(\xi - \xi_1, t) \otimes \hat{u}(\xi_1, t) d\xi_1\|_{L^2} \\ & \leq \frac{\sqrt{V}}{(2\pi)^3} \|\hat{\chi}_k \hat{u}\|_{L^2} \|\xi \hat{\chi}_k\|_{L^2} \left| \int \hat{u}(\xi - \xi_1, t) \otimes \hat{u}(\xi_1, t) d\xi_1 \right|_{L^\infty}, \end{aligned}$$

where we have used the property of incompressibility that $\hat{u}(\xi - \xi_1) \cdot \xi_1 = \xi \cdot \hat{u}(\xi - \xi_1)$. Furthermore on the support of $\hat{\chi}_k$, $|\xi| \leq 3|k|/2$ therefore

$$\|\xi \hat{\chi}_k\|_{L^2} \leq \frac{3|k|}{2} (2\delta)^{3/2}, \quad \left| \int \hat{u}(\xi - \xi_1, t) \otimes \hat{u}(\xi_1, t) d\xi_1 \right|_{L^\infty} \leq \|\hat{u}\|_{L^2}^2 \leq \frac{(2\pi)^3 R^2(t)}{V}.$$

The third term of the RHS of (32) is not present without a force. When there is a force, it admits an upper bound

$$\begin{aligned} & \left| \text{re} \int \overline{(\hat{\chi}_k(\xi)\hat{u}(\xi, t))} \hat{\chi}_k(\xi) \hat{f}(\xi, t) d\xi \right| \leq \|\xi \hat{\chi}_k \hat{u}\|_{L^2} \|\xi^{-1} \hat{\chi}_k \hat{f}(\cdot, t)\|_{L^2} \\ & \leq \frac{3|k|}{2} \|\hat{\chi}_k \hat{u}\|_{L^2} \|f(\cdot, t)\|_{\dot{H}^{-1}}. \end{aligned}$$

An estimate of the RHS is therefore

$$\text{RHS} \leq -\frac{\nu}{4} |k|^2 e_2^2(k, t) + \frac{3(2\delta)^{3/2}}{2} \frac{1}{\sqrt{V}} R^2(t) |k| e_2(k, t) + \frac{3}{2} f_2(k, t) |k| e_2(k, t).$$

This is the situation from which the proof of Theorem 2.4 proceeds. Consider the set $B_{R_1} = \{e : e \leq (R_1/|k|)\}$, and suppose that the inequality holds

$$\frac{(2\delta)^{3/2}}{\sqrt{V}} R^2(t) + f_2(k, t) < \frac{\nu}{6} R_1(t). \quad (33)$$

When $e = e_2$ is on the boundary of B_{R_1} , that is when $e_2 = R_1(t)/|k|$, then

$$\begin{aligned} \text{RHS} & \leq -\frac{\nu}{4} |k|^2 e_2^2(k, t) + \frac{3(2\delta)^{3/2}}{2} \frac{1}{\sqrt{V}} R^2(t) |k| e_2(k, t) + \frac{3}{2} f_2(k, t) |k| e_2(k, t) \\ & \leq \left(-\frac{\nu}{6} R_1 + (2\delta)^{3/2} \frac{1}{\sqrt{V}} R^2(t) + f_2(k, t) \right) \frac{3}{2} R_1 < 0. \end{aligned}$$

That is $\dot{e}_2(k, t) < 0$, and thus B_{R_1} is an attracting set for $e_2(k, t)$. If initially $e_2(k, 0) \leq R_1(0)/|k|$, then for all $t \in \mathbb{R}^+$, $e_2(k, t) < R_1(t)/|k|$. This proves the lemma. \square

Lemma 2.10. *Given $k \in \mathbb{R}^3$, suppose that for some $2 \leq p < +\infty$ there is a nonincreasing function $R_1(t)$ which satisfies*

$$(2\delta)^{3/p} \frac{R^2(t)}{\sqrt{V}} + f_p(k, t) < \frac{\nu}{6} R_1(t), \quad (34)$$

for some $\delta < |k|/2\sqrt{3}$. If a solution to (7) initially satisfies $e_p(k, 0) < R_1(0)/|k|$, then for all $t \in \mathbb{R}^+$

$$e_p(k, t) < \frac{R_1(t)}{|k|}. \quad (35)$$

Proof. The principle is to show that the local L^p norms of $\hat{u}(\xi, t)$ are bounded, using the same strategy as the proof of Lemma 2.9. Since $e_p^p(k, t)$ is Lipschitz continuous for each $k \in \mathbb{R}^3$, one calculates

$$\begin{aligned}
\frac{d}{dt} e_p^p(k, t) &= \partial_t \int |\hat{\chi}_k \hat{u}|^p d\xi \\
&= \operatorname{re} \int p |\hat{\chi}_k \hat{u}|^{p-2} \overline{(\hat{\chi}_k(\xi) \hat{u}(\xi, t))} \partial_t (\hat{\chi}_k(\xi) \hat{u}(\xi, t)) d\xi \\
&= \operatorname{re} \int p |\hat{\chi}_k \hat{u}|^{p-2} \overline{(\hat{\chi}_k(\xi) \hat{u}(\xi, t))} \left(-\nu |\xi|^2 (\hat{\chi}_k(\xi) \hat{u}(\xi, t)) \right. \\
&\quad \left. - i \hat{\chi}_k(\xi) \frac{\sqrt{V}}{(2\pi)^3} \Pi_\xi \int \hat{u}(\xi - \xi_1, t) \cdot \xi_1 \otimes \hat{u}(\xi_1, t) d\xi_1 \right) d\xi \\
&\quad + \operatorname{re} \int p |\hat{\chi}_k \hat{u}|^{p-2} \overline{(\hat{\chi}_k(\xi) \hat{u}(\xi, t))} \hat{\chi}_k(\xi) \hat{f}(\xi, t) d\xi .
\end{aligned} \tag{36}$$

The first term of the RHS of (36) is negative,

$$-p\nu \int |\xi|^2 |\hat{\chi}_k \hat{u}|^p d\xi \leq -p\nu \frac{|k|^2}{4} e_p^p(k, t) .$$

Using the assumptions of the lemma and the Hölder inequality, the second term has an estimate

$$\begin{aligned}
&\left| \operatorname{im} \int p |\hat{\chi}_k \hat{u}|^{p-2} \overline{(\hat{\chi}_k(\xi) \hat{u}(\xi, t))} \left(\hat{\chi}_k(\xi) \frac{\sqrt{V}}{(2\pi)^3} \Pi_\xi \xi \cdot \int \hat{u}(\xi - \xi_1, t) \otimes \hat{u}(\xi_1, t) d\xi_1 \right) d\xi \right| \\
&\leq p \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{(p-1)/p} \left(\int |\xi|^p |\chi_k(\xi)|^p d\xi \right)^{1/p} \frac{\sqrt{V}}{(2\pi)^3} \left| \int \hat{u}(\xi - \xi_1, t) \otimes \hat{u}(\xi_1, t) d\xi_1 \right|_{L^\infty} \\
&\leq |k| e_p^{p-1} \frac{3p}{2} (2\delta)^{3/p} \frac{1}{\sqrt{V}} R^2(t) .
\end{aligned}$$

