

Bounds on Kolmogorov spectra for the Navier Stokes equations

Walter Craig

Department of Mathematics & Statistics



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joint work with:

Andrei Biryuk

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Abstract

- ▶ The Kolmogorov & Obukov laws for isotropic turbulence
- ▶ A new estimate on Leray weak solutions
- ▶ Estimates on Kolmogorov spectra
- ▶ Restrictions on the spectral behavior of weak solutions

Navier – Stokes equations

The **equations of motion** of an incompressible viscous fluid

$$\begin{aligned}\partial_t u + (u \cdot \nabla)u &= \nu \Delta u - \nabla p + f \\ \nabla \cdot u &= 0 \\ u(x, 0) = u_0(x), \quad \nabla \cdot u_0 &= 0 \quad \text{initial data}\end{aligned}\tag{1}$$

Forcing term $f : \nabla \cdot f = 0$ Take f to be zero at present
Space-time domain

$$D = \mathbb{R}^3 \quad (x, t) \in D \times \mathbb{R}^+ := Q$$

Alternatively $D = \mathbb{T}^3$ and

$$(x, t) \in \mathbb{T}^3 \times \mathbb{R}^+ = Q$$

A bounded smooth domain $D \subseteq \mathbb{R}^3$; we leave this open.

Weak solutions

The usual definition of a **weak solution** over $t \in [0, T]$ is that:

1. **Integrability conditions**

$$\begin{aligned} u &\in L^\infty([0, T]; L^2(D)) \cap L^2([0, T]; \dot{H}^1(D)) , \\ p &\in L_{loc}^{5/3}(Q) \end{aligned} \tag{2}$$

2. The pair (u, p) is a **distributional solution** of (1)

3. The **energy inequality** is satisfied

$$\frac{1}{2} \int_D |u(x, t)|^2 dx + \nu \int_0^t \int_D |\nabla u(x, s)|^2 dx ds \leq \frac{1}{2} \int_D |u_0(x)|^2 dx \tag{3}$$

The existence of weak solutions

Theorem (Leray (1934))

Given $u_0 \in L^2(D)$ divergence free, then there exists *at least one weak solution* to (1) globally in time. Weak solutions satisfy

$$u \in L_t^\infty(L_x^2)$$

as well as

$$u \in C_t(L_x^2 : \text{weak topology})$$

A lot is known about such solutions, for example that

$$u \in L_t^s(L_x^p), \quad \frac{3}{p} + \frac{2}{s} = \frac{3}{2}$$

Uniqueness and global regularity are unknown

Fourier transforms

- ▶ The Fourier transform of $u(x, t)$ exists *a.e. t*

$$\hat{u}(k, t) = \frac{1}{(2\pi)^{3/2}} \int e^{-ik \cdot x'} u(x', t) dx'$$

and $\hat{u}(\cdot, t) \in L_t^\infty(L_x^2)$

- ▶ The Fourier transform is smooth in t

Theorem

The function $\hat{u}(k, t)$ is C^1 as a function of t for every k (when $D = \mathbb{T}^3$ at least).

- ▶ Define the **energy spectrum** as the spherical integrals

$$E(\kappa, t) := \frac{1}{V} \int_{|k|=\kappa} |\hat{u}(k, t)|^2 dS(k), \quad 0 \leq \kappa < +\infty \quad (4)$$

where V is a characteristic unit volume.

Power spectrum

- ▶ Plancherel's identity

$$\int_0^\infty E(\kappa, t) d\kappa = \frac{1}{V} \|u(\cdot, t)\|_{L^2}^2$$

- ▶ Sobolev norms

$$\int_0^\infty \kappa^2 E(\kappa, t) d\kappa = \frac{1}{V} \|\nabla u(\cdot, t)\|_{L^2}^2$$

- ▶ **dimensional analysis**, where $[*]$ denotes dimension

$$[u] = \frac{L}{T}, \quad [|\hat{u}|^2] = \frac{L^{2(d+1)}}{T^2}, \quad [\nu] = \frac{L^2}{T}, \quad [E(\cdot, t)] = \frac{L^3}{T^2}$$

Reynold's number

There is a literature on the **Reynold's number** defined in terms of the energy spectrum

- ▶ $Re := UL/\nu$ a dimensionless parameter
- ▶ Intrinsic Reynold's number (Gammond & Gage)

$$Re_1 := \frac{\Lambda}{\eta_K}, \quad \Lambda := \frac{\int_0^\infty \kappa^{-1} E(\kappa) d\kappa}{\int_0^\infty E(\kappa) d\kappa}, \quad \eta_K := \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

for $\varepsilon := 2\nu \int_0^\infty \kappa^2 E(\kappa) d\kappa$, the rate of energy dissipation

- ▶ Proposal for a mathematical Reynold's number

$$Re_2 := \frac{\|u\|_{\dot{H}^{1/2}}}{\nu}$$

in the light of the classical Fujita - Kato existence theorem

Kolmogorov's hypotheses

For high Reynolds number flows which exhibit fully developed turbulence, Kolmogorov supposed:

- ▶ The small scale turbulent motions of a fluid are **statistically isotropic**
- ▶ The statistics of these motions are determined by the two parameters ν and ε

