Hamiltonian Long Wave Expansions for Free Surfaces and Interfaces

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Abstract

The theory of internal waves between two bodies of immiscible fluid is important both for its interest to ocean engineering, and as a source of numerous interesting mathematical model equations which exhibit nonlinearity and dispersion. In this paper we derive a Hamiltonian formulation of the problem of a dynamic free interface (with rigid lid upper boundary conditions), and of a free surface and a free interface, this latter situation occurring more commonly in experiment and in nature. From the formulation, we develop a Hamiltonian perturbation theory for the long wave limits, and we carry out a systematic analysis of the principal long wave scaling regimes. This analysis provides a uniform treatment of the classical works of Peters and Stoker [28], Benjamin [3, 4], Ono [26] and many others. Our considerations include the Boussinesq and KdV regimes over finite depth fluids, the Benjamin-Ono regimes in the situation in which one fluid layer is infinitely deep, and the intermediate long wave regimes. In addition we describe a novel class of scaling regimes of the problem, in which the amplitude of the interface disturbance is of the same order as the mean fluid depth, and the characteristic small parameter corresponds to the slope of the interface. Our principal results are that we highlight the discrepancies between the case of rigid lid and of free surface upper boundary conditions, which in some circumstances can be significant. Motivated by the recent results of Choi and Camassa [6, 7], we also derive novel systems of nonlinear dispersive long wave equations in the large amplitude/small slope regime. Our formulation of the dynamical free surface/free interface problem is shown to be very effective for perturbation calculations, and as well it holds promise as a basis for numerical simulations.

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1 Introduction

Internal waves in a fluid body occur in a sharp interface between two fluids of different densities. Scientific interest in internal waves includes the need to quantify induced loads on submerged engineering constructions (such as oil platforms and rail and road tunnels lying on the sea bed), as well as the mathematical interest in the variety of nonlinear dispersive evolution equations that occur in the discipline of free surface hydrodynamics. In nature they are observed in the pycnocline induced by an abrupt jump in salinity, often occurring in fjords, and in thermoclines found in relatively common situations in tropical seas. Observations report amplitudes of internal waves greater than 100m in fluid bodies of depth less than 1000m with wavelength of one to ten kilometers [16, 2]. This is a highly nonlinear regime of wave motion, characterized by large amplitudes which are nevertheless of small slope. Additionally, in oceanographic observations, waves on the sea surface are affected in a nontrivial manner by the presence of disturbances in the interface. Indeed one characteristic signature of internal waves can be a change in the smaller scale wave patterns in the surface, giving rise to a differential reflectancy property under oblique lighting. Our own interest in the topic is motivated in part by the NASA photographs of internal waves taken from an orbiting space shuttle. The effect is particularly striking in views of tidally induced internal waves in the Andaman Sea, which have been imaged under the highly oblique incident light of the late afternoon sun. We are also motivated by the recent work of Choi and Camassa [6, 7] on internal waves, and their models for larger amplitude long wave motion.

In this paper we give a formulation for the equations of motion of a system of one or several ideal fluids with a dynamic free surface, free interfaces, or both, as Hamiltonian systems with infinitely many degrees of freedom. The top surface of the upper layer is either subject to rigid lid boundary conditions, or else it is itself a free surface. We confine our considerations to two dimensional fluid motions, which are valid approximations for long-crested waves. In principle our methods extend to the fully three dimensional case. We then develop a basic framework of perturbation theory for Hamiltonian PDE, and using it, provide a systematic treatment of the long wavelength perturbation regimes for the problem. This includes (1) the Boussinesq and the Korteweg-deVries (KdV) scaling regimes in the setting of a body of fluid bounded below by a horizontal bottom and above either by a rigid lid or by a free surface; (2) the Benjamin-Ono (BO) regime, in which one of the fluid layers is infinite; and (3) the intermediate long wave (ILW) regime, characterized as in situation (1) but with one layer asymptotically thin. We have focused in particular on quantifying the difference between the choice of rigid lid boundary conditions, most often used in mathematical modeling, and the setting of a free surface top boundary, which is the physically most relevant case. There are a surprising number of important differences, affecting even the linear dispersion relation and the linear wavespeed, but as well the character of the dispersion and the nonlinearity. In some situations the sign of the term governing the principal nonlinear effects is reversed in the two cases. To emphasize the importance of the distinction between the two upper boundary conditions, we mention in particular that the influence on the free surface of the presence of large-scale disturbances in an interface is not modeled in the case of the rigid lid.