The third term of the RHS of (36) is bounded by

$$\begin{aligned}
&\left| \operatorname{re} \int p \left(|\hat{\chi}_k \hat{u}|^{p-2} \overline{(\hat{\chi}_k(\xi) \hat{u}(\xi, t))} \right) \hat{\chi}_k \hat{f} d\xi \right| \leq p \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{(p-1)/p} \frac{3|k|}{2} \left(\int \frac{|\hat{\chi}_k \hat{f}|^p}{|\xi|^p} d\xi \right)^{1/p} \\
&\leq \frac{3p}{2} |k| e_p^{p-1} f_p .
\end{aligned}$$

An estimate of the RHS of (36) is thus

$$RHS \leq p \left(-\frac{\nu}{4} |k|^2 e_p^p(k, t) + \frac{3(2\delta)^{3/p}}{2} \frac{1}{\sqrt{V}} R^2(t) |k| e_p^{p-1}(k, t) + \frac{3}{2} f_p(k, t) |k| e_p^{p-1}(k, t) \right) .$$

Consider again the set $B_{R_1} = \{e : 0 \leq e \leq R_1/|k|\}$. When $e = e_p$ is on the boundary, that is when $e_p = R_1(t)/|k|$, then

$$\begin{aligned}
RHS &\leq p \left(-\frac{\nu}{4} \frac{R_1^p}{|k|^{p-2}} + \frac{3(2\delta)^{3/p}}{2} \frac{1}{\sqrt{V}} R^2(t) \frac{R_1^{p-1}}{|k|^{p-2}} + \frac{3}{2} f_p \frac{R_1^{p-1}}{|k|^{p-2}} \right) \\
&= \left(-\frac{\nu}{6} R_1 + (2\delta)^{3/p} \frac{1}{\sqrt{V}} R^2(t) + f_p \right) \frac{3p R_1^{p-1}}{2 |k|^{p-2}} .
\end{aligned}$$

Supposing that (33) holds, the RHS is negative for $e = e_p$ on the boundary, and the set B_{R_1} is attracting for the quantity $e_p(k, t)$ for $t \in \mathbb{R}^+$. \square

These are estimates which are uniform in the parameter p . We are now prepared to complete the proof of Theorem 2.7, indeed we note that $\lim_{p \rightarrow +\infty} e_p(k, t) = |\hat{\chi}_k \hat{u}|_{L^\infty}$. The quantity $e_p(k, t)$ is given a uniform upper bound in Lemma 2.10 under the stated hypotheses, and hence the theorem follows.

Proof of Theorem 2.8. Start with the identity in (36) for $e_p(k, t)$, which we read as

$$\partial_t e_p^2 = \frac{2}{p} e_p^{p(2/p-1)} \partial_t e_p^p. \quad (37)$$

Because of the support properties of the cutoff functions $\hat{\chi}_k(\xi)$, there is the comparison $|k|/2 \leq |\xi| \leq 3|k|/2$ on the support of $\hat{\chi}_k$, thus one has upper and lower bounds

$$\frac{|k|^2}{4} \int |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi \leq \int |\xi|^2 |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi \leq \frac{9|k|^2}{4} \int |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi. \quad (38)$$

We therefore can rewrite the RHS of (37) as

$$\begin{aligned} & 2e_p^{p(2/p-1)} \left(-\nu \int |\xi|^2 |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi \right. \\ & \quad + \frac{\sqrt{V}}{(2\pi)^3} \text{im} \int |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^{p-2} \overline{\hat{\chi}_k(\xi) \hat{u}(\xi, t)} \\ & \quad \quad \times \hat{\chi}_k(\xi) \left(\Pi_\xi \int \hat{u}(\xi - \xi_1, t) \cdot \xi_1 \otimes \hat{u}(\xi_1, t) d\xi_1 \right) d\xi \\ & \quad \quad + \text{re} \int |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^{p-2} \overline{\hat{\chi}_k(\xi) \hat{u}(\xi, t)} \hat{\chi}_k(\xi) \hat{f}(\xi, t) d\xi \left. \right) \\ & := -I_1 + I_2 + I_3 \end{aligned} \quad (39)$$

The first of the three terms of the RHS is

$$I_1 = 2\nu \left(\frac{\int |\xi|^2 |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi}{\int |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi} \right)^{1-2/p} \left(\int |\xi|^2 |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi \right)^{2/p},$$

which is well-defined because of (38), is positive, and through a lower bound will give us the result of the theorem. The second term of the RHS is

$$\begin{aligned} I_2 &= \frac{2\sqrt{V}}{(2\pi)^3} \text{im} \int |\hat{\chi}_k \hat{u}|^{p-2} \overline{(\hat{\chi}_k \hat{u})} \hat{\chi}_k \left(\Pi_\xi \int \hat{u}(\xi - \xi_1, t) \cdot \xi_1 \otimes \hat{u}(\xi_1, t) d\xi_1 \right) d\xi \\ & \quad \times \left(\int |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi \right)^{-1+2/p}, \end{aligned}$$

for which one uses the Hölder inequality (with $(p-2)/p + 1/p + 1/p = 1$) to obtain an upper bound;

$$\begin{aligned} |I_2| &\leq \frac{2\sqrt{V}}{(2\pi)^3} \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{(p-2)/p} \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{1/p} \left(\int |\hat{\chi}_k|^p d\xi \right)^{1/p} \\ & \quad \times \left| \Pi_\xi \int \hat{u}(\xi - \xi_1, t) \cdot \xi_1 \otimes \hat{u}(\xi_1, t) d\xi_1 \right|_{L^\infty} \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{-1+2/p}. \end{aligned}$$

The third term of the RHS is

$$\begin{aligned} I_3 &= 2\text{re} \int |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^{p-2} \overline{\hat{\chi}_k(\xi) \hat{u}(\xi, t)} \hat{\chi}_k(\xi) \hat{f}(\xi, t) d\xi \\ & \quad \times \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{-1+2/p}, \end{aligned}$$

which satisfies an estimate of similar form, namely

$$\begin{aligned} |I_3| &\leq 2 \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{(p-2)/p} \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{1/p} \left(\int |\hat{\chi}_k \hat{f}|^p d\xi \right)^{1/p} \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{-1+2/p} \\ &= 2 \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{1/p} \left(\int |\hat{\chi}_k \hat{f}|^p d\xi \right)^{1/p}. \end{aligned}$$

Integrating (37) over the interval $[0, T]$,

$$\begin{aligned} &2 \int_0^T \nu \left(\int |\xi|^2 |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi \right)^{2/p} \left(\frac{\int |\xi|^2 |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi}{\int |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi} \right)^{1-2/p} dt \\ &\leq e_p^2(k, 0) - e_p^2(k, T) + \int_0^T |I_2(t)| + |I_3(t)| dt. \end{aligned} \quad (40)$$

Because of the properties of $\hat{\chi}_k$, we have $|k|/2 \leq |\xi| \leq 3|k|/2$ in the support of the integrand, and therefore

$$|k|^{2-4/p} \leq \left(2 \frac{\int |\xi|^2 |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi}{\int |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi} \right)^{1-2/p},$$

which means that the LHS of (40) gives an upper bound for the quantity

$$\int_0^T \nu |k|^2 \left(\int |\hat{\chi}_k(\xi) \hat{u}(\xi, t)|^p d\xi \right)^{2/p} dt.$$