Kolmogorov's scaling law

- ▶ **Prediction:** For high Reynold's number fully developed turbulent flows, for κ in an inertial range, the energy spectrum has universal behavior

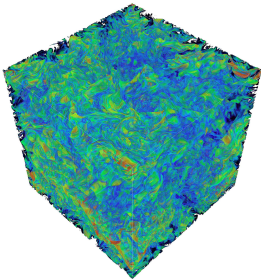
$$E_K(\kappa) = C_0 \varepsilon^{2/3} \kappa^{-5/3} \quad (5)$$

Proof : These are the unique exponents for which the dimensions match

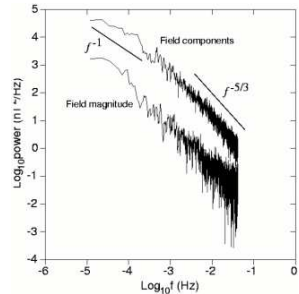
In fact the exponents are independent of space dimension

- ▶ Considerable experimental and numerical evidence has been garnered to support this conjecture.
- ▶ **Goal:** to give mathematically rigorous upper bounds on the range of validity of the Kolmogorov scaling law

SDSC simulation by Chowasia, Donzis and Yeung



Turbulence Spectrum



Estimates on weak solutions

- ▶ The energy inequality (3) can be viewed as the statement that the ball $B_R(0) \subseteq L_x^2$ is an **invariant set** for Navier – Stokes flow

$$u_0(\cdot) \in B_R(0) \quad \implies \quad \forall t > 0, u(\cdot, t) \in B_R(0)$$

- ▶ Another invariant set. Define

$$A := \{(\hat{u}(k))_{k \in \mathbb{R}^3} : |k| |\hat{u}(k)| < R_1\} \cap B_R(0)$$

Theorem (A. Biryuk (2003))

If $R^2 < \nu R_1$ then A is an invariant set for Navier – Stokes flow.

Proof given at end of talk if there is time

Global estimates on weak solutions

- ▶ **Bounds** on $L_t^\infty(L_k^{(\infty,1)}(\mathcal{F}u))$, supposing that the initial data lies in the set \mathbf{A} , then for all $k \in \mathbb{R}^3$,

$$\sup_{t \geq 0} |\hat{u}(k, t)| \leq \frac{R_1}{|k|} \quad (6)$$

- ▶ **Time average quantities** obey better estimates:

Corollary

For all $k \in \mathbb{R}^3$ and all $T \geq 0$, then $\nu \int_0^T |\hat{u}(k, s)|^2 ds \leq \frac{R_1^2}{|k|^4}$

- ▶ The quantity $\sup_t \| |k| \hat{u}(\cdot, t) \|_{L^\infty}$ scales like the **BV** norm $\sup_t \| \partial_x u(\cdot, t) \|_{L^1}$ (for which there are no known bounds).
P. Constantin (1992) has a global bound on $\sup_t \| \nabla_x \times u(\cdot, t) \|_{L^1}$

Estimates on spectra

Proposition (1)

The spectrum of a weak solution with initial data $u_0 \in A$ satisfies a *global upper bound*

$$E(\kappa, t) = \frac{1}{V} \int_{|k|=\kappa} |\hat{u}(k, t)|^2 dS(k) \leq \frac{R_1^2}{V\kappa^2} 4\pi\kappa^2 = \frac{4\pi R_1^2}{V}$$

Proposition (2)

Time averages of energy spectra have a *uniform decay rate*. Weak solutions with initial data $u_0 \in A$ satisfy

$$\frac{1}{T} \int_0^T E(\kappa, t) dt = \frac{1}{VT} \int_0^T \int_{|k|=\kappa} |\hat{u}(k, t)|^2 dS(k) dt \leq \frac{4\pi\kappa^2}{T} \frac{R_1^2}{\nu V\kappa^4} = \mathcal{O}(\kappa^{-2})$$

rate of decay

How does the **energy spectrum** of a solution compare to the Kolmogorov prediction.

Theorem

The exponent 2 is larger than 5/3.

Is this a problem with the theory?

- ▶ One resolution could be that Navier – Stokes flows which exhibit spectral behavior like the Kolmogorov law are in the support of a probability measure \mathbf{P} on $L^2(D)$ -divergence-free.
- ▶ And it could be the case that for all R, R_1 , $\text{supp}\mathbf{P} \cap A = \emptyset$.

