Mathematics 742:
Final Exam

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Due date: Thursday April 28, 2016.

Problem 1. *Liouville and Bernstein theorems.*
Consider harmonic functions $u(x)$ defined for all $x \in \mathbb{R}^n$;

$$\Delta u = 0 \ .$$

(a) Suppose that $u(x)$ is bounded;

$$|u(x)| \leq C_0 \ .$$

Show that $u(x) = \beta$ a constant.
(b) Suppose that $u(x)$ has bounded linear growth;

$$|u(x)| \leq C_1(|x| + 1) \ .$$

Show that $u(x) = \omega \cdot x + \beta$ for some constant vector ω and some constant β.
(c) Give a suggestion for an extension of these results, with a sketch of the proof.

Problem 2. *Nonlinear heat equations and singularity formation.*
Let $B_1(0) \subseteq \mathbb{R}^n$ and consider a nonlinear heat equation on $B_1(0) \times [0, T]$ of the form

$$\partial_t u = \Delta u + u^p \ , \quad u(t, x) = 0 \quad \text{for} \quad x \in S_1(0) \ ,$$

where $p > 1$.
(a) Suppose the initial data satisfies $f(x) \geq 0$; show that for $t > 0$ then $u(t, x) > 0$ (or else $f(x) = 0$ and $u(t, x) = 0$).
(b) Show that for all $f(x)$ not identically zero there exists a time $T = T(f) < +\infty$ such that

$$\lim_{t \to T} \|u(t, \cdot)\|_{L^\infty} = +\infty \ .$$

Problem 3. *Regularity in time of solutions of the heat equation.* We have shown in lecture that a solution $u(t, x)$ of the heat equation on $\mathbb{R}^n \times \mathbb{R}_+$ is *analytic* in x for all $t > 0$ if we ask for reasonable conditions on the initial data such as $f \in L^1(\mathbb{R}^n)$. Being analytic means that for a point $(t_0, x_0) : t_0 > 0$ there exist constants C_0 and N such that

$$|(\partial_x^k u)(t_0, x_0)| \leq C_0 N^k k!$$

Given a point $(t_0, x_0) : t_0 > 0$ show that there are constants C_1 and M such that

$$|(\partial_t^m u)(t_0, x_0)| \leq C_1 M^m (2M)! \ .$$

This bound does not imply analyticity, but it shows that as a function of t, $u(t, \cdot)$ is in the Gevrey class.

Problem 4. *The Cauchy problem for the wave equation in the Friedman – Robertson – Walker space-time.*

The metric for a Friedman – Robertson – Walker (FRW) space-time is given in terms of the line element in the form

$$ds^2 = -dt^2 + S^2(t) d\sigma^2 \ , \quad S(0) = 0 \ ,$$

defined on the half space-time $\mathbb{R}^1_+ \times \mathbb{R}^3_x$, where $d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2$ is the Euclidian metric of each space-like hypersurface $\{(t, x) : t = \text{Const.}\} \simeq \mathbb{R}^3$. This metric describes an emerging space-time from a Big Bang at $t = 0$. Changing time variable

$$\frac{dt}{d\tau} = S(\tau) = \tau^2$$

the metric becomes

$$ds^2 = S^2(\tau)(-d\tau^2 + d\sigma^2) \ .$$

Consider the wave equation on $\mathbb{R}^1_+ \times \mathbb{R}^3_x$ in this metric,

$$\Box u := \frac{1}{S^2} \partial_\tau^2 u - \frac{2\dot{S}}{S^3} \partial_\tau u - \frac{1}{S^2} \Delta_\sigma u = 0 \ .$$

Initial data is given on the Cauchy hypersurface $\{(\tau_0, x)\} \simeq \mathbb{R}^3$,

$$u(\tau_0, x) = g(x) \ , \quad \partial_\tau u(\tau_0, x) = h(x) \ .$$

(a) Make the change of variables

$$v(\tau, x) = \frac{1}{\tau} \partial_\tau (\tau^3 u)$$

show that $v(\tau, x)$ satisfies the usual wave equation in Minkowski space

$$\partial_t^2 v = \Delta_\sigma v \quad \tau > 0 \ .$$

Show that the initial data for v at $\tau = \tau_0 > 0$ is given by

$$\begin{cases} v(\tau_0, x) = 3\tau_0 g(x) + \tau_0^2 h(x) := \phi(x) \\ \partial_\tau v(\tau_0, x) = 3g(x) + \tau_0^2 \Delta g(x) + \tau_0 h(x) := \psi(x) \ . \end{cases}$$

Thus the solution can be given in terms of spherical means; state this expression for the solution.

(b) The inverse of the transformation is given by

$$\tau^3 u(\tau, x) = \int_0^{\tau-\tau_0} (r+\tau_0) v(r+\tau_0, x)\, dr + \tau_0^3 g(x) \ .$$

Assume (for simplicity) that $h(x) = 0$. Give an expression for $u(\tau, x)$ in terms of $g(x)$ using the spherical means expression for v and the above inverse.

(c) The above expression is for fixed $\tau_0 > 0$ defining the Cauchy surface, and it gives the solution at time $\tau > 0$. Now consider the solution expression at a fixed time $\tau > \tau_0$, and take the limit as $\tau_0 \to 0$. What do you get?

(d) Make a sketch of the light cone structure of this problem in the original variables (t, x).

Problem 5. *H. Lewy's example of nonexistence.*

There is a basic question as to whether every linear partial differential equation has a solution, at least locally. If the equation has constant coefficients the answer is affirmative, given by the Malgrange – Ehrenpreis theorem. And the case of analytic coefficients and analytic data is addressed by the Cauchy – Kowalevsky theorem. However if the equation has variable coefficients, even analytic coefficients, there are cases for which there is no solution if data is C^∞ but not analytic.