Cancelling terms, one uses Cauchy – Schwartz to estimate the two time integrals on the RHS of (40). Using that

$$\left| \Pi_\xi \int \hat{u}(\xi - \xi_1, t) \cdot \xi_1 \otimes \hat{u}(\xi_1, t) d\xi_1 \right|_{L^\infty} \leq \|\hat{u}(\cdot, t)\|_{L^2} \|\xi \hat{u}(\cdot, t)\|_{L^2},$$

we estimate the first time integral as follows:

$$\begin{aligned} \int_0^T |I_2(t)| dt &\leq \frac{2\sqrt{V}}{(2\pi)^3} \left(\int |\hat{\chi}_k|^p d\xi \right)^{1/p} \left(\int_0^T \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{2/p} dt \right)^{1/2} \\ &\quad \times \left(\int_0^T \|\hat{u}(\cdot, t)\|_{L^2}^2 \|\xi \hat{u}(\cdot, t)\|_{L^2}^2 dt \right)^{1/2} \\ &\leq \frac{2\sqrt{V}}{(2\pi)^3} (2^3 \delta^3)^{1/p} \left(\int_0^T \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{2/p} dt \right)^{1/2} \\ &\quad \times \left(\frac{(2\pi)^3}{\sqrt{\nu V}} \left(\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^2} \right) \left(\int_0^T \nu \|\nabla u(\cdot, t)\|_{L^2}^2 dt \right)^{1/2} \right). \end{aligned}$$

We have used the Plancherel identity and its constant, as well as the fact that $\int |\hat{\chi}_k|^p d\xi \leq (2\delta)^3$. Thus

$$\int_0^T |I_2| dt \leq \frac{2R^2(T)}{\nu\sqrt{V}} (2^3 \delta^3)^{1/p} \left(\int_0^T \nu \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{2/p} dt \right)^{1/2}.$$

Under similar considerations,

$$\int_0^T |I_3| dt \leq 2 \left(\int_0^T \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{2/p} dt \right)^{1/2} \left(\int_0^T \left(\int |\hat{\chi}_k \hat{f}|^p d\xi \right)^{2/p} dt \right)^{1/2}.$$

Now multiply the inequality (40) by $|k|^2$ and use the above estimates with the fact that $|k|/2 \leq |\xi| \leq 3|k|/2$ on the support of $\hat{\chi}_k$.

$$\begin{aligned} \int_0^T \nu |k|^4 \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{2/p} dt &\leq |k|^2 e_p^2(0) \\ &+ 2((2\delta)^3)^{1/p} \frac{R^2(T)}{\nu\sqrt{V}} \left(\int_0^T \nu |k|^4 \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{2/p} dt \right)^{1/2} \\ &+ \frac{2}{\sqrt{\nu}} \left(\int_0^T \left(\int |\hat{\chi}_k \hat{f}|^p d\xi \right)^{2/p} dt \right)^{1/2} \left(\int_0^T \nu |k|^4 \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{2/p} dt \right)^{1/2}. \end{aligned} \quad (41)$$

From our hypotheses on the initial data we know that $|k|^2 e_p^2(0) \leq R_1^2(0)$. Defining

$$I_p^2(k, T) := \int_0^T \nu |k|^4 \left(\int |\hat{\chi}_k \hat{u}|^p d\xi \right)^{2/p} dt,$$

the inequality (41) states that

$$I_p^2(k, T) \leq R_1^2(0) + \left(2(2\delta)^{3/p} \frac{R^2(T)}{\nu\sqrt{V}} + \frac{2F_p(T)}{\sqrt{\nu}} \right) I_p(k, T), \quad (42)$$

where we define $F_p(T) := \left(\int_0^T \left(\int |\hat{\chi}_k \hat{f}|^p d\xi \right)^{2/p} dt \right)^{1/2}$. As we have argued before, this implies that $I_p(k, t)$ cannot exceed the largest positive root $R_{2,p}$ of the associate quadratic equation, resulting in the statement that

$$I_p(k, t) \leq R_{2,p}(T)$$

where the constant $R_{2,p}(T)$ is given by

$$R_{2,p}(T) = \frac{1}{2} \left(R_{5,p}(T) + \sqrt{4R_1^2(0) + R_{5,p}^2(T)} \right), \quad (43)$$

where in turn

$$R_{5,p}(T) = 2(2\delta)^{3/p} \frac{R^2(T)}{\nu\sqrt{V}} + \frac{2F_p(T)}{\sqrt{\nu}}, \quad F_p^2(T) = \left(\int_0^T \left(\int |\hat{\chi}_k(\xi) \hat{f}(\xi, t)|^p d\xi \right)^{2/p} dt \right).$$

The result of the theorem will follow by taking the limit of large $p \rightarrow +\infty$, recovering the estimate on $|\hat{\chi}_k \hat{u}|_{L^\infty}$. \square

3. Estimates of energy spectra

The *energy spectral function* is the main concern of the present paper. For the problem (7) posed on $D = \mathbb{R}^3$ this is defined by the spherical integrals

$$E(\kappa, t) = \int_{|k|=\kappa} |\hat{u}(k, t)|^2 dS(k), \quad (44)$$

where $0 \leq \kappa < +\infty$ is the radial coordinate in Fourier transform variables. When considering the case of a periodic domain $D = \mathbb{T}^3$ the Fourier transform is defined over the dual lattice, and therefore to avoid questions of analytic number theory one defines the energy spectral function to be a sum over Fourier space annuli of given thickness a ;

$$E(\kappa, t) = \frac{1}{a} \sum_{\kappa \leq |k| < \kappa+a} |\hat{u}(k, t)|^2 |\Gamma'|. \quad (45)$$

The classical Sobolev space norms of the function u can be defined in terms of the energy spectral function, via the Plancherel identity. Indeed in the case $D = \mathbb{R}^3$ the L^2 norm is given as

$$\|u\|_{L^2}^2 = \frac{V}{(2\pi)^3} \int_0^{+\infty} E(\kappa) d\kappa, \quad (46)$$

and the H^r Sobolev norms are

$$\|u\|_{H^r}^2 = \frac{V}{(2\pi)^3} \int_0^{+\infty} (\kappa^2 + 1)^r E(\kappa) d\kappa. \quad (47)$$

Analogous definitions hold for the case $x \in \mathbb{T}^3$.

3.1. Kolmogorov spectrum

There is considerable lore and a large literature on the behavior of the spectral function, particularly for large Reynolds number flows, the most well known statement being due to Kolmogorov and Obukov. The prediction depends upon a parameter ε , which is interpreted as the *average rate of energy transfer per unit volume*. Assuming that a flow exhibiting fully developed and isotropic turbulence has a regime of wavenumbers over which $E(\kappa, \cdot)$ depends only upon ε and κ , The famous argument of Kolmogorov and Obukov states that over an *inertial range* $\kappa \in [\kappa_1, \kappa_2]$,

$$E(\kappa, \cdot) \sim C_0 \varepsilon^{2/3} \kappa^{-5/3}, \quad (48)$$

for a universal constant C_0 . Their reasoning is through a dimensional analysis. The actual history of this prediction, which is well documented in [6] among other references, includes a number of statements of Kolmogorov as to the small scale structure of the fluctuations in a turbulent flow [7, 8, 9], and an interpretation of his results by Obukhov [14] in terms of the Fourier transform, as is stated in (48)¹. Some of the issues surrounding this statement are whether the Kolmogorov scaling law (48) should hold for an individual flow at every instant in time, whether it should hold on time average, or whether it is a statement for the average behavior for a statistical ensemble of flows with the probability measure for this ensemble being given by some natural invariant measure for solutions of the Navier – Stokes equations. The bounds given below have implications on the energy spectral function in all of these cases.

3.2. Bounds on energy spectra

The estimates given in Section 2 on the Fourier transform of solutions translate into estimates on the energy spectral function for such solutions. Bounds which are pointwise in time are given in the following theorem.