Better resolution of this dilemma

Bounds on the **inertial range** where this spectral behavior is manifest

Theorem

The upper and lower bounds for the inertial range $[\kappa_1, \kappa_2]$ over which the Kolmogorov spectral function E_K does not violate our estimates

$$\kappa_1 = \left(\frac{C_0 V}{4\pi R_1} \right)^{3/5} \varepsilon^{2/5} \quad (7)$$

$$\kappa_2 = \frac{1}{T} \left(\frac{R_1^2}{\nu C_0 V} \right)^3 \frac{1}{\varepsilon^2} \quad (8)$$

Maximum time for which this behavior persists is $T_0 : \kappa_1 = \kappa_2(T)$

$$T_0 = \frac{R_1^{11/5}}{\nu \varepsilon^{4/5}} \left(\frac{4\pi}{C_0^6 V^6} \right)^{1/5}$$

Comparison with the classical quantities

- ▶ Kolmogorov **lengthscale** $\eta_K := (\nu^3/\varepsilon)^{1/4}$

$$\frac{2\pi}{\eta_K} = 2\pi \left(\frac{\varepsilon}{\nu^3}\right)^{1/4} < \kappa_2 = \left(\frac{R_1^2}{\nu C_0 VT}\right)^3 \frac{1}{\varepsilon^2}$$

- ▶ Kolmogorov **timescale** $\tau_K := \left(\frac{\nu}{\varepsilon}\right)^{1/2}$

$$\tau_K = \left(\frac{\nu}{\varepsilon}\right)^{1/2} \ll T_0 = \frac{R_1^{11/5}}{\nu \varepsilon^{4/5}} \left(\frac{4\pi}{(C_0 V)^6}\right)^{1/5}$$

This is as it should, since these two times signify different things

Comparison with a Navier – Stokes velocity field

Various definitions of **proximity** to $\hat{u}_K(k) \simeq \varepsilon^{1/3} |k|^{11/6}$

- ▶ **Definition 1:** $\|u_K - u(\cdot, t)\|_{L_x^2} \leq C_1$.

Since $\hat{u}_K \notin L^2$ this is not a satisfactory criterion.

- ▶ Dyadic decomposition $u = \sum_j \Delta_j u$ with support

$$\text{supp}(\widehat{\Delta_j u}(k)) \subseteq A_j$$

where $A_j := \{2^{j-1} < |k| < 2^{j+1}\}$.

Definition 2: $\|\Delta_j(u - u_K)\|_{L^1} 2^{11j/3} \leq C_2$ for all j in the range $\log_2(\kappa_1) \leq j \leq \log_2(\kappa_2)$

continued comparison

- ▶ There is the question as to whether $E(\kappa, t)$ has spectral behavior for individual solutions, or does it hold in an **average** sense, over a statistical ensemble of solutions with probability measure \mathbf{P} .
- ▶ Therefore study the **ensemble averages**

$$\langle E(\kappa, t) \rangle := \int_{|k|=\kappa} \langle |\hat{u}(k, t)|^2 \rangle dS(k)$$

Definition 3: Use Definition 2 for ensemble averages of solutions.

- ▶ In fact \mathbf{P} should be ergodic with regard to NS flow, so that asymptotically the \mathbf{P} average should approximate the time average

$$\langle E(\kappa, t) \rangle \simeq \frac{1}{T} \int_0^T E(\kappa, t) dt$$

Theorem (Bounds on Kolmogorov spectra)

In order that $u(x, t)$ exhibit Kolmogorov-like behavior of its spectral energy function, in either of the senses of Definition 2 or Definition 3 over an inertial range $[\kappa_1, \kappa_2]$, then the three constants

$$\kappa_1, \quad \kappa_2, \quad T_0$$

must satisfy the above three relations, up to a constant.

proof of the $L_t^\infty(L_k^{(\infty,1)}(\mathcal{F}u))$ estimate

- ▶ For fixed k the field $\hat{u}(k) \in \mathbb{C}_k^2 \subseteq \mathbb{C}^3$
Because of incompressibility $k \cdot \hat{u}(k) = 0$

Suppose that $\|u(\cdot)\|_{L^2} \leq R$

- ▶ The Fourier transform satisfies

$$\begin{aligned} \partial_t \hat{u}(k) &= -\nu |k|^2 \hat{u}(k) - ik \Pi_k \int \hat{u}(k - k_1) \cdot \hat{u}(k_1) dk_1 + \hat{f}(k, t) \\ &:= X(u)_k \end{aligned}$$

- ▶ Consider the vector field $X(u)$ when $|\hat{u}(k)| = R_1/|k|$. Then

$$\operatorname{re}(\hat{u}(k) \cdot X(u)_k) < -\nu |k|^2 (R_1/|k|)^2 + (R_1/|k|) |k| R^2 + |\hat{f}|(R_1/|k|)$$

which is **negative** when $R^2 + |\hat{f}(k)|/|k| < \nu R_1$

proof of corollary

- ▶ A fact about the vector field $X(\hat{u})$ is that solutions obey

$$\begin{aligned} & |\hat{u}(k, T)|^2 - |\hat{u}_0(k)|^2 + 2\nu \int_0^T |k|^2 |\hat{u}(k, t)|^2 dt \\ &= 2\text{im} \left[\int_0^T \bar{\hat{u}}(k) \cdot \int \hat{u}(k - k_1) \cdot k_1 \hat{u}(k_1) dk_1 dt \right] \end{aligned}$$

(setting $f = 0$ for simplicity)

- ▶ Writing $I^2(k) = (2\nu)^3 \int_0^T |k|^4 |\hat{u}(k, t)|^2 dt$
this gives an inequality

$$I^2(k) - 2R^2 I(k) - (2\nu R_1)^2 \leq 0$$

Thank you