# Mathematical aspects of surface water waves

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The theory of the motion of a free surface over a body of water is a fascinating subject, with a long history in both applied and pure mathematical research, and with a continuing relevance to the enterprises of mankind having to do with the sea. Despite the recent advances in the field (some of which we will hear about during this Workshop on Mathematical Hydrodynamics at the Steklov Institute), and the current focus of the mathematical community on the topic, many fundamental mathematical questions remain. These have to do with the evolution of surface water waves, their approximation by model equations and by computer simulations, the detailed dynamics of wave interactions, such as would produce rogue waves in an opean ocean, and the theory (partially probabilistic) of approximating wave fields over large regions by averaged 'macroscopic' quantities which satisfy essentially kinetic equations of motion. In this note, we would like to point out open problems and some of the directions of current research in the field. We believe that the introduction of new analytical techniques and novel points of view will play an important rôle in the future development of the area.

**Keywords: nonlinear surface water waves**

## 1. Equations of motion

The problem of surface water waves generally refers to the dynamics of a fluid which satisfies the Euler equations of motion which occupy a space-time region with a free surface, and which move under the influence of a 'body force' which is the acceleration of gravity. In addition the fluid is assumed to be incompressible, which is a very reasonable hypothesis for water under normal conditions, and irrotational, which is a much less reasonable hypothesis in general. Still, these hypotheses are quite reasonable for the description of wave propagation on the surface of bodies of water such as an ocean or a lake, and it is in general use by physical oceanographers today. The equations of motion in Eulerian coordinates describe a velocity field $\mathbf{u}(x,t)$ which satisfies

$$\nabla \cdot \mathbf{u} = 0 \ , \qquad \nabla \times \mathbf{u} = 0 \ . \tag{1.1}$$

Because of these two conditions the velocity field can be described as the gradient of a harmonic function

$$\mathbf{u} = \nabla\varphi \ , \qquad \Delta\varphi = 0 \ . \tag{1.2}$$

We will take our fluid region to occupy some space-time domain, which for the purpose of this description will be a subset of Euclidian space $\Sigma \subseteq \mathbb{R}^{d+1}$, $d = 3$ or 2 normally being chosen. Unless we are describing waves of a global extent, such as a tsunami, for our purposes we will assume a 'flat' earth, with spatial coordinates $\{(x, y); x \in \mathbb{R}^{d-1}, y \in \mathbb{R}\}$ and the force of gravity being $F = -gy$. The fluid body is normally assumed to have a bottom boundary (the ocean floor) given by $\{(x, y) : y = -b(x)\}$, and a classical problem is to consider the case of an infinitely flat region, namely $b(x) = h$ a constant. On the bottom, as well as on any solid components of the boundary of the fluid region, one assumes that the flow has no normal component;

$$\mathbf{u} \cdot N = 0 \ ,$$

while on the free surface itself one poses the two classical free surface boundary conditions, the kinematic condition and the Bernoulli condition. In case the free surface is given as a graph $\{y = \eta(x, t)\}$ (but everyone from California knows that the more interesting situation of surf is not covered by this case), these two boundary conditions are expressed as

$$\partial_t\eta = \partial_y\varphi - \partial_x\eta \cdot \partial_x\varphi \ , \qquad \partial_t\varphi = -g\eta - \tfrac{1}{2}|\nabla\varphi|^2 \ . \tag{1.3}$$

Adding the effect of surface tension modifies the second equation by adding a term proportional to the mean curvature of the free surface. The situation where the surface is not a graph can also be treated, using different and suitably chosen coordinates. The initial value problem consists of finding the solution $(\eta(x, t), \varphi(x, y, t))$ for the fluid domain $S(\eta(\cdot, t)) = \{(x, y) : x \in \mathbb{R}^{d-1}, -b(x) < y < \eta(x, t)\}$ over a period of time $t \in [-T, T]$, given prescribed data $(\eta_0(x), \varphi_0(x, y))$ such that $\varphi_0(x, y)$ is a harmonic function on the initial fluid domain $S(\eta_0)$ determined by the initial free surface $\{y = \eta_0(x)\}$.

It is evident that in the free boundary conditions (1.3), the constant determining the acceleration of gravity appears in a lower order term, and therefore should be of lower order concern in a theory of the evolution of solutions. Yet to anyone who has carried a container of water, it is clear the difference the sign of $g$ makes to the dynamics. It turns out that the problem (1.3) is hyperbolic, but with multiple characteristics. This accounts for the fact that the sign of a lower order term can play such an important rôle in the behavior of solutions of the initial value problem.