Theorem 3.1. *Suppose that $f = 0$ and that the initial data satisfies $u_0 \in A_{R_1} \cap B_R(0)$, where R and R_1 satisfy (15). Then for all κ and all times t ,*

$$E(\kappa, t) \leq 4\pi R_1^2. \quad (49)$$

In the case of non-zero forcing f , then there is a finite but possibly growing upper bound given by

$$E(\kappa, t) \leq 4\pi R_1^2(t) \quad (50)$$

Bounds which concern the time average of the energy spectral function are derived from Theorem 2.5.

¹In fact Obukhov formulated an integral version of this result, which he called the ‘two-thirds law of energy distribution’.

Theorem 3.2. *Again suppose that the initial data satisfies $u_0 \in A_{R_1} \cap B_R(0)$, where R and R_1 satisfy (15), and the force $f \in L_{loc}^\infty([0, +\infty); H^{-1}(D) \cap L^2(D))$ is bounded as in (25)(29). Then for every T the energy spectral function satisfies*

$$\frac{1}{T} \int_0^T E(\kappa, t) dt \leq \frac{4\pi R_2^2(T)}{\nu T} \frac{1}{\kappa^2}. \quad (51)$$

In particular, under the hypotheses of Theorems 2.3, 2.4, and 2.5, the energy spectrum must decay with an upper bound of order $\mathcal{O}(\kappa^{-2})$ for every T . In case that the solution is such that $\limsup_{T \rightarrow +\infty} R_2^2(T)/T$ is finite, then the time average behavior in (51) has an upper bound which is uniform in T . In any case, this is evidently faster than the Kolmogorov power law (48) and thus merits a further discussion.

Proof of Theorems 3.1 and 3.2. In the case of spatially periodic solutions, the definition of the energy spectral function gives that

$$\begin{aligned} E(\kappa, t) &= \frac{1}{a} \sum_{\kappa \leq |k| < \kappa+a} |\hat{u}(k, t)|^2 |\Gamma'| \\ &\leq \frac{1}{a} \sum_{\kappa \leq |k| < \kappa+a} \frac{R_1^2}{|k|^2} |\Gamma'| \\ &\leq \frac{1}{a} \frac{4\pi \kappa^2 a R_1^2}{|\Gamma'|} |\Gamma'| \leq 4\pi R_1^2(t). \end{aligned}$$

We have used that the lattice point density of Γ' is $|\Gamma'|^{-1}$. The inequalities of Theorem 3.1 follow. In the case in which the spatial domain $D = \mathbb{R}^3$, the proof is similar.

To prove Theorem 3.2, consider first the case of $D = \mathbb{R}^3$, where

$$\begin{aligned} \frac{1}{T} \int_0^T E(\kappa, t) dt &= \int_{|k|=\kappa} \left(\frac{1}{T} \int_0^T |\hat{u}(k, t)|^2 dt \right) dS(k) \\ &\leq 4\pi \kappa^2 \left(\frac{R_2^2(T)}{\nu T \kappa^4} \right). \end{aligned}$$

This is the stated estimate. The periodic case is similar. \square

3.3. Estimates on the inertial range

The two theorems 3.1 and 3.2 have implications on the inertial range of a solution of (7). In particular the inequalities (49)(50) give uniform upper bounds for $E(\kappa, t)$, while (51) estimates its time averages from above with a decay rate $C\kappa^{-2}$. For direct comparison we define the idealized Kolmogorov energy spectral function with parameter ε to be

$$E_K(\kappa) = C_0 \varepsilon^{2/3} \kappa^{-5/3}; \quad (52)$$

this is to be considered to be stationary in time so that it also represents the idealized time average. These bounds and the idealized energy spectral function are illustrated in Figure 1. The first constraint implied by (49)(50) and (51) is that a spectral regime with parameter ε is incompatible with the situation in which $E_K(\kappa)$ lies entirely above the permitted set $S := \{E \leq 4\pi R_1^2\} \cap \{E \leq 4\pi R_2^2(T)/\nu \kappa^2 T\}$.

Proposition 3.3. *In order that the graph of $E_K(\kappa)$ intersect the set S , the parameters must satisfy the relation*

$$\nu^{5/6} C_0 \varepsilon^{2/3} \leq 4\pi \left(\frac{R_2(T)}{\sqrt{T}} \right)^{5/3} R_1^{1/3}(T). \quad (53)$$

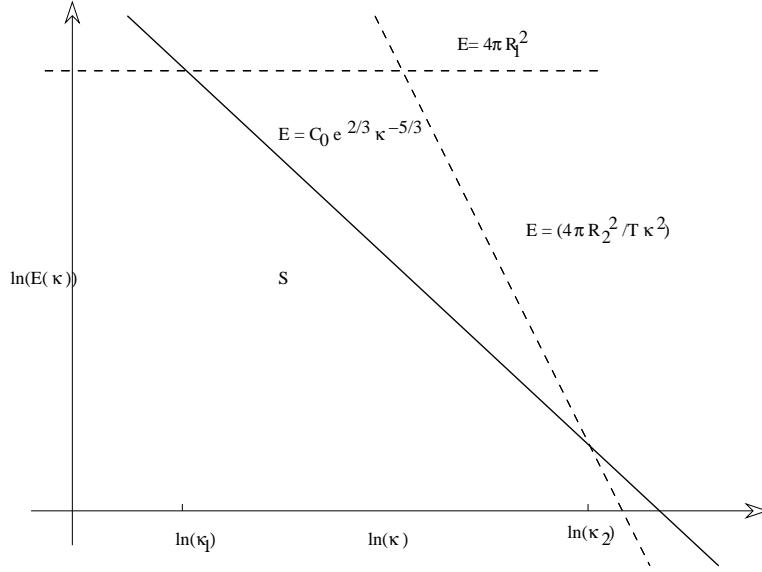


Figure 1: The accessible set S and a spectral function $E_K(\kappa)$.

The proof is elementary. This gives an upper bound on the parameter ε , in fact on the quantity $C_0 \nu^{5/6} \varepsilon^{2/3}$, in terms of quantities that are determined by the initial data and the inhomogeneous forces. In the setting of statistically stationary solutions, $R_2^2(T)/T \leq \bar{R}_2^2$, where \bar{R}_2 and R_1 are constant. In order that a spectral regime is achieved, the relation

$$C_0 \nu^{5/6} \varepsilon^{2/3} \leq 4\pi \bar{R}_2^{5/3} R_1^{1/3}$$

must hold. This constrains the values of the parameter ε for any solution regime that exhibits spectral behavior.

We now take up the question of the endpoints of the inertial range $[\kappa_1, \kappa_2]$, assuming a given value of ε . We will produce an interval $[\bar{\kappa}_1, \bar{\kappa}_2]$ such that upper and lower limits of the inertial range, respectively κ_1 and κ_2 must necessarily satisfy $\bar{\kappa}_1 \leq \kappa_1 \leq \kappa_2 \leq \bar{\kappa}_2$. First of all, the function $E_K(\kappa)$ will violate the estimate (49) (if the force is not present) or (50) (when there is a force) unless $\kappa \geq \bar{\kappa}_1$, where

$$C_0 \varepsilon^{2/3} \bar{\kappa}_1^{-5/3} = 4\pi R_1^2, \quad (54)$$

which gives a bound from below for the lower endpoint of the inertial range.