In addition we develop new model systems of equations for perturbation regimes in which wave profiles have small slope, allowing amplitudes that are fully of the same order as the mean depth of the fluid layers. This regime reflects the realities of the observed interfacial waves in the ocean, where the ratio of amplitude to layer depth may be of order $O(1)$, while the ratio of amplitude to wavelength remains small. In this scaling regime we have found several unusual and interesting Hamiltonian PDE which have nonlinear rational coefficients of dispersion.

Throughout our analysis we have made a point to extend our perturbation calculations in a systematic manner to at least one order higher than the Boussinesq and KdV level of approximation. This serves as a natural stabilization procedure for the Boussinesq systems, provides higher order corrections to the KdV equation which are useful in degenerate cases, and in any case it exhibits the power and flexibility of our Hamiltonian approach to the perturbation analysis.

The history of the problem of free surface water waves viewed as a Hamiltonian system dates to Zakharov's article [30] on surface waves in deep water, who derived a Hamiltonian for the problem which involved the Dirichlet integral for the fluid domain. A Hamiltonian formulation of the problem of a free interface between two ideal fluids, under rigid lid boundary conditions for the upper fluid, is given by Benjamin and Bridges [5]. Zakharov's formulation has been reworked numerous times, including by Craig and Sulem [15], who posed the Hamiltonian for the water waves problem in terms of the Dirichlet-Neumann operator and the trace of the velocity potential on the free surface, giving it a theoretically straightforward and calculationally efficient expression. The paper by Craig and Groves [10] gives a similar expression for Benjamin and Bridges' Hamiltonian for the free interface problem, using the Dirichlet-Neumann operators for both the upper and lower fluid domains. Ambrosi [1] addressed the Hamiltonian formulation of the problem of a free interface with an upper free surface, however his expression for the Hamiltonian misses some interaction terms between the surface and the interface. Our present formulation of the problem is complete, with the Hamiltonian being given in terms of the deformations of the free surface and the free interface, the traces of the velocity potential functions on them, and the Dirichlet-Neumann operators for the upper and lower fluid domains. This formulation also has implications for the convenience of perturbation calculations in these variables.

There is a longer history of long wave modeling of free interface motion. Peters and Stoker [28] considered the case of steady waves in a system of two immiscible fluid layers of finite depth possessing a free surface as well as a free interface. In this article they give a criterion for the sign of the solitary wave disturbance at the interface. Benjamin [3] considered the analogous system of two layer fluids of finite depth, with rigid lid boundary conditions on the upper fluid boundary, giving an analysis of steady solutions as well as deriving evolution equations in the long wave approximation. Subsequent to this, Benjamin [4] and Ono [26] considered the case in which one layer is infinite, and Joseph [19] and Kubota, Ko and Dobbs [22] studied the regime in which the intermediate long wave equations appear. Kawahara [21] derived higher order dispersive equations as corrections to the KdV equation, which are particularly relevant in a degenerate case that is pointed out in Benjamin [3]. Though Benjamin indicated in this paper the important differences between imposing free surface boundary conditions and rigid lid boundary conditions on the upper fluid surface, the majority of the above references consider rigid lid conditions alone. In case the upper fluid boundary is a free surface, Gear and Grimshaw [17] and Matsuno [24] derived long wave approximate equations, describing coupled KdV-like systems for the evolution of the interface and the free surface. Our own work on this problem is partially motivated by two recent papers of Choi and Camassa on larger amplitude evolution equations for the interface. Indeed we recover their model equations from [6] in the BO scaling regime, providing it with a Hamiltonian formulation as is automatic from our point of view. We furthermore extend it to higher order in perturbation theory. In the new scaling regime of large amplitude/small slope, our model equations are not dissimilar to the model equations in Choi and Camassa [7] and Ostrovsky and Grue [27], although they differ significantly in many details and they are in particular Hamiltonian PDE.