## 2. Zakharov's Hamiltonian

A beautiful paper of V. E. Zakharov [47] reformulated the problem of surface water waves as a Hamiltonian system with infinitly many degrees of freedom, much in the way that the KdV equation, the nonlinear Schrödinger equation and the nonlinear wave equations can be viewed. The total energy functional is easy to predict as the Hamiltonian, indeed

$$H = \int_{\mathbb{R}^{d-1}} \int_{-b(x)}^{\eta x} \tfrac{1}{2}|\nabla\varphi|^2 \ dydx + \int_{\mathbb{R}^{d-1}} \tfrac{g}{2}\eta^2(x) \ dx \ . \tag{2.1}$$

The more subtle question is as to the choice of canonically conjugate variables. Zakharov's statement is that the quantities $\eta(x)$ and $\xi(x) := \varphi(x, \eta(x))$ are the appropriate variables with which the surface water waves problem can be written in the form of Hamilton's canonical equations

$$\partial_t \eta = \mathrm{grad}_\xi H \ , \qquad \partial_t \xi = -\mathrm{grad}_\eta H \ . \tag{2.2}$$

Because of the nature of potential flow, it suffices to know the domain given by $\eta(x,t)$ and the boundary values of the velocity potential $\xi(x,t)$ at any particular time $t$, in order to recover the flow throughout the fluid region at that time; indeed given a (sufficiently regular) domain defined through $\eta$, the boundary data $\xi(x,t)$ is enough to determine the velocity potential $\varphi(x,y)$ as a harmonic function defined on the fluid domain $S(\eta)$, and therefore the data $(\eta, \xi)$ determine the fluid velocity field $\mathbf{u}(x,y,t) = \nabla\varphi(x,y)$ throughout the fluid region at time $t$.

Formalizing the comment above, we define the Dirichlet – Neumann operator for the fluid domain $S(\eta)$ through the relationship

$$\xi(x) \mapsto \varphi(x,y) \mapsto N \cdot (\nabla\varphi)(x, \eta(x))\, dS_\eta := G(\eta)\xi \, dx \ . \tag{2.3}$$

This operator is linear in $\xi(x)$, but quite nonlinear and certainly non-local in its dependence upon the fluid domain defined through the free surface given by $\eta(x)$. The normalization ensures that the operator $G(\eta)$ is self-adjoint on (an appropriate domain in) $L^2(\mathbb{R}^{d-1})$. Rewriting the Hamiltonian (2.1) using this operator [15], we have

$$H(\eta, \xi) = \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\xi G(\eta)\xi + \tfrac{g}{2}\eta^2 \, dx \ . \tag{2.4}$$

Hamilton's canonical equations (2.2) are equivalent to the equations of motion and free boundary conditions (1.3). Indeed the first equation of (1.3) is an immediate consequence of the definition of the Dirichlet – Neumann operator; the second equation follows from a calculation which is closely related to the variational formula of Hadamard.

The case of a quiescent fluid surface $\eta = 0$, when the bottom of the fluid region is flat, gives rise to an explicit expression for the Dirichlet – Neumann operator. The Fourier transform expression is that

$$G(0)\xi(x) = \tfrac{1}{2\pi^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} e^{ik\cdot x} |k| \tanh(h|k|)\hat{\xi}(k)\, dk = |D| \tanh(h|D|)\xi(x) \ , \tag{2.5}$$

where as usual $D := -i\partial_x$. Using this in the Hamiltonian (2.4) and taking its quadratic part (the linearized equations about the zero solution), one finds the classical dispersion relation

$$\omega^2(k) = g|k| \tanh(h|k|) \ .$$

Hence the phase and group velocities of solutions are

$$c_p(k) = \sqrt{\frac{g\tanh(h|k|)}{|k|}}\frac{k}{|k|} \ , \qquad c_g(k) = \partial_k \sqrt{g|k|\tanh(h|k|)} \ .$$

The limit of large $|k| \to \infty$ of the group velocity gives the velocity of the characteristics. In the case of surface water waves,

$$\lim_{|k|\to\infty} c_g(k) = 0 \ ,$$

hence the characteristics are real, but with zero characteristic velocity for any direction $k/|k|$. This is a fact that every swimmer knows; in a field of waves in the water surface, the short waves travel more slowly than the long waves. Without further effects such as capillarity, there is nothing more to influence their speed, and the high wavenumber limits of $c_g$ and $c_p$ are zero. This is the phenomenon of hyperbolicity with multiple characteristics, mentioned in the first section, and is the mathematical reason for the sensitivity of the initial value problem to the sign of $g$ in the subprincipal symbol of the RHS of (1.3). Surface tension, on the other hand, is a dispersive effect, which in many cases regularizes the problem to some extent.