Proposition 3.4. *An absolute lower bound for the inertial range is given by*

$$\bar{\kappa}_1 = \frac{C_0^{3/5} \varepsilon^{2/5}}{(4\pi R_1^2)^{3/5}}. \quad (55)$$

It is an amusing exercise to check that the RHS has the appropriate units of L^{-1} , for which we note that the units of ε are L^2/T^3 . In the case of a nonzero forcing, $R_1(t)$ may be increasing, in which case $\bar{\kappa}_1(t)$ would decrease. In the case of bounded forces, $R_1(t)$ may increase linearly in t , implying that $\bar{\kappa}_1(t) \sim t^{-6/5}$. For a statistically stationary solution as described above, R and R_1 , and therefore $\bar{\kappa}_1$ are constant.

The upper bound for the inertial range comes from comparing time averages of E_K with the upper bound (51). Indeed,

$$\frac{1}{T} \int_0^T E_K(\kappa) dt = C_0 \varepsilon^{2/3} \kappa^{-5/3} \leq \frac{4\pi R_2^2(T)}{\nu T \kappa^2}. \quad (56)$$

Proposition 3.5. *The inequality (56) holds only over an interval of κ bounded above by*

$$\bar{\kappa}_2 = \frac{(4\pi)^3}{(C_0 \nu)^3} \frac{1}{\varepsilon^2} \frac{R_2^6(T)}{T^3}. \quad (57)$$

It is again amusing to check that the RHS has units of L^{-1} , noting that ν has units of L^2/T . When there is a force present, the constant $R_2^2(T)$ may grow in T . When $f \in L^\infty(\mathbb{R}^+; H^{-1} \cap L^2)$ the constant $R_2(T)$ grows at most linearly in T . When considering the case of a bounded and statistically stationary forcing term, for example, the ratio $R_2^2(T)/T$ is expected to have a limit as T grows large, $\lim_{T \rightarrow +\infty} R_2^2(T)/T = \bar{R}_2^2$, which gives rise to a fixed upper bound for $\bar{\kappa}_2$. Indeed in any case in which the constant $\bar{R}_2^2 := \limsup_{T \rightarrow +\infty} (R_2^2(T)/T)$ is finite, this argument gives an upper bound for $\bar{\kappa}_2$.

However with no force present, or with a force which decays in time, then $R_2^2(T)$ will be bounded, or may grow sublinearly, which results in the bound for $\bar{\kappa}_2 = \bar{\kappa}_2(T)$ which is decreasing in time. Supposing that at some time T_0 we have that for $T > T_0$ then $\bar{\kappa}_2(T) \leq \bar{\kappa}_1$, implying that the interval consisting of the inertial range is necessarily empty. The explicit bound for T_0 in the case of no force present is as follows.

Proposition 3.6. *Suppose that the force $f = 0$, so that R_1 and R_2 are constant in time. Then $\bar{\kappa}_2(T) \leq \bar{\kappa}_1$ for all $T \geq T_0$, where*

$$T_0 = \frac{(4\pi)^{6/5} R_1^{2/5} R_2^2}{\varepsilon^{4/5} C_0^{6/5} \nu}. \quad (58)$$

The RHS has units of time. If there is a nonzero force present, then $R_1 = R_1(T)$ and $R_2 = R_2(T)$, so that the expressions (55)(57) for $\bar{\kappa}_1 = \bar{\kappa}_1(T)$ and $\bar{\kappa}_2 = \bar{\kappa}_2(T)$ depend on time. It nonetheless could happen that

$$\limsup_{T \rightarrow +\infty} \bar{\kappa}_2(T) < \liminf_{T \rightarrow +\infty} \bar{\kappa}_1(T), \quad (59)$$

then again there is a maximum time T_0 for the existence of spectral behavior of solutions.

The above three estimates give lower and upper bounds on the inertial range, and an upper bound of the time of validity of a spectral description of a solution to (7), if indeed it behaved exactly like the Kolmogorov power spectrum profile over its inertial range.

As discussed in the introduction, when additional physical assumptions are made as to the behavior of a solution of (7), then further information is available about the energy transfer rate ε . The classical hypotheses are stated in [14] among other places, describing the character of flows in a regime of fully developed turbulence. Specifically, they are that (1) the flow is in a (statistically) steady state of energy transfer from the inertial range $|k| \leq \kappa_\nu$ to the dissipative range $|k| > \kappa_\nu$; (2) the support of the spectrum $E(\kappa, t)$ lies essentially in inertial range; and (3) a certain scale invariant form is assumed for the transport of energy $T(\kappa)$ at wavenumber scale κ which assumes a form of homogeneity of the flow. Under hypotheses (1)(2) and (3), one concludes that the energy dissipation rate ε_1 in (6) is equal to the energy transfer rate ε , and that the upper end of the inertial range $\kappa_2 = \kappa_\nu = 2\pi(\varepsilon/\nu^3)^{1/4}$. Alternatively, one can simply work under the hypothesis that the energy spectral function depends only upon the two quantities κ and the energy dissipation rate ε_1 . Assuming that the force is stationary, and that $\varepsilon = \varepsilon_1$, the conclusion of [4] is that

$$\varepsilon \leq c_1 \nu \frac{\langle \|u(\cdot)\|_{L^2}^2 \rangle}{V^{5/3}} + c_2 \frac{\langle \|u(\cdot)\|_{L^2}^2 \rangle^{3/2}}{V^{11/6}},$$

where $\langle \cdot \rangle$ denotes either time or ensemble averaging, and in any case $\|u(\cdot)\|_{L^2} \leq R$. This is to be compared with the general estimate (53), and it behaves better for small ν . On the other hand, the estimates in the present paper hold for all weak solutions of (7), and they give an upper bound on the energy transfer rate ε , essentially independently of the Obukhov hypotheses. Furthermore, the upper and lower bounds on the inertial range are conclusions of the analysis rather than assumptions of the theory. In particular this work gives a lower bound on the inertial range, which is the first such known, either with mathematically rigorous arguments or under physical hypotheses, at least to the authors.

3.4. Limits on spectral behavior

The endpoints $\bar{\kappa}_1, \bar{\kappa}_2$ of the interval which bounds the inertial range, and the temporal upper bound T_0 are given in terms of the idealized Kolmogorov spectral function $E_K(\kappa)$, rather than one given by an actual solution of the Navier – Stokes equations. In order to have a relevance to actual solutions, one must quantify the meaning of *spectral behavior* of a solution. This can have a number of interpretations, several of which we have mentioned in Section 3.1. It could be that we define an individual solution to have spectral behavior if its energy spectral function $E(\kappa, t)$ is sufficiently close to the idealized Kolmogorov spectral function $E_K(\kappa)$, uniformly over a time period $[0, T]$. Since the solution $u(\cdot, t) \in L^2$ and is indeed in \dot{H}^1 for almost all times t , while E_K is neither (*i.e.* neither $E_K(\kappa)$ nor $\kappa^2 E_K(\kappa) \in L^1(\mathbb{R}_\kappa^+)$), this already implies that the inertial range must be finite.