This paper is structured as follows. In Section 2, we derive the Hamiltonian formulation of the free interface problem and the problem of a free surface above a free interface, using the description of the Dirichlet integral for the velocity potentials in terms of the Dirichlet-Neumann operators on the fluid domain boundaries. This derivation is posed from the 'first principles of mechanics' in that the canonical conjugate variables are deduced from the Lagrangian under a Legendre transform. In principle the Hamiltonian formulation can be extended to the setting of multiple fluid layers, separated by free interfaces. In Section 3 we describe a perturbation theory for Hamiltonian PDE, and develop the basic transformation theory that is relevant to the problem of perturbation analysis in the long wave and/or small amplitude scaling regime. Section 4 gives the analysis of two linearized problems; the free interface case with rigid lid boundary conditions on the upper surface, and the free interface with free surface boundary conditions on the upper fluid surface. We quantify the behavior of the dispersion relations of the two problems, and indicate a number of significant differences even at the linear level. The asymptotic analysis of the long wave regime for the free interface problem with an upper rigid lid appears in Section 5. There are six basic regimes. (i) The KdV regime occurs when there are two layers of finite depth, and one seeks long waves of small amplitude. (ii) The finite stepness regime describes the setting of large interface deviations of small slope, again between finite upper and lower layers. (iii) The BO regime appears when one of the fluid layers is infinite (we choose this to be the bottom layer, however the other case involves only changes of sign in the resulting model equations). (iv) The regime of small steepness in the presence of an infinite lower layer, (v) the ILW regime in which one of the two finite layers is very shallow, and (vi) the small steepness regime in the ILW setting. The descriptions of the settings (ii), (iv) and (vi) are new, as far as we know. In Section 6 we describe the long wave analysis of the problem of a free surface above a free interface. In the regime of two finite layers we give the analogous Boussinesq system and KdV equation, we compare the coefficients of dispersion and nonlinearity with those of the KdV regime of Section 5, and we quantify a number of significant differences. We chose not to pursue the nonlinearly coupled free surface and free interface case for the KdV regime, as in Gear and Grimshaw [17], as the linear velocity of the surface mode does not coincide with that of the interface mode, and therefore we judge that the timescale of nonlinear interaction of localized disturbances is too short to be significant. The appendix contains a full Taylor expansion of the Dirichlet-Neumann operators for the upper and lower fluid domains; this is used at the heart of our perturbation analysis, but it is also potentially useful for future analysis and numerical computations of free surface and free interface water waves.

2 Formulation of the problem

2.1 Equations of motion

The fluid domain is the region consisting of the points (x,y) such that $-h < y < h_1 + \eta_1(x,t)$, and it is divided into two regions $S(t;\eta) = \{(x,y) : -h < y < \eta(x,t)\}$ and $S_1(t;\eta,\eta_1) = \{(x,y) : \eta(x,t) < y < h_1 + \eta_1(x,t)\}$ by the interface $\{y = \eta(x,t)\}$. The two regions are occupied by two immiscible fluids, with ρ the density of the lower fluid and ρ_1 the density of the upper fluid. The system is in a stable configuration, in that $\rho > \rho_1$. In such a configuration, the fluid motion is assumed to be potential flow, namely in Eulerian coordinates the velocity is given by a potential in each fluid region, $\mathbf{u}(x,y,t) = \nabla\varphi(x,y,t)$ in $S(t;\eta)$, and $\mathbf{u}_1(x,y,t) = \nabla\varphi_1(x,y,t)$ in $S_1(t;\eta,\eta_1)$, where the two potential functions satisfy

$$\begin{aligned} \Delta\varphi &= 0\,, \quad \text{in the domain } S(t;\eta) \\ \Delta\varphi_1 &= 0\,, \quad \text{in the domain } S_1(t;\eta,\eta_1)\,. \end{aligned} \tag{2.1}$$

The boundary conditions on the fixed bottom $\{y = -h\}$ of the lower fluid are that

$$\nabla\varphi \cdot N_0(x,-h) = -\partial_y\varphi(x,-h) = 0\,, \tag{2.2}$$

where N_0 is the exterior unit normal, enforcing that there is no fluid flux across the boundary.

