Nonlinear Water Waves in Random Bathymetry

Walter Craig

Department of Mathematics & Statistics

McMaster University

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joint work with:

Anne deBouard	Université de Paris - Sud
Oliver Dìaz-Espinosa	McMaster University
Philippe Guyenne	University of Delaware
Catherine Sulem	University of Toronto

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Overview of the hydrodynamic problem

- Goal: asymptotic description of water waves over a variable bottom, in the long wave limit.
- Basic assumption: large scale topography of the bottom of the fluid region is known, but details of the topography are unknown (and therefore subject to modeling).
- A homogenization problem of separation of scales (or not). Our work is a reappraisal of [Rosales & Papanicolaou (1983)]

Overview of the hydrodynamic problem: conclusions

1. Periodic bottom topography; the problem homogenizes fully to give a KdV equation with effective coefficients [Craig, Guyenne, Nicholls & Sulem (2005)]
2. Random bottom topography given by a stationary ergodic process which is mixing, *i.e.* which decorrelates with spatial distance.
3. There remain realization dependent effects in the equations as important as the nonlinearity and the dispersion, given by a canonical process $\sigma_\beta \partial_X B_\omega(X)$ (white noise).
4. The solution has both transmitted (nonlinear) and scattered (linear) components. Skewness of the statistics of the random process can stabilize or destabilize solutions.

Euler's equations

- Euler's equations $\Delta\varphi = 0$

$$N \cdot \nabla\varphi = 0 \quad \text{on the variable bottom} \quad y = -h + \beta(x)$$

Nonlinear boundary conditions on the free surface

$$\left.\begin{aligned} \partial_t\varphi + \tfrac{1}{2}(\nabla\varphi)^2 + g\eta = 0 \\ \partial_t\eta + \partial_x\eta \cdot \partial_x\varphi - \partial_y\varphi = 0 \end{aligned}\right\} \quad \text{on} \quad y = \eta(x,t)\,,$$

Figure: Cartoon of a fluid domain with a varying bottom boundary

- Hamiltonian form: variables $\eta(x,t)$, $\xi(x,t) = \varphi(x,\eta(x,t),t)$

$$H(\eta,\xi) = \frac{1}{2}\int_{-\infty}^{\infty} \xi G(\eta,\beta)\xi + g\eta^2 \, dx \tag{1}$$

- $G(\eta,\beta)$ is the Dirichlet – Neumann operator for the fluid domain

- Conservation laws

$$M = \int \eta(x)\,dx \;, \qquad \text{mass}$$
$$H = \int \int_{b(x)}^{\eta} \tfrac{1}{2}|\nabla\varphi|^2 \,dydx + \tfrac{g}{2}\int \eta^2(x)\,dx \qquad \text{energy}$$

Momentum is not conserved, due to the bottom variations $\beta(x)$

Scaling regime

- long wave scaling regime:

$$X = \varepsilon x \quad , \qquad \beta(x) = \varepsilon \beta'(X/\varepsilon)$$
$$\varepsilon \xi'(X) = \xi(x) \quad , \qquad \varepsilon^2 \eta'(X) = \eta(x)$$

- Hamiltonian for this problem (Boussinesq regime)

$$H(\eta', \beta'; \varepsilon) = \tag{2}$$
$$\frac{1}{2}\int (h + \varepsilon\beta'(X/\varepsilon) - \varepsilon^2 \Big(\beta' D \tanh(hD)\beta'\Big)(X/\varepsilon))|D_X \xi'|^2 \, dX$$
$$+ \frac{1}{2}\int g\eta'^2 + \varepsilon^2 (\xi' D\eta' D_X \xi' - \frac{h^3}{3}\xi' D_X^4 \xi')\, dX + \mathcal{O}(\varepsilon^3)$$

Stationary ergodic processes

Realizations of the bottom topography $\beta = \beta(x, \omega)$ are taken from a statistical ensemble $\omega \in \Omega$

- Probability space $(\Omega, \mathcal{M}, \mathsf{P})$
- Ergodic one parameter group of measure preserving transformations $\{\tau_y\}_{y\in\mathbb{R}}$ such that $\beta(x, \tau_y\omega) = \beta(x-y, \omega)$ (and a filtration of the measurable sets, $\mathcal{M}_y$, $y \in \mathbb{R}$ adapted to $\{\tau_y\}_{y\in\mathbb{R}}$)
- Case 1: periodic $\beta(x+p) = \beta(x)$
- Case 2: mixing conditions

$$|\mathsf{P}(A \cap \tau_y(B)) - \mathsf{P}(A)\mathsf{P}(B)| < \varphi(y)\sqrt{\mathsf{P}(A)\mathsf{P}(B)}\ , \qquad (3)$$

for sets $A \in \mathcal{M}_{\{y\le 0\}}$ and $B \in \mathcal{M}_{\{y\ge 0\}}$

Mixing rate

In case 2, we require that the mixing rate satisfy

$$\int_0^\infty \varphi^{1/2}(y)\, dy < +\infty \tag{4}$$

The variance of the process $\beta(x, \omega)$ is defined to be

$$\sigma_\beta^2 := 2 \int_0^\infty \mathsf{E}(\beta(0,\omega)\beta(0,\tau_y\omega))\, dy \ ,$$