Definition 3.7. A solution (u, p) to (7) is said to have the spectral behavior of $E_K(\kappa)$, uniformly over the range $[\kappa_1, \kappa_2]$ and for the time interval $[0, \bar{T}]$ if its energy spectral function $E(\kappa, t)$ satisfies

$$\sup_{\kappa \in [\kappa_1, \kappa_2], t \in [0, \bar{T}]} (1 + \kappa^{5/3}) |E(\kappa, t) - E_K(\kappa)| < C_1 \ll C_0 \varepsilon^{2/3} . \quad (60)$$

An alternate version of this specification would be to replace the criterion (60) with a weaker one, for instance asking that a Sobolev space norm be controlled, which for $D = \mathbb{R}^3$ could be the statement that

$$\sup_{t \in [0, \bar{T}]} \int_{\kappa_1}^{\kappa_2} (1 + \kappa^{5/3}) |E(\kappa, t) - E_K(\kappa)| d\kappa < C_1 \ll C_0 \varepsilon^{2/3} . \quad (61)$$

Or else one could specify a criterion which respected the metric of a Besov space. For example, one could use the Fourier decomposition $\Delta_j = \{k : 2^{j-1/2} < |k| \leq 2^{j+1/2}\}$, and ask that over a time period $[0, \bar{T}]$ a solution satisfy

$$\left| \int_{\Delta_j} |\hat{u}(k, t)|^2 dk - C_0 \varepsilon^{2/3} 3 \sinh((\ln 2)/3) 2^{-2j/3} \right| < C_1 2^{-5j/3} \ll C_0 \varepsilon^{2/3} 2^{-5j/3} \quad (62)$$

for all $j_1 \leq j \leq j_2$, where j_1, j_2 are such that $j_1 < \log_2 \kappa_1$, and $\log_2 \kappa_2 < j_2$. In any of these cases, Theorems 3.1 and 3.2 imply bounds on the inertial range given by the interval $[\bar{\kappa}_1, \bar{\kappa}_2]$.

Theorem 3.8. *Suppose that an individual solution $(u(x, t), p(x, t))$ is such that $u_0(x) \in A_{R_1} \cap B_R(0)$, where R and R_1 satisfy (15) and if a force is present, it satisfies (18). If $u(x, t)$ exhibits the spectral behavior of E_K uniformly over the range $[\kappa_1, \kappa_2] \times [0, \bar{T}]$, then*

$$\bar{\kappa}_1 \leq \kappa_1 , \quad \kappa_2 \leq \bar{\kappa}_2 , \quad (63)$$

and if $f = 0$ then

$$\bar{T} \leq T_0 , \quad (64)$$

with possibly different constants C_0 in (55)(57)(58).

Thus the spectral behavior of solutions whose initial data $u_0(x)$ lie in one of the sets $A_{R_1} \cap B_R(0)$ is limited by the bounds given in (63). The proof will show that the same constraints hold for solutions which exhibit spectral behavior over a given nonzero proportion of the measure of the time interval $[0, \bar{T}]$.

However it could be argued that the behavior of an individual solution is less important, and that spectral behavior is a property of a statistical ensemble of solutions. Members of this ensemble should have their spectral behavior considered in terms of the ensemble average, rather than individually as above. Theorems 3.1 and 3.2 are relevant to this situation as well. Suppose there were a probability measure \mathbf{P} defined on a statistical ensemble $\Omega \subset L^2(D) \cap \{u : \nabla \cdot u = 0\}$ which is invariant under the solution map of the Navier – Stokes equations, however this has been chosen to be defined, with force f (also possibly stationary, taken from a family of realizations which have their own statistics). Using the standard notation, define the ensemble average of a functional $F(u)$ defined and \mathbf{P} -measurable on Ω by $\langle F(u) \rangle$. Without loss of generality we can take \mathbf{P} to be ergodic with respect to the Navier – Stokes solution map.

The ergodicity of the invariant measure \mathbf{P} tells us two things. The first is that space averages are *a.e.* time averages, so that

$$\langle E(\kappa, \cdot) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T E(\kappa, t) dt \leq \frac{4\pi}{\nu \kappa^2} \lim_{T \rightarrow \infty} \frac{R_2^2(T)}{T} \quad (65)$$

for \mathbf{P} -*a.e.* initial data u_0 . The second thing is that whenever R, R_1 satisfy (18) then $\mathbf{P}(A_{R_1} \cap B_R(0))$ is either zero or one, as $A_{R_1} \cap B_R(0)$ is an invariant set.

Definition 3.9. A statistical ensemble (Ω, \mathbf{P}) is said to exhibit the spectral behavior of $E_K(\kappa)$ on average over the range $[\kappa_1, \kappa_2]$ when the ensemble average of its energy spectral function,

$$\langle E(\kappa, t) \rangle := \int_{|k|=\kappa} \langle |\hat{u}(k, t)|^2 \rangle dS(k) \quad (66)$$

satisfies the estimate

$$\sup_{\kappa \in [\kappa_1, \kappa_2], t \in [0, \bar{T}]} (1 + \kappa^{5/3}) |\langle E(\kappa, t) \rangle - E_K(\kappa)| < C_1 \ll C_0 \varepsilon^{2/3} \quad (67)$$

over the range $[\kappa_1, \kappa_2]$.

Let us suppose that the force f satisfies

$$|\hat{f}(k, t)| \leq \nu |k| R_1 \quad \text{and} \quad \left(\int_0^t |\hat{f}(k, s)|^2 ds \right)^{1/2} \leq F_\infty(t), \quad (68)$$

as in (18), and we are to examine the spectral behavior of the statistical ensemble of solutions $\{u(\cdot)\}$.

Theorem 3.10. *Suppose that the ensemble (Ω, \mathbf{P}) has the spectral behavior of $E_K(\cdot)$ over the range $[\kappa_1, \kappa_2]$. Then either*

$$\mathbf{P}(A_{R_1} \cap B_R(0)) = 0 \quad (69)$$

for all R, R_1 , or else

$$\bar{\kappa}_1 \leq \kappa_1, \quad \kappa_2 \leq \bar{\kappa}_2, \quad (70)$$

with a possibly different constant C_0 in (55)(57)(58).

Proof of Theorems 3.8 and 3.10. We will give the argument in the case of Euclidian space $D = \mathbb{R}^3$, the torus case is similar. Start with the proof of Theorem 3.8 with the criterion of (60), and suppose that $\kappa_1 < \bar{\kappa}_1$. Using the estimate of Theorem 3.1 and the form (52) of $E_K(\kappa_1)$ we have

$$C_0 \varepsilon^{2/3} \kappa_1^{-5/3} - 4\pi R_1^2 \leq |E_K(\kappa_1) - E(\kappa_1)| \leq o(1) C_0 \varepsilon^{2/3}.$$

Because of the identity (54), this implies

$$C_0 \varepsilon^{2/3} (\kappa_1^{-5/3} - \bar{\kappa}_1^{-5/3}) \leq o(1) C_0 \varepsilon^{2/3} .$$

A lower bound for the LHS is given by

$$\kappa_1^{-5/3} - \bar{\kappa}_1^{-5/3} = \int_{\kappa_1}^{\bar{\kappa}_1} \frac{5}{3} \kappa^{-8/3} d\kappa \geq (\bar{\kappa}_1 - \kappa_1) \frac{5}{3} \bar{\kappa}_1^{-8/3} .$$

Therefore

$$0 \leq (\bar{\kappa}_1 - \kappa_1) \leq o(1) \frac{3}{5} \bar{\kappa}_1^{8/3} ,$$

that is, κ_1 is a bounded distance from $\bar{\kappa}_1$. Furthermore the defining relation (54) for the left endpoint $\bar{\kappa}_1$ of the bounds on the inertial range can be rewritten

$$4\pi R_1^2 = C_0 \varepsilon^{2/3} \bar{\kappa}_1^{-5/3} = C_0 \varepsilon^{2/3} \kappa_1^{-5/3} \left(\frac{\kappa_1}{\bar{\kappa}_1} \right)^{5/3} ,$$

and since $(1 - o(1))(3/5)\bar{\kappa}_1^{5/3} \leq \kappa_1/\bar{\kappa}_1 \leq 1$, then (54) continues to hold for κ_1 , with only a small change in the constant C_0 .

