Math 742: Problem Set #1

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Problem 1. Suppose that $f(x,u) = b(x)u$ is the flux function for a linear scalar conservation law, and consider a solution $u(t,x) \in C^1(\mathbb{R}^2)$;

$$\partial_t u + \partial_x(b(x)u) = 0 \ .$$

Suppose that $x = X_1(t)$ and $x = X_2(t)$ are two characteristic curves such that $X_1(t) < X_2(t)$. Show that for all $t \in \mathbb{R}^1$

$$\int_{X_1(t)}^{X_2(t)} u(t,x)\,dx = \int_{X_1(0)}^{X_2(0)} u(0,x)\,dx \ .$$

Problem 2. Solve the linear scalar conservation law with flux function $f(x,u) = x^3 u$;

$$\partial_t u + \partial_x(x^3 u) = 0 \ , \qquad u(0,x) = h(x) \ .$$

Is the solution unique? Does $h(x) \in C^\infty$ imply that $u(t,x) \in C^\infty$?

Problem 3. (i) Give the general (entropy condition satisfying) solution to Burger's equation with the Riemann data

$$u(0,x) = h(x) \ ; \tag{1}$$

$$h(x) = h_- \quad x < 0 \ , \qquad h(x) = h_+ \quad x \geq 0 \ . \tag{2}$$

(ii) Work out the more complicated but still explicit (entropy condition satisfying) solution to the family of problems with piecewise constant initial data

$$u(0,x) = h(x) \ ; \tag{3}$$

$$h(x) = h_- \quad x < -1 \ , \qquad h(x) = h_0 \quad -1 \leq x < 1 \ , \qquad h(x) = h_+ \quad x \geq 1 \ . \tag{4}$$

Note: These solutions are global in $(t,x) \in \mathbb{R}^1_+ \times \mathbb{R}^1$. There are 8 cases, depending on the relative inequalities between h_-, h_0, h_+.

Problem 4. We have shown in class that on a domain $\Omega \subseteq \mathbb{R}^2$ in which $u(t,x) \in C^1$ solves a general scalar conservation law with convex flux $f(u)$, the problem is equivalent to Burger's equation, up to transformation of dependent variables. Suppose now that $u(t,x)$ is only a piecewise C^1 weak solution of this conservation law, with C^1 shock curves. Under transformation does it satisfy the jump condition for Burger's equation? If not, what condition does it satisfy?