On the interface $\{(x,y) : y = \eta(x,t)\}$ it is natural to impose three boundary conditions, two kinematic conditions which are essentially geometrical, and a physical condition of force balance. The kinematical conditions assume that there is no cavitation in the interface between the fluids, and therefore the function $\eta(x,t)$ whose graph defines the interface satisfies simultaneously

$$\partial_t \eta = \partial_y \varphi - \partial_x \eta \, \partial_x \varphi = \nabla\varphi \cdot N (1 + |\partial_x \eta|^2)^{1/2} \tag{2.3}$$

where N is the unit exterior normal on the interface for the lower domain, and

$$\partial_t \eta = \partial_y \varphi_1 - \partial_x \eta \, \partial_x \varphi_1 = -\nabla\varphi_1 \cdot (-N)(1 + |\partial_x \eta|^2)^{1/2} \ . \tag{2.4}$$

The third boundary condition imposed on the interface is the Bernoulli condition, which states that

$$\rho(\partial_t \varphi + \tfrac{1}{2}|\nabla\varphi|^2 + g\eta) = \rho_1(\partial_t \varphi_1 + \tfrac{1}{2}|\nabla\varphi_1|^2 + g\eta) \ . \tag{2.5}$$

Finally, in assigning boundary conditions for the upper boundary in the problem, we are interested in considering two situations. The first is where $\eta_1 = 0$ and the top surface is considered a solid boundary (a rigid lid). In this case the boundary condition

$$\nabla\varphi_1 \cdot N_1(x, h_1) = \partial_y \varphi_1(x, h_1) = 0 \tag{2.6}$$

is appropriate, where N_1 is the unit exterior normal to the upper fixed surface. The problem is therefore to find the evolution of a single free interface $\{(x, \eta(x,t))\}$. We allow $0 < h, h_1 \leq +\infty$, and either h or h_1 or both are specifically allowed to be infinite.

The second situation that we consider is where the top surface is itself a free surface $\{(x,y) : y = h_1 + \eta_1(x,t)\}$, on which the velocity potential φ_1 and the function η_1 satisfy a surface kinematic condition

$$\partial_t \eta_1 = \partial_y \varphi_1 - \partial_x \eta_1 \, \partial_x \varphi_1 = \nabla\varphi_1 \cdot N_1 (1 + |\partial_x \eta_1|^2)^{1/2} \tag{2.7}$$

and a Bernoulli condition

$$\partial_t \varphi_1 + \tfrac{1}{2}|\nabla\varphi_1|^2 + g\eta_1 = 0 \ . \tag{2.8}$$

The problem then is to describe the simultaneous evolution of the free surface $\{(x, h_1 + \eta_1(x,t))\}$ and the free interface $\{(x, \eta(x,t))\}$.

2.2 Lagrangian for free interfaces

It is straightforward to derive useful expressions for the kinetic energy and the potential energy for the first system above, consisting of one free interface separating two otherwise confined fluid regions. From these one can pose a Lagrangian for the system. In an analogy with classical mechanics the Hamiltonian for the system and the form of the canonically conjugate variables can be derived. In this way we deduce from the 'first principles' of mechanics the form of the canonical variables that were originally given in Benjamin and Bridges [5].

The kinetic energy is given by the weighted sum of Dirichlet integrals of the two velocity potentials,

$$K = \tfrac{1}{2} \int_{\mathbb{R}} \int_{-h}^{\eta(x)} \rho |\nabla\varphi(x,y)|^2 \, dydx + \tfrac{1}{2} \int_{\mathbb{R}} \int_{\eta(x)}^{h_1} \rho_1 |\nabla\varphi_1(x,y)|^2 \, dydx \ , \tag{2.9}$$

and the potential energy is

$$
\begin{aligned}
V &= \int_{\mathbb{R}} \int_{-h}^{\eta(x)} g\rho y\, dy dx + \int_{\mathbb{R}} \int_{\eta(x)}^{h_1} g\rho_1 y\, dy dx \\
&= \tfrac{1}{2} \int_{\mathbb{R}} g\rho \eta^2(x)\, dx - \tfrac{1}{2} \int_{\mathbb{R}} g\rho_1 \eta^2(x)\, dx + C\ . \qquad (2.10)
\end{aligned}
$$