Now suppose that $\bar{\kappa}_2 \leq \kappa_2$. The criterion (60) implies that

$$\frac{1}{\bar{T}} \int_0^{\bar{T}} \kappa^{5/3} |E_K(\kappa) - E(\kappa, t)| dt \leq o(1) C_0 \varepsilon^{2/3} .$$

Therefore using (51) and (52), we have

$$C_0 \varepsilon^{2/3} - \frac{4\pi}{\nu} \frac{R_2^2}{T} \kappa^{-1/3} \leq o(1) C_0 \varepsilon^{2/3} .$$

for $\kappa \in [\kappa_1, \kappa_2]$ and $T \leq \bar{T}$. This applies in particular to $\kappa = \bar{\kappa}_2$, therefore

$$C_0 \varepsilon^{2/3} (1 - o(1)) \leq \frac{4\pi}{\nu} \frac{R_2^2}{T} \frac{1}{\kappa_2^{1/3}} .$$

Hence

$$\kappa_2^{1/3} \leq \frac{4\pi}{\nu} \frac{R_2^2 (1 + o(1))}{C_0 \varepsilon^{2/3} T} = (1 + o(1)) \bar{\kappa}_2^{1/3} ,$$

where we have used (57). Therefore

$$(1 - o(1)) \leq \frac{\bar{\kappa}_2}{\kappa_2} \leq 1 ,$$

and (57) holds for κ_2 with only a change of the overall constant C_0 .

Similar considerations give the analog result to Theorem 3.8 if we accept the Sobolev or Besov criteria for spectral behavior. For instance, suppose it is considered that the estimate (61) is the indicator of spectral behavior. If $\kappa_1 < \bar{\kappa}_1$ then

$$\int_{\kappa_1}^{\bar{\kappa}_1} (C_0 \varepsilon^{2/3} \kappa^{-5/3} - 4\pi R_1^2) d\kappa \leq \int_{\kappa_1}^{\bar{\kappa}_1} |E_K(\kappa) - E(\kappa)| d\kappa \leq o(1) C_0 \varepsilon^{2/3} .$$

Using (54), this implies that

$$C_0 \varepsilon^{2/3} \int_{\kappa_1}^{\bar{\kappa}_1} (\kappa^{-5/3} - \bar{\kappa}_1^{-5/3}) d\kappa \leq o(1) C_0 \varepsilon^{2/3} ,$$

which in turn implies (by convexity) that

$$C_0 \varepsilon^{2/3} \frac{(\bar{\kappa}_1 - \kappa_1)^2}{2} \frac{5}{3} \bar{\kappa}_1^{-8/3} \leq o(1) C_0 \varepsilon^{2/3} .$$

This controls $\bar{\kappa}_1 - \kappa_1$ and also their ratio. Suppose that $\bar{\kappa}_2 < \kappa_2$. Then the criterion (61) implies that

$$\int_{\bar{\kappa}_2}^{\kappa_2} \kappa^{5/3} (C_0 \varepsilon^{2/3} \kappa^{-5/3} - \frac{4\pi R_2^2}{\nu T} \kappa^{-2}) d\kappa \leq \int_{\bar{\kappa}_2}^{\kappa_2} \kappa^{5/3} |E_K(\kappa) - E(\kappa)| d\kappa \leq o(1) C_0 \varepsilon^{2/3} .$$

Therefore, using (57)

$$(\kappa_2 - \bar{\kappa}_2) - \frac{3}{2} \bar{\kappa}_2^{1/3} (\kappa_2^{2/3} - \bar{\kappa}_2^{2/3}) \leq o(1) .$$

Define $a(\kappa) := \kappa - \frac{3}{2} \bar{\kappa}_2^{1/3} \kappa^{2/3}$, which is increasing and convex for $\kappa \geq \bar{\kappa}_2$. The last estimate states that $a(\kappa_2) - a(\bar{\kappa}_2) \leq o(1)$, which controls the quantity $\kappa_2 - \bar{\kappa}_2$.

The proof of the analogous statements of Theorem 3.10 are similar, except that the upper bounds on κ_2 are easier as the ensemble average already subsumes the time average due to the ergodicity hypothesis. \square

4. Conclusions

The global estimates given in theorems 2.3, 2.4 and 2.5 for the domain $D = \mathbb{T}^3$, and theorems 2.7 and 2.8 in the case $D = \mathbb{R}^3$, provide control in L^∞ of the Fourier transform of weak solutions of the Navier – Stokes equations. These are in terms of constants R , R_1 and R_2 which depend only upon the initial data and the inhomogeneous forces. These results in turn give estimates of the energy spectral function, which show that $E(\kappa, t)$ is bounded from above, and its time averages are bounded above by $\mathcal{O}(1/\kappa^2)$. These upper bounds constrain the ability for a weak solution to exhibit spectral behavior in the manner of the idealized Kolmogorov spectral function $E_K(\kappa) = C_0 \varepsilon^{2/3} \kappa^{-5/3}$. The constraints extend to the case of a statistical ensemble forces and solutions, applying to the ensemble averages $\langle E(\kappa, t) \rangle$ of the energy spectral function. We remark that the estimates, and the subsequent constraints on spectral behavior, are valid for weak solutions of the Navier – Stokes equations, and our considerations are separate from the physical assumptions of Obukhov on flows exhibiting fully developed turbulence, or the question of possible formation of singularities.

It is natural to compare the above constraints with the physical quantities describing spectral behavior and the inertial range that come from the Kolmogorov – Obukhov theory of turbulence. The first of these is the Kolmogorov length scale $\eta_\nu = (\nu^3/\varepsilon)^{1/4}$, or rather its associated wavenumber $\kappa_\nu = 2\pi/\eta_\nu$. On physical grounds, dissipation is expected to dominate the behavior of $E(\kappa, t)$ for $\kappa > \kappa_\nu$. Comparing κ_ν to our upper bounds on the inertial range, we find that

$$\kappa_\nu = 2\pi \left(\frac{\varepsilon}{\nu^3} \right)^{1/4} \leq \left(\frac{4\pi R_2^2(T)}{C_0 \nu T} \right)^3 \frac{1}{\varepsilon^2} = \bar{\kappa}_2$$

for sufficiently small ε and ν . Indeed, with everything else fixed, κ_ν is decreasing in ε while $\bar{\kappa}_2$ is increasing, and furthermore while both κ_ν and $\bar{\kappa}_2$ are increasing as $\nu \rightarrow 0$, however $\kappa_\nu \ll \bar{\kappa}_2$. It seems clear that $\bar{\kappa}_2$ is an absolute, but not necessarily a very sharp, estimate of the upper limit of the inertial range and the start of the dissipative regime for solutions that is expected on physical grounds.

As described in section 3.3, if one assumes a number of physical hypotheses, as Obukhov does, on the form of the energy transfer rate, from which one deduces that κ_ν gives an upper bound on the spectral regime and that $\varepsilon = \varepsilon_1$ is given by the energy dissipation rate, then there are better upper bounds available for ε than our result (53). This assumption however is based on physical assumptions on the character of solutions of the Navier – Stokes equations in a statistically stationary regime.