## 3. The initial value problem

It is natural to start with a discussion of the initial value problem, as (1.3), or equivalently (2.2) are posed as such. We shall refer to the solution which at time $t = 0$ is in the state $(\eta_0(x), \xi_0(x))$ in terms of the flow $\Phi_t(\eta_0, \xi_0)$, for the latter equation at least, on a phase space which we have yet to specify. The question can be succinctly phrased as to whether the flow exists, in what space of functions taken to be the phase space, for how long of a time interval, and whether data should be restricted to a small neighborhood of the origin, or can be chosen more globally.

Rigorous mathematical study of the initial value problem for $d = 2$ date from the work of Ovsiannikov [], where the phase space is taken to be a space of analytic functions, and Nalimov [] whose work is in the category of functions with finite Sobolev norms. Apparently working independently, results in the analytic category were published by Kano & Nishida []. Reeder & Shinbrot [] have obtained the first results for the full $d = 3$ problem, working again in the analytic category, and the paper of Sulem, Sulem Bardos & Frisch [] also gives a result for the analytic case as the limit of an interface problem with zero upper fluid density.

The initial value problem posed in the category of Sobolev space is the more important case; in fact results in the category of analytic functions are insensitive to the sign of the acceleration of gravity $g$. Extensions of Nalimov's work have been given by Yosihara for variable bottom topographies [] and for the problem with surface tension [], and by Craig [] for existence over long intervals of time and the Boussinesq and KdV scaling regimes. More recent results include S. Wu's work [] on existence for quite arbitrarily shaped initial fluid domains (for short time intervals, of course), and the progress by Wu [] and by Lannes [] on the initial value problem in three (and higher) dimensions.

**Problem 1.** *It seems as though it is a natural question to show that, at least for sufficiently small initial data, solutions exist globally in time. That is, the flow $\Phi_t(\eta, \xi)$ should be shown to exist for all $t \in \mathbb{R}$ on a ball $B_r(0)$ of possibly small radius $r$ about zero in an appropriate function space.*

None of the results cites above gives a result which is global in time, but it is known that there do exist some such solutions, the periodic traveling waves given by Levi-Civita [], Nekrasov [] and Struik [], for example, or the solitary waves of Lavrentiev [] and Friedrichs & Hyers []. Three dimensional (and $d \geq 3$) multiply periodic examples also exist. One suggestion for an approach, at least for $d = 3$, would be to perform a normal forms transformation on the water waves Hamiltonian (2.4) to eliminate cubic terms, and then to use time decay estimates to give a global bound on a supremum norm of the solution. That is, one expects that solutions of the linearized problem decay at a rate

$$\|(\eta(\cdot,t),\xi(\cdot,t))\|_{L^\infty} \leq \frac{C}{\langle t\rangle^{(d-1)/2}} \tag{3.1}$$

and the square of the RHS is integrable in $t$ for $d \geq 3$. It is not integrable for $d = 2$, but related estimates may lead to very long time existence theorems by similar considerations. A Strichartz inequality appropriate to the water waves problem may be an alternative to using precisely the statement (3.1).

**Problem 2.** *Use the coordinates in which the water waves problem has a Hamiltonian structure, in an essential way for the initial value problem.*

The various approaches to the initial value problem use a variety of coordinates, including Lagrangian coordinates (Nalimov []), coordinates given through the conformal mapping of the fluid domain (for $d = 2$, as in Levi Civita [], and Kano & Nishida []), and Eulerian coordinates (Lannes []). These coordinates influence the character of the analysis to a large extent. Given the analytic techniques and canonical transformation theory available through the analogy to Hamiltonian mechanics, it seems an anomaly that they have so far not been exploited in the initial value problem in an essential way. For example, one might pursue a Birkhoff normal forms analysis, with the goal of obtaining a long time or even a global existence theorem.