Define the linear differential operator on a neighborhood of $\mathbb{R}^3$;

$$Lu = -\partial_x u - i\partial_y u + 2i(x+iy)\partial_z u \ ,$$

and consider the problem $Lu = h(x, y, z)$.

(a) If $h = h(z)$ is real valued, show that a C^1 solution of $Lu = h(x, y, z)$ exists only if $h \in C^\omega$, i.e. it is real analytic.

Hint: Write $(x, y) = (\sqrt{r}\cos(\theta), \sqrt{r}\sin(\theta))$ in modified polar coordinates, and change variables to

$$v(r, \theta, z) = \sqrt{r} e^{i\theta} u(\sqrt{r}\cos(\theta), \sqrt{r}\sin(\theta), z) \ ,$$

which is C^1 for $0 < r \le R$. The equation is transformed to

$$Lv = -2\partial_r v - \frac{i}{r}\partial_\theta v + 2i\partial_z v = h(z) \ .$$

In polar coordinates $v(r, \theta, z)$ is 2π-periodic in θ; denote its average

$$V(r, z) = \frac{1}{2\pi}\int_0^{2\pi} v(r, \theta, z)\, d\theta \ ,$$

which satisfies

$$(\partial_z + i\partial_r)V = -ih(z) \ .$$

Letting $H(z)$ be such that $\partial_z H(z) = h(z)$, show that the function $W(r,z) = V + iH(z)$ satisfies the Cauchy – Riemann equations, and furthermore that it is continuous at $r = 0$. Extending $W(-r,z) = -\overline{W(r,z)}$ by reflection to negative values of r, the point $r = 0$ is a removable singularity, and $W(r,z)$ is analytic. This implies that the solution $u(x,y,z)$ must be analytic in (r,z) as well. Conclude that $H(z)$ must originally have been analytic.

(b) The *symbol* of L is

$$\sigma(L) = -i\xi + \eta + 2i(x+iy)\zeta = (\eta - 2y\zeta) + i(-\xi + 2x\zeta) := \sigma_R + i\sigma_I \ .$$

The terms σ_R and σ_I are the real and imaginary parts of the symbol $\sigma(L)$. Define the *Poisson bracket* between two symbols α and β to be the expression

$$\{\alpha,\beta\} := \partial_x\alpha\partial_\xi\beta - \partial_\xi\alpha\partial_x\beta + \partial_y\alpha\partial_\eta\beta - \partial_\eta\alpha\partial_y\beta + \partial_z\alpha\partial_\zeta\beta - \partial_\zeta\alpha\partial_z\beta \ .$$

Show that

$$\{\sigma_R, \sigma_I\} \neq 0 \ .$$

In Hörmander's theory of C^∞ solvability, the vanishing of the Poisson bracket $\{\sigma_R,\sigma_I\}$ is a necessary and essentially a sufficient condition.

Problem 6.* *Invariant norm Sobolev inequality.*

In some instances it is more useful to define a Sobolev space with respect to vector fields that are not simply derivatives in the Euclidian coordinate directions. Namely define the infinitessimal rotations and dilations as, respectively

$$\Omega_{k\ell} = x_k\partial_{x_\ell} - x_\ell\partial_{x_k} \ , \quad \Lambda = \sum_{m=1}^{n} x_m\partial_{x_m} = r\partial_r \ .$$

An example is the Sobolev space $H_s(\mathbb{S}^{n-1})$ of functions on the unit sphere,

$$\|u\|^2_{H_s(\mathbb{S}^{n-1})} = \sum_{|\alpha|\le s} \int_{\mathbb{S}^n} |\Omega^\alpha u(\varphi)|^2 \, dS_\varphi \ ,$$

which involves the vector fields of the Lie algebra of rotations in $\mathbb{R}^n$ restricted to the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$; of course α is a $n(n-1)/2$ component multiindex. A second example of this is to take Sobolev spaces built on differentiation with respect to the Lie algebra of rotations and dilations, and form the Sobolev spaces Z_{ab} as the closure of Schwartz class $\mathcal{S}(\mathbb{R}^n)$ in the following norms:

$$\|u\|^2_{Z_{ab}} := \sum_{|\alpha|\le a, |\beta|\le b} \int_{\mathbb{R}^n} |\Lambda^\beta\Omega^\alpha u(x)|^2 \, dx \ .$$

(a) Show that the vector fields $\Omega_{k\ell} = -\Omega_{\ell k}$ and Λ form a Lie algebra, and in fact their commutators satisfy the relations

$$[\Omega_{k\ell}, \Omega_{mp}] = \begin{cases} \Omega_{kp} \ , & \text{when} \quad \ell = m \ , k \neq p \\ 0 & \text{when} \quad \{k,\ell\} = \{m,p\} \\ 0 & \text{when} \quad \{k,\ell\} \cap \{m,p\} = \emptyset \end{cases}$$

and

$$[\Omega_{k\ell}, \Lambda] = 0 \ .$$

(b) Prove the Sobolev lemma on $\mathbb{S}^{n-1}$, that for $s > (n-1)/2$ then

$$|u(\varphi)| \leq C_n \|u\|^2_{H_s(\mathbb{S}^{n-1})} \ .$$

(c) Prove the invariant norm Sobolev inequality on $\mathbb{R}^n$ for $a > (n-1)/2$, in the form

$$|u(x)| \leq \frac{C_n}{|x|^{\frac{n}{2}}} \|u\|^{1/2}_{Z_{a0}} \|u\|^{1/2}_{Z_{a1}} \ .$$

This gives a weighted estimate on the absolute value of the function $u(x)$ as $|x| \to +\infty$.