The constant term C is superfluous to the dynamics, and it is able to be normalized to zero. Following the analogy with mechanics, the Lagrangian of the system is given by

$$L = K - V\ .$$

To place the kinetic energy in a more convenient expression for analysis, we introduce the Dirichlet-Neumann operators for the two fluid domains. Let N be the unit exterior normal to the lower fluid domain $S(\eta)$ along the free interface. Given $\Phi(x) = \varphi(x,\eta(x))$ and $\Phi_1(x) = \varphi_1(x,\eta(x))$ the boundary values of the two velocity potentials on the free interface $\{(x,\eta(x,t))\}$, and following Craig and Sulem [15], define the operators

$$G(\eta)\Phi = \nabla\varphi \cdot N(1+|\partial_x\eta|^2)^{1/2}\ , \qquad (2.11)$$

which is the Dirichlet-Neumann operator for the fluid domain $S(\eta)$, and

$$G_1(\eta)\Phi_1 = -\nabla\varphi_1 \cdot N(1+|\partial_x\eta|^2)^{1/2}\ , \qquad (2.12)$$

the Dirichlet-Neumann operator for the fluid domain $S_1(\eta)$. These operators are linear in the quantities Φ and Φ_1, however they are nonlinear and reasonably complicated in their dependence on $\eta(x)$ which determines the two fluid domains. Using Green's identities, the kinetic energy (2.9) can be rewritten as

$$K = \tfrac{1}{2} \int_{\mathbb{R}} \rho \Phi G(\eta)\Phi\, dx + \tfrac{1}{2} \int_{\mathbb{R}} \rho_1 \Phi_1 G_1(\eta)\Phi_1\, dx\ . \qquad (2.13)$$

Under the conditions of no cavitation at the interface, the kinetic boundary conditions (2.3)(2.4) read

$$\partial_t \eta = G(\eta)\Phi = -G_1(\eta)\Phi_1\ . \qquad (2.14)$$

Solving (2.14) for $\Phi(x) = G^{-1}(\eta)\dot\eta(x)$ and $\Phi_1(x) = -G_1^{-1}(\eta)\dot\eta(x)$ and substituting into the quantity (2.13) one obtains a reasonable expression for the Lagrangian

$$L(\eta,\dot\eta) = \tfrac{1}{2} \int_{\mathbb{R}} \rho\dot\eta G^{-1}(\eta)\dot\eta + \rho_1\dot\eta G_1^{-1}(\eta)\dot\eta\, dx - \tfrac{1}{2} \int_{\mathbb{R}} g(\rho-\rho_1)\eta^2(x)\, dx\ .$$

From this Lagrangian, which depends upon $(\eta,\dot\eta)$, we are in a position to deduce from the principles of classical mechanics the Hamiltonian and the canonically conjugate variables with respect to which the system (2.1)–(2.6) is formally a Hamiltonian dynamical system. Namely, we define

$$
\begin{aligned}
\xi(x) &= \delta_{\dot\eta} L = \rho G^{-1}(\eta)\dot\eta + \rho_1 G_1^{-1}\dot\eta \\
&= \rho\Phi(x) - \rho_1\Phi_1(x)\ , \qquad (2.15)
\end{aligned}
$$

which is precisely the expression of Benjamin and Bridges [5] for the variable conjugate to $\eta(x)$.

The Hamiltonian for the system is given by $K+V$ since L is a quadratic form in $\dot{\eta}$. Using (2.14) and (2.15), one finds that $(\rho_1 G(\eta)+\rho G_1(\eta))\Phi = G_1(\eta)\xi$ and $(\rho G_1(\eta)+\rho_1 G(\eta))\Phi_1 = -G(\eta)\xi$, whereupon the Hamiltonian can be written

$$H(\eta,\xi) = \tfrac{1}{2}\int_{\mathbb{R}} \xi G_1(\eta)(\rho_1 G(\eta)+\rho G_1(\eta))^{-1} G(\eta)\xi\, dx + \tfrac{1}{2}\int_{\mathbb{R}} g(\rho-\rho_1)\eta^2\, dx\ . \tag{2.16}$$

This expression for the Hamiltonian has appeared in [10]. The system of equations of motion for the interface takes the form of a classical Hamiltonian system, namely

$$\partial_t \eta = \delta_\xi H\ , \qquad \partial_t \xi = -\delta_\eta H\ , \tag{2.17}$$

which is equivalent to (2.1) subject to the boundary conditions (2.2)(2.6) and the free interface conditions (2.3)(2.4)(2.5).

We note that by setting $\rho_1 = 0$ the expressions (2.10) and (2.13) reduce to the ones for a single free surface alone, the canonical conjugate variables (2.15) state that $\xi(x) = \rho\Phi(x)$ which is precisely the choice of Zakharov in [30], and the sum $K+V$ is the Hamiltonian for the system given in [30].