**Problem 3.** *How do solutions break down?*

There are several versions of this question, including 'What is the lowest exponent of Sobolev space $H^s$ in which one can produce an existence theorem local in time?' Or one could ask 'For which $\alpha$ is it true that, if one knows *a priori* that $\sup_{[-T,T]} \|(\eta(\cdot,t),\xi(\cdot,t))\|_{C^\alpha} < +\infty$, then $C^\infty$ data $(\eta_0,\xi_0)$ implies that the solution is $C^\infty$ over the time interval $[-T,T]$. At present, the answer to the first question is that one can take any $s > 4$, but it is not so satisfying to say that the solution fails to exist after $t = T$ because it is no longer lies in $H^{4+}$. It would be more satisfying to say that it fails to exist because the 'curvature of the surface has diverged at some point', or a related geometrical and/or physical statement. Furthermore, although it is deemed 'obvious' that solutions don't exist for very long after they overturn (that is, after the slope of the free surface becomes infinite, and no longer given as the graph of a smooth function $\eta(x,t)$), there is no rigorous mathematical proof of this fact. Nor are there even special examples, such as self similar solutions, to prove the standard intuition is correct.

The free surface problem without the presence of gravity is a related problem, which is in fact more delicate than the case with gravity as a restoring force. It does turn out that the condition of incompressibility is enough to maintain the well-posedness of the initial value problem, locally in time, for nearly stationary fluid configurations; this is work of Christodoulou & Lindblad []. If the fluid is in violent motion, for example if it is rapidly spinning, the initial value problem is known to be ill posed; see Ebin [].

## 4. The long wave and modulational limits

The nonlinear equations which have been used to model solutions of the free surface water waves problem are possibly more well known then the full problem of Euler's equations. These equations include the Boussinesq and the Korteweg – deVries (KdV) equations, the cubic nonlinear Schrödinger equation, the Davey – Stewartson system, the Kadomtsev – Petviashvili (KP) equations, as well as their higher order and/or more detailed analogues. Certainly for some of these, the structure of their orbits in an infinite dimensional phase space is very well studies, due to the fact that they possess the structure of a completely integrable Hamiltonian dynamical system. Furthermore, there is a lot of algebraic structure to their integrals, while the full equations of water waves are generally thought to possess many fewer integrals of motion (see Benjamin & Olver []), and much less of a rigid structure of their phase space. In the considerations of this note, we will not take on a survey of results for these approximate equations, rather we focus on the question of the mathematical justification of their use in approximating solutions of Euler's equations (1.3).

The paper of Ovsiannikov [] and the papers by Kano & Nishida [] considered the free surface problem in the shallow water scaling limit, for $d = 2$. Further work of of Kano & Nishida addressed the Friedrichs expansion [], and Craig [] and again Kano & Nishida studied the Boussinesq and the KdV scaling regimes. Note that the slow time that is characteristic of the Boussinesq and KdV scaling regimes implies that it is very relevant to understand the initial value problem over long time intervals to obtain a correspondence between solutions of the Euler equations and solutions of the appropriate long wave limit. This subject has had a recent re-emergence, due to the work of Schneider & Wayne [], which addresses solitary wave interactions and includes the effects of surface tension. Higher order extensions of this work are given by Wright []. All of the above adresses the case $d = 2$.

The question of a justification of the modulational limit is also an outstanding issue, where the goal is to show that solutions of Euler's equations which satisfy a modulational Ansatz (i) exist for a sufficiently long time interval, and then (ii) approach solutions described by the nonlinear Schrödinger equation. Preliminary work on this question when $d = 2$ appears in Craig, Sulem & Sulem [], and when $d = 3$ in Craig, Schanz & Sulem []. However these two references did not produce an existence theorem for long intervals of time. The question when $d = 2$ has been recently solved by Schneider & Wayne, using in part a normal forms transformation of Euler's equations (but not necessarily canonical transformations).

**Problem 4.** *Justify the $d = 3$ dimensional water wave models, as limits of the equations of free surface water waves.*

Another direction of work that would be very striking would be to use the structure of the equations for water waves as a Hamiltonian system to elucidate the long wave and the modulational scaling regimes. This would possibly involve normal forms transformations, and rigorous analysis along the lines of the formal approach of transformation theory and expansions of a Hamiltonian system with respect to a small parameter, as in Craig & Groves [] and Craig, Guyenne, Nicholls & Sulem [].