The Taylor length scale $\kappa_\lambda = 2\pi(\varepsilon V/\nu R^2)^{1/2}$ is another indicator of the lower limit of the dissipative regime, one which incidentally is independent of the form of the Kolmogorov idealized energy spectral function E_K . The quantity R^2/V is a bound on the energy per unit volume of the solution. We again see that $\bar{\kappa}_2$ is an overly pessimistic upper bound for κ_λ for small ε and ν , since in such a case

$$\kappa_\lambda = 2\pi \left(\frac{\varepsilon V}{\nu R^2} \right)^{1/2} \leq \left(\frac{4\pi}{C_0\nu} \frac{R_2^2(T)}{T} \right)^3 \frac{1}{\varepsilon^2} = \bar{\kappa}_2 .$$

In both of these comparisons the quantities $R_2^2(T)/T$ are to be replaced by \bar{R}_2^2 in the case of a statistically stationary ensemble of solutions.

In a flow regime of fully developed turbulence, it is generally expected that $\kappa_\lambda < \kappa_\nu$, an inequality which is worthwhile to discuss. Calculate

$$\frac{\kappa_\lambda}{\kappa_\nu} = \frac{\sqrt{V}}{R} \left(\frac{\varepsilon}{\nu} \right)^{1/2} \left(\frac{\nu^3}{\varepsilon} \right)^{1/4} = \frac{\sqrt{V}}{R} (\varepsilon\nu)^{1/4}$$

where $(\varepsilon\nu)^{1/4} = u_\nu$ is the Kolmogorov velocity scale. For solutions that we consider, (15) holds, so that in particular

$$\frac{\sqrt{V}}{R} (\varepsilon\nu)^{1/4} \geq \frac{R}{\nu R_1} (\varepsilon\nu)^{1/4} = \frac{R}{R_1} \frac{\kappa_\nu}{2\pi} .$$

The implication is that

$$\kappa_\lambda > \frac{R}{2\pi R_1} \kappa_\nu^2 ,$$

which indicates, for fixed data R, R_1 , that κ_λ cannot be too much smaller than κ_ν , and is very possibly much larger. An inequality in the other sense does not seem to arise from this or similar considerations. We do find an upper bound for the Kolmogorov velocity scale

$$u_\nu = (\varepsilon\nu)^{1/4} = \frac{R}{\sqrt{V}} \frac{\kappa_\lambda}{\kappa_\nu} \leq \frac{1}{\nu^{1/16}} \left(\frac{4\pi}{C_0} \right)^{3/8} \left(\left(\frac{R_2}{\sqrt{T}} \right)^5 R_1 \right)^{1/8} .$$

Considering the case $f = 0$, which is as in the original papers of Kolmogorov [8], the constraint of Theorem 3.8 is that $\bar{T} \leq T_0$, with the latter given by the expression in (58). This is to be compared with the Kolmogorov timescale $\tau_\nu = (\nu/\varepsilon)^{1/2}$. It is clear, for R_1 and R_2 fixed constants, that

$$\frac{\tau_\nu}{T_0} = \varepsilon^{3/10} \nu^{3/2} \left(\frac{C_0^{6/5}}{(4\pi)^{6/5} R_1^{2/5} R_2^2} \right)$$

which is of course small for small ε, ν . This is again as it should be, allowing large multiples of the eddy turnover time before one runs into the upper allowed limit for the persistence of spectral behavior of solutions.

It is also natural, given the bounds $\bar{\kappa}_1, \bar{\kappa}_2$ on the inertial range, to introduce the dimensionless parameter

$$r_\nu := \frac{\bar{\kappa}_2}{\bar{\kappa}_1} = \frac{1}{\varepsilon^{12/5} \nu^3} \left(\frac{4\pi}{C_0} \right)^{18/5} \left(\frac{R_1^{2/5} R_2^2(T)}{T} \right)^3 , \quad (71)$$

which governs the extent of the possibility of spectral behavior of solutions. It is somewhat similar to a Reynold's number; when $r_\nu < 1$ then solutions satisfying (15)(18) (respectively, (25)) are disallowed from exhibiting spectral behavior. For $r_\nu > 1$ an inertial range is permitted, although it is not guaranteed by the analysis of this paper. The larger r_ν the larger the permitted inertial range, although again it is not the case that the actual interval of κ over which solutions exhibit spectral behavior will necessarily extend through a significant proportion of the interval $\bar{\kappa}_1, \bar{\kappa}_2$. In the situation of a statistical ensemble of forces and solutions, the form of r_ν is somewhat more compelling,

$$r_\nu := \frac{\bar{\kappa}_2}{\bar{\kappa}_1} = \frac{1}{\varepsilon^{12/5} \nu^3} \left(\frac{4\pi}{C_0} \right)^{18/5} \left(R_1^{2/5} R_2^2 \right)^3 . \quad (72)$$

This quantity is a stand-in for the ratio of the integral scale to the Kolmogorov scale, which is itself often used as an indicator of the Reynolds number of a flow.

This paper does not address the corrections to the Kolmogorov – Obukhov theory of Navier – Stokes flows in a turbulent regime, along the lines proposed in Kolmogorov (1962) [10]. This is focused on the deviations from Gaussian nature of the moments of the structure function for such flows, and it has been a very active area of research over the past decades. We will reserve our own thoughts on this matter for a future publication.

References

- [1] A. Biryuk, W. Craig & S. Ibrahim. Construction of suitable weak solutions of the Navier-Stokes equations. *Contemporary Math.* **429**, (2007) 1–18.
- [2] J.-Y. Chemin. unpublished lectures on the Navier - Stokes equations. (2008).
- [3] P. Constantin. Navier-Stokes equations and area of interfaces. *Commun. Math. Physics* **129**, (1990) 241–266.
- [4] C. R. Doering & C. Foias. Energy dissipation in body-forced turbulence. *J. Fluid Mech.* **467**, (2002) 289–306.
- [5] C. R. Doering & J. D. Gibbon, *Applied analysis of the Navier – Stokes equations* Cambridge University Press, Cambridge 1995.
- [6] U. Frisch, *Turbulence: The legacy of A. N. Kolmogorov* Cambridge University Press, Cambridge 1995.
- [7] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous flow for very large Reynolds' numbers. *Dokl. Akad. Nauk SSSR* **30**, (1941) 299–303.
- [8] A. N. Kolmogorov. On degeneration (decay) of isotropic turbulence in an incompressible viscous liquid. *Dokl. Akad. Nauk SSSR* **31**, (1941) 538–541.
- [9] A. N. Kolmogorov. Dissipation of energy in the locally isotropic turbulence. *Dokl. Akad. Nauk SSSR* **32**, (1941) 19–21.
- [10] A. N. Kolmogorov. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. *J. Fluid Mech.* **13**, (1962) 82–85.
- [11] S. Kuksin. Spectral properties of solutions for nonlinear PDEs in the turbulent regime. *Geom. Funct. Anal.* **9** (1999) no. 1, 141–184.

- [12] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* **63**, (1934) 193–248.
- [13] J. Leray. Essai sur les mouvements plans d'un liquide visqueux que limitent des parois. *Journal de Math.* **8**, (1934) 331–418.
- [14] A. M. Obukhov. On the energy distribution in the spectrum of a turbulent flow. *Dokl. Akad. Nauk SSSR* **32** (1), (1941) 22–24.
- [15] H. Sohr & W. von Wahl. On the regularity of the pressure of weak solutions of Navier–Stokes equations. *Arch. Math.* **46**, (1986) 428–439.