Lemma

If $\beta(x,\omega) = \partial_x \gamma(x,\omega)$ *for some stationary process* $\gamma \in C^1$ *then*

$$\sigma_\beta = 0$$

separation of scales: periodic case

Theorem

Suppose that $\gamma(x)$ *is periodic, and that* $f(X) \in \mathcal{S}$. *Then*

$$\int \gamma(X/\varepsilon) f(X)\, dX = \mathsf{E}(\gamma) \int f(X)\, dX + \mathcal{O}(\varepsilon^N)$$

for any N, *where* $\mathsf{E}(\gamma) = \frac{1}{p}(\int_0^p \gamma\, dx)$.

In the periodic case (Case 1) the contributions of the new terms homogenizes, giving a perturbation to the effective wavespeed, due to the following result.

effective coefficients: periodic case

- Since $\mathsf{E}(\beta') = 0$ the two new terms are therefore
  - $$\int \beta'(X/\varepsilon)|D_X\xi'|^2 dX = 0 + \mathcal{O}(\varepsilon^N)$$
  - $$\int \Big(\beta'(X/\varepsilon) D \tanh(hD) \beta'(X/\varepsilon)\Big) |D_X\xi'|^2 dX$$
    $$= \mathsf{E}(\beta' D \tanh(hD)\beta') \int |D_X\xi'|^2 dX + \mathcal{O}(\varepsilon^N)$$
- the effective wavespeed is
  $\overline{c}^2 = g(h - \varepsilon^2 \mathsf{E}(\beta'(x) D \tanh(hD) \beta'(x)))$

Lemma

There is a strict inequality $\overline{c}^2 < c^2$ for nonzero β'.

- The analysis extends to the situation where $\beta(x) = \mathcal{O}(1)$

separation of scales: random case

In the mixing case (Case 2) there are realization dependent contributions which are significant. In particular

Theorem

Let $\gamma(x;\omega)$ *be a stationary mixing ergodic process with rate* $\varphi(y)$ *satisfying* (4). *Then as* $\varepsilon \to 0$

$$\int \gamma(X/\varepsilon,\omega) f(X)\,dX = \int \big(\mathsf{E}(\gamma) + \sqrt{\varepsilon}\sigma_\gamma \partial_X B_\omega(X)\big) f(X)\,dX + o(\sqrt{\varepsilon})\ ,$$

where $B_\omega(X)$ *is normal Brownian motion.*

This principally affects the wavespeed and can affect the stability in the KdV regime

random KdV equation

This results in the following long wave system with random coefficients

$$\begin{aligned}
\partial_T r &= -\partial_X \big(c_0(X,\varepsilon,\omega) r + \varepsilon^2 b r + \varepsilon^2 (c_1 \partial_X^2 r + c_2 r^2) \big) \\
\partial_T s &= \partial_X \big(c_0(X,\varepsilon,\omega) s \big) - \frac{1}{4}\sqrt{g/h} \big(\sigma_\beta \partial_X^2 B_\omega(X) r \big)
\end{aligned}$$

where the random wavespeed is

$$c_0^2(X,\varepsilon,\omega) = g(h - \varepsilon^{3/2}\sigma_\beta \partial_X B_\omega(X) - \varepsilon^2 a_{KdV})$$

constants c_1 and c_2 are effective (homogenized) coefficients of dispersion and nonlinearity, a_{KdV} is a wavespeed adjustment, and $b = c_3 \mathsf{E}(\beta_x^3)$ is statistics dependent and determines stability.

random KdV equation

This results in the following long wave system with random coefficients

$$\begin{aligned}
\partial_T r &= -\partial_X\big(c_0(X,\varepsilon,\omega)r + \varepsilon^2 br + \varepsilon^2(c_1\partial_X^2 r + c_2 r^2)\big) \\
\partial_T s &= \partial_X\big(c_0(X,\varepsilon,\omega)s\big) - \frac{1}{4}\sqrt{g/h}\big(\sigma_\beta \partial_X^2 B_\omega(X) r\big)
\end{aligned}$$

where the random wavespeed is

$$c_0^2(X,\varepsilon,\omega) = g(h - \varepsilon^{3/2}\sigma_\beta \partial_X B_\omega(X) - \varepsilon^2 a_{KdV})$$

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The form of a random solution

- Solve the deterministic equation

$$\partial_\tau Q = c_1 \partial_Y^3 Q + c_2 \partial_Y Q^2 + bQ$$

- Introduce random characteristic coordinates

$$X = X(Y,T;\omega) = (Y + T\sqrt{gh} - \varepsilon^2 a_{KdV} T) - \frac{\varepsilon^{3/2}\sigma_\beta}{2}\sqrt{\frac{g}{h}} B_{\omega(Y)}(T\sqrt{gh})$$

- The solution $r(X,T)$ is given up to $o(\varepsilon^2)$ by

$$\begin{aligned} r(X,T) &= \partial_X Q(Y(X,T;\omega), \varepsilon^2 T) \\ &= Q_Y(Y(X,T;\omega), \varepsilon^2 T)\partial_X Y(X,T;\omega) \end{aligned}$$

- Consequences: the phase undergoes Brownian motion, the amplitude is modulated by white noise.