## 5. Traveling waves

Two types of problems are the usual candidates for study, either (i) periodic lateral (in $x \in \mathbb{R}^{d-1}$) boundary conditions over some periodic fundamental domain, and (ii) the solitary wave problem. The distinguished history of this problem contains again several stories of independent research in Russia and in the west. Namely, the first results on the existence of periodic traveling wave solutions of the problem of free surfaces in deep water occurs in the work of Nekrasov [] in 1921, which was essentially unknown in the west until 1967. Levi-Civita studied the problem in a 1925 paper [], which was extended to the case of finite depth bt Struik []. Surface tension was not considered until Zeidler's 1971 article on the subject [].

Because the problem can be posed in a frame of reference traveling with the velocity of the solution, it is posed as an elliptic boundary value problem with nonlinear boundary conditions on the free surface. There are a variety of ways to coordinatize the equations in order to handle the boundary, either through conformal mappings (as in Levi-Civita's work), Lagrangian coordinates, or more *ad hoc* methods. The literature to date is very extensive, and cannot be reviewed in detail in this short paragraph; we will limit ourselves to the following remarks.

The problem of the solitary wave for $d = 2$ was considered by Lavrentiev [] (1943) and Friedrichs & Hyers [] (1954). Global considerations of the solitary wave problem, which is in essence a nonlinear elliptic bifurcation problem, but with its own essential character and difficulties, appeared in work of Amick & Toland [] and Amick, Fraenkel & Toland []. The latter reference addresses one aspect of what is called the 'Stokes conjecture' on the highest solitary wave, or at least the solitary wave 'of extremal form'; namely that it possesses a non-differentiable crest having a Lipschitz singularity, of open angle $\pi/3$. Special methods have been introduced by Kirchgässner [] and Mielke [] to study solitary waves with and without surface tension, which have been sucessful at answering a number of further questions on solitary waves with possible oscillating asymptotic behavior at $x \mapsto \pm\infty$.

In case $d = 3$ (or more), surface tension plays an important rôle. Periodic wave patterns with the symmetry of a 'symmetric diamond' were shown to exist in free surfaces with surface tension by Reeder & Shinbrot []. Only recently has the general picture of multiply periodic solutions been described; Craig & Nicholls []. What is seen is a connection between resonant interactions between Fourier modes, and the multiplicity of solutions to the nonlinear problem. Because of a fifth order resonant interaction for a particular fundamental domain, two types of traveling waves exist, with quite different geometries; hexagonal patterns and crescent shaped patterns. The hexagonal patterns have a shape which depends in a particular way upon the depth of the fluid domain.

**Problem 5.** *Elucidate the nature of the crescent wave patterns.*

The above paragraph refers to traveling wave patterns with nonzero surface tension. Recently there has been a beautiful series of results by Iooss, Plotnikov & Toland [] on standing waves for $d = 2$. It is conceivable that the techniques they develop can address the problem of three-dimensional traveling wave patterns without surface tension.

When $d = 2$ (and without surface tension) it is known [] that all solitary waves are of positive elevation, symmetric, and monotone decreasing on either side of the (unique) crest. The paper [] on the Stokes conjecture proves that every solitary wave of extremal form has a Lipschitz continuous singularity at the crest. This singularity also is present in periodic traveling wave patterns, when the height of the crest reaches its maximum allowed by the Bernoulli condition. It is natural to ask about singularities in the free surface for traveling waves for $d = 3$ which attain their maximum allowable amplitude.

**Problem 6.** *What is the nature of the singularity of crests of extremal traveling waves for the three-dimensional problem?.*

For the problem with $d = 3$ (and without surface tension), nonnegative solitary waves which decay at spatial infinity (that is, for which $|\eta(x)| \mapsto 0$ as $|x| \mapsto +\infty$) are necessarily the trivial solution $\eta = 0$ []. That is, no truly three dimensional and nonnegative solitary waves exist.

**Problem 7.** *Do truly three dimensional solitary waves exist? These must of course change sign.*

A variant of this problem is related to the Di Georgi problem for nonlinear elliptic PDE; namely to show that any three dimensional solitary wave solution is in fact the trivial extension to three dimensions of a two dimensional solitary wave (this problem was brought to our attention by H. Brezis). When surface tension is present, and the Bond number is sufficiently big, then genuine solitary waves in $d = 3$ do exist []; they are solitary waves of depression.

What is of practical importance is the theory of stability of these traveling wave patterns, and there has been much attention paid to this point in the applied mathematics literature. The paper of Plotnikov [] describes a connection between the stability of the $d = 2$ solitary wave and secondary bifurcations in its principal branch of solutions.

However it is not common to see periodic wave patterns in the ocean, and it seems likely that most if not all doubly periodic solutions in three dimensions are subject to linear instabilities.

**Problem 8.** *Give a Bloch theory for the stability of doubly periodic water wave patterns.*

It would be also rewarding to have a rigorous theory of the stability of the solitary wave; when $d = 2$ one can still call for its stability under perturbations which are three dimensional.

## 6. Invariant structures in phase space

The linearization of the system of equations (2.2) is performed in an elegant way through the Taylor expansion of the Hamiltonian (2.4), and then retaining only the quadratic terms in $(\eta, \xi)$. Restrict ourselves for the moment to the problem of periodic boundary conditions, which is to say $x \in \mathbb{R}^{d-1}/\Gamma := \mathbb{T}^{d-1}$, where $\Gamma \subseteq \mathbb{R}^{d-1}$ is the imposed lattice of translations. The quadratic part of the Hamiltonian in question is thus

$$H^{(2)}(\eta,\xi) = \int_{\mathbb{T}^{d-1}} \tfrac{1}{2}\xi(x)G(0)\xi(x) + \tfrac{g}{2}\eta^2(x)\ dx\ , \tag{6.1}$$

which can be rewritten via the Plancherel identity in terms of the Fourier transform (a canonical transformation) and the Fourier multiplier expression for the Dirichlet – Neumann operator, as

$$H^{(2)}(\eta,\xi) = \sum_{k\in\Gamma'} \tfrac{1}{2}|k|\tanh(h|k|)|\hat{\xi}(k)|^2 + \tfrac{g}{2}|\hat{\eta}(k)|^2\ . \tag{6.2}$$

Hamilton's equations are expressed succinctly as

$$\partial_t \begin{pmatrix}\eta\\ \xi\end{pmatrix} = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}\partial_\eta H^{(2)}\\ \partial_\xi H^{(2)}\end{pmatrix}\ , \tag{6.3}$$

and in particular

$$\partial_t\hat{\eta}(k,t) = |k|\tanh(h|k|)\hat{\xi}(k,t)\ , \qquad \partial_t\hat{\xi}(k,t) = -g\hat{\eta}(k,t)\ , \tag{6.4}$$

whose solutions are given by the linear flow

$$\Phi_t^0(\eta,\xi) = \sum_{k\in\Gamma'} \begin{pmatrix}\cos(\omega(k)t) & \sin(\omega(k)t)/\omega(k)\\ -\omega(k)\sin(\omega(k)t) & \cos(\omega(k)t)\end{pmatrix}\begin{pmatrix}\hat{\eta}(k)\\ \hat{\xi}(k)\end{pmatrix}\ , \tag{6.5}$$

where $\omega(k) = \sqrt{g|k|\tanh(h|k|)}$. It is evident that (except for perhaps $\hat{\xi}(0,t)$) all solutions of the linear equations evolve on invariant tori in phase space, given by the closure of their orbit; $\mathbb{T}^m = \overline{\{\Phi_t^0(\eta(x),\xi(x))\ :\ t\in\mathbb{R}\}}$. This is to say, all solutions of the linearized equations are *periodic* (when $m=1$, and these are invariant circles in phase space), or *quasi-periodic* (the case that $m<+\infty$) or respectively *almost periodic* ($m=\infty$) as functions of time. Resonant tori give rise to parameter families of such solutions, and the tori themselves are parametrized by the action variables $I(k) = ((\omega(k)/2g)|\hat{\eta}(k)|^2 + (g/2\omega(k))|\hat{\xi}(k)|^2)$. It is a natural question as to whether any of the solutions of the nonlinear equations share these strong recurrence properties. From the analogy to Hamiltonian systems, one expects that the perturbation theory for quasi-periodic solutions involves a small divisor problem. This is even more true for almost periodic solutions. But because this is a PDE, the small divisor problem occurs in the case of periodic solutions as well.