Random characteristic coordinates

- The characteristic equations

$$\frac{dX}{dt} = c_0(X,\varepsilon;\omega)\,, \qquad X(0,\varepsilon;\omega) = Y$$
$$\simeq \sqrt{gh}\Big(1 - \frac{\varepsilon^{3/2}\sigma_\beta}{2h}\partial_X B_\omega(X) - \varepsilon^2 a_{KdV}\Big)$$

  Too singular in general for solutions of SDEs.

- Regularized problem

$$\frac{dX}{dt} = c_\varepsilon(X,\varepsilon;\omega) \simeq \sqrt{gh}\Big(1 - \frac{\varepsilon\beta(X/\varepsilon)}{2h} - \varepsilon^2 a_{KdV}\Big)$$

  where $\beta \in C^1$ implies that the characteristic speed $c_\varepsilon(X,\varepsilon;\omega)$ is bounded uniformly in C^1.

- Orbits are uniformly bounded in C^1, and by Donsker's theorem the limit points of orbits distribute like Brownian motion (15)

Scattering

- The scattered wavefield is solved for $s(X,T)$ by the method of characteristics. Set $T' := T + (X-\theta)/\sqrt{gh}$.

$$s(X,T) = s_0(X+\sqrt{gh}T) + \frac{\sigma_\beta}{4h}\int_X^{X+\sqrt{gh}T} \partial_X^2 B_\omega(\theta)\partial_X Q(Y(\theta,T';\omega),\varepsilon^2 T')\,d\theta$$

- Notice the irregularity in the scattered wavefield due to the multiple derivatives of Brownian motion $\partial_X^2 B_\omega(X)$ which interact with the random solution $r(X,T;\omega)$.

Instabilities

The coefficient $b = c_3 \mathsf{E}(\beta_x^3)$ determines stability

Proposition

If the statistics of the ensemble $(\Omega, \mathcal{M}, \mathsf{P})$ *are reversible, then* $b = 0$

However there are reasonable situations in which this is not the case

Figure: Cartoon of a fluid domain with a bottom boundary with $b > 0$

Bottom variations on multiple scales

- Fluid domains bounded by a bottom with large scale as well as small scale variations with $\mathsf{E}(\beta_1(\cdot, X)) = 0$

$$\beta(x, X, \varepsilon, \omega) = \beta_0(X, \varepsilon) + \beta_1(x, X, \varepsilon, \omega)$$

- Furthermore, the statistics of the bottom may vary

$$\sigma^2_{\beta_1}(X) = 2 \int_0^\infty \mathsf{E}(\beta_1(0, X; \omega)\beta_1(0, X; \tau_y\omega))\, dy$$

Figure: Cartoon of a fluid domain with a random bottom boundary with varying statistics

Stationary arrays of mixing processes

Theorem

Let $\gamma(x, X; \omega)$ *be a smooth (in* X*) family of stationary mixing ergodic processes with uniform mixing rate* $\varphi(y)$ *satisfying* (4)*. Define the local variance to be*

$$\sigma_\gamma^2(X) = \int_{-\infty}^{\infty} \mathsf{E}(\gamma(\cdot, X; \omega)\gamma(\cdot, X; \tau_y\omega))\, dy$$

Then as $\varepsilon \longrightarrow 0$

$$\int \gamma(X/\varepsilon, X; \omega) f(X)\, dX = \int \big(\mathsf{E}(\gamma(\cdot, X)) + \sqrt{\varepsilon}\sigma_\gamma(X)\partial_X B_\omega(X)\big) f(X)\, dX + o(\varepsilon^{1}$$

where $B_\omega(X)$ *is normal Brownian motion.*

Random characteristics

- Random characteristic coordinates

$$\frac{dX}{dt} = c_0(X, \varepsilon; \omega) \ , \qquad x(0, \varepsilon; \omega) = Y$$

has trajectories which describe a diffusion

$$\begin{aligned} X(t, Y, \varepsilon; \omega) &= (Y + \sqrt{gh}t) \\ &- \frac{1}{2}\sqrt{\frac{g}{h}} \int_0^t \varepsilon\beta_0(Y + \sqrt{gh}s) + \varepsilon^2 a_{KdV}(Y + \sqrt{gh}s)\, ds \\ &- \epsilon^{3/2} \frac{1}{2h} \int_0^t \sigma_{\beta_1}(Y + \sqrt{gh}s) dB_\omega(s) \end{aligned}$$

Thank you