In the two-dimensional case ($d=2$) there has been recent progress on the problem of time periodic solutions in the form of standing waves, by Plotnikov & Toland [] in the case of a fluid body of finite depth, and by Iooss, Plotnikov & Toland [] in case $h=+\infty$. Both of these papers use a version of the Nash – Moser scheme to overcome the problem of small divisors.

**Problem 9.** *Prove that there exist parameter families of quasi-perodic and/or almost periodic solutions for the water waves problem.*

This problem includes the case of time periodic solutions in higher spatial dimensions as well. A simple version of a periodic solution is a spatially periodic traveling wave solution; to avoid these simpler solutions one can either impose

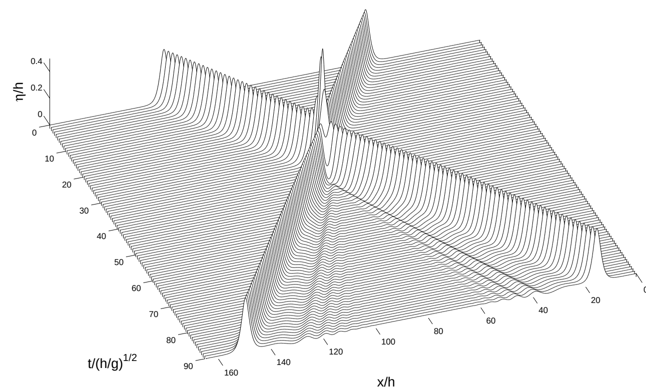


Figure 1. Head-on collision of two solitary waves of equal height $S/h = 0.4$, showing the run-up, the phase lag and the dispersive tail generated by the interaction. The amplitude of the post-collision solitary waves is slightly less than before the collision, being measured as $S^+/h = 0.3976$ at time $t/\sqrt{h/g} = 90$. The post-collision solitary waves are also delayed slightly from their pre-collision asymptotically linear trajectories. This phase lag is measured to be $(a_j - a_j^+)/h = 0.3257$, where $a_j$ (respectively $a_j^+$) are the $t$-axis intercepts of the asymptotically linear trajectories before (respectively after) the collision.

Neumann boundary conditions in $x \in \mathbb{R}$ (standing waves), which is the strategy in [] and []. Or else one can otherwise seek to guarantee that solutions that one finds are more general than those which are time invariant in an appropriately chosen moving frame. One expects to find that there are Cantor parameter families of solutions, which are Whitney smooth but not smooth in the classical sense; as is the conclusion in the case of a finite dimensional Hamiltonian system in the presence of small divisors.

Now consider the problem for $d = 2$ and in the non-compact setting $x \in \mathbb{R}^1$. It is an interesting question how solitary waves interact. For special equations such as the KdV, in addition to exact solitary waves there is a special class of multi-soliton solutions, which are asymptotic to a finite number of solitary waves as $t \mapsto \pm\infty$, and which interact cleanly with each other in collisions. These however are solutions of model equations. It is expected that solitary wave solutions of the free surface water wave equations do not interact cleanly, but it is quite remarkable how small a residual there is from collisions of even quite large solitary waves. Numerical and experimental results [] show that solitary water waves are quite stable, and that following collisions between two solitary waves, the solution resolves itself into two slightly modified solitary waves, plus a small trailing residual, and all three components separate from each other and asymptotic evolve independently, see the numerical simulations in Figure 1. More generally, one might predict that any localized initial data for the water waves problem would resolve itself for large time into two sequences of solitary waves (left-propagating and right-propagating), which evolve ahead of a dispersing residual component of the solution.

**Problem 10.** *Prove this scattering picture occurs. That is, general initial data,* *suitable localized, resolves itself under time evolution into two trains of solitary waves plus a residual, and the latter is described by a scattering operator related to the linear problem.*

This will imply that for large times the solution is determined by (i) a finite number of parameters (the amplitudes and phases of the asymptotic solitary waves) and (ii) a linear scattering map for the residual. At present it is not even known whether a solution of the water waves problem exists of the form of a superposition of $N$ many solitary waves of different speeds, existing over $t \in [0, +\infty)$ and asymptotically separating as $t \mapsto +\infty$ (a problem brought up by J. Moser and R. Sachs).

## 7. Wave turbulence

While it is useful to study the dynamics of individual water wave interactions in detail, it is impractical both numerically and theoretically to study the fine structure of fields of water waves of the dimensions of an ocean. Furthermore, predictions of precisely this nature are of great importance to forecasting of sea state which is used by the worlds' shipping. An approach to quantify the behavior of certain macroscopic quantities having to do with the evolution of ocean waves involves an averaging process, and results in kinetic theory-like evolution equations for them. This approximation process merits consideration on a rigorous mathematical level. The two derivations of a theory along these lines are by Hasselman [] and Zakharov [], the former being somewhat *ad hoc* while the latter is based on formal physical considerations and a Hamiltonian formulation of the equations of motion. To adopt our own point of view, describe the canonical variables for the water wave problem in complex symplectic coordinates as

$$a(x) := \sqrt{\frac{g}{2\omega(D)}}\eta(x) + i\sqrt{\frac{\omega(D)}{2g}}\xi(x) \ , \tag{7.1}$$

where $D = -i\partial_x$. Under this transformation the quadratic Hamiltonian (giving the linearized equations of motion) becomes

$$H^{(2)} = \int \overline{a}(x)(\omega(D)a(x)) \ dx \ ,$$

which recovers (6.4). Following Zakharov and working on a formal mathematical level, we take normal forms transformations of the Hamiltonian (2.4) up to a certain order (say, order $N = 4$). Denoting the transformed variables $b = \tau_N(a)$, consider their Wigner transform

$$W_\varepsilon[b](x,k,t) := \frac{1}{\varepsilon^{d-1}} \int_{\mathbb{R}_y^{d-1}} e^{ik\cdot y}\, \overline{b}(x - y/\varepsilon)b(x + y/\varepsilon)\, dy \ . \tag{7.2}$$

Up to higher powers of $\varepsilon$, the Wigner transform $W = W_\varepsilon[b]$ will satisfy a kinetic equation reminiscent of the Boltzmann equation (considerations of Bardos, Craig & Panferov)

$$\partial_t W + \partial_k \omega(k) \cdot \partial_x W = S_{nl} \ , \tag{7.3}$$

where the nonlinear interaction term $S_{nl}$ is derived from the quartic and higher terms of the water waves Hamiltonian $H$ in its normal form expansion. These must be expressed in terms of the Wigner transform itself, which is one reason for the normal form. Often at this point one takes an average of $W$ over a statistical ensemble of wave fields, possibly with some random phase closure assumptions. All of these manipulations are formal, however.

**Problem 11.** *Elucidate this approach as a kinetic theory of wave field propagation and interaction. Provide, to the extent possible, a mathematically rigorous derivation of* (7.3)*.*

The mathematical analysis of the normal forms transformation that is implicit in the above statement is already a good question.

Just as the stationary distributions of the collision operator (the Maxwellian distributions) play a special rôle for the Boltzmann equation, the stationary distributions for $S_{nl}$ should play a rôle in the theory of solutions of (7.3). Indeed they should be parametrized by the principal macroscopic variables for the problem, whose dynamics will be determined by certain macroscopic equations of motion. In the case of wave fields which are homogeneous in $x \in \mathbb{R}^{d-1}$, solutions of $S_{nl} = 0$ in the form of a power law have been found by Zakharov and co-workers, and it is reasonable to call these solutions *Kolmogorov – Zakharov* spectra. Similar spectral behavior of averages of wave fields has been seen in experimental observations, the first being in Lake Ontario. By analogy with the Maxwellian distributions in the theory of Boltzmann's equation, parameter families of such spectra lead to a definition of macroscopic variables, which then vary in $x, t$ in the non-homogeneous problem. The Wigner transform $W(x, k, t)$ is essentially a microlocalization of the wave field given by $b(x, t)$. However the power law distributions may not be the only homogeneous wave fields which are solutions of $S_{nl} = 0$, and it could well be that there are different parameter families of such solutions in different nonlinear regimes. The different parameter families of such solutions would give rise to different sets of macroscopic variables in different regimes. This leads us to pose our final problem.

**Problem 12.** *Prove that power law Kolmogorov – Zakharov spectra exist, and that they govern the large time asymptotics of wave fields, at least in certain situations. Find the full set of solutions of $S_{nl} = 0$, and their respective macroscopic variables, and derive the appropriate macroscopic equations which determine their evolution.*

**Acknowledgements:** The research of WC has been supported in part by the Canada Research Chairs Program, the NSERC under operating grant #238452, and the NSF under grant #DMS-0070218. The research of CEW has been supported by the NSF. The numerical simulations in this note were performed by P. Guyenne.

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