2.3 Lagrangian for free surfaces and interfaces

In the second situation described above, the system of interest involves the coupled evolution of the free interface and a free surface lying over the upper fluid. This problem can also be described in terms of a Lagrangian, which will depend upon both the deformations $\eta_1(x,t)$ of the free surface, as well as those of the free interface $\eta(x,t)$. Again the 'first principles' of mechanics can be cited in deriving the natural canonically conjugate variables for a Hamiltonian description of the problem, and for a convenient expression for the Hamiltonian function. This choice of variables has been previously given by Ambrosi [1], however the form of the Hamiltonian is to our knowledge new.

As in the first case, the kinetic energy is again given as a weighted sum of the Dirichlet integrals of the two velocity potentials, namely

$$K = \tfrac{1}{2}\int_{\mathbb{R}}\int_{-h}^{\eta(x)} \rho|\nabla\varphi(x,y)|^2\, dydx + \tfrac{1}{2}\int_{\mathbb{R}}\int_{\eta(x)}^{h_1+\eta_1(x)} \rho_1|\nabla\varphi_1(x,y)|^2\, dydx\ . \tag{2.18}$$

In a manner similar to (2.10), the potential energy is

$$V = \tfrac{1}{2}\int_{\mathbb{R}} g(\rho-\rho_1)\eta^2(x)\, dx + \tfrac{1}{2}\int_{\mathbb{R}} g\rho_1\eta_1^2(x) + 2g\rho_1 h_1\eta_1(x)\, dx + C\ , \tag{2.19}$$

where again we may take $C = 0$. The analogy with mechanics implies that the Lagrangian of the system is given by

$$L = K - V\ .$$

Following (2.13), we express the Dirichlet integrals in terms of the boundary values for the two velocity potentials and the Dirichlet-Neumann operators for the two fluid domains. We define the quantities $\Phi(x) = \varphi(x, \eta(x))$, $\Phi_1(x) = \varphi_1(x, \eta(x))$ as above, and $\Phi_2(x) = \varphi_1(x, h_1 + \eta_1(x))$ on the free surface. The Dirichlet-Neumann operator for the lower domain is the same as in the first case, namely $G(\eta)\Phi(x) = \nabla\varphi \cdot N(1 + (\partial_x \eta)^2)^{1/2}$. For the upper fluid domain $S_1(\eta, \eta_1)$ both $\Phi_1(x)$ and $\Phi_2(x)$ contribute to the exterior unit normal derivative of φ_1 on each boundary. That is, the Dirichlet-Neumann operator is a matrix operator which takes the form

(2.20)
$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix} = \begin{pmatrix} -(\nabla\varphi_1 \cdot N)(x, \eta(x))(1 + (\partial_x \eta(x))^2)^{1/2} \\ (\nabla\varphi_1 \cdot N_1)(x, h_1 + \eta_1(x))(1 + (\partial_x \eta_1(x))^2)^{1/2} \end{pmatrix} .$$

Using Green's identities, and expressing the normal derivatives of the velocity potentials on the boundaries in terms of Dirichlet-Neumann operators, the kinetic energy takes the form

$$K = \tfrac{1}{2} \int_{\mathbb{R}} \rho \Phi G(\eta) \Phi \, dx + \tfrac{1}{2} \int_{\mathbb{R}} \rho_1 \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}^T \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} dx \tag{2.21}$$

In terms of the Dirichlet-Neumann operators (2.11)(2.20), the kinematic boundary condition (2.14) for $\Phi(x)$ is unchanged, while (2.4) and (2.7) become

$$\begin{aligned} \dot{\eta} &= -(G_{11}\Phi_1 + G_{12}\Phi_2) \\ \dot{\eta}_1 &= G_{21}\Phi_1 + G_{22}\Phi_2 \, . \end{aligned} \tag{2.22}$$

Using (2.14) and (2.22) we rewrite the kinetic energy in terms of the variables $(\eta, \eta_1, \dot{\eta}, \dot{\eta}_1)$, giving the following expression for the Lagrangian for the free surface/free interface problem;

$$\begin{aligned} L = {} & \tfrac{1}{2} \int_{\mathbb{R}} \rho \dot{\eta} G^{-1}(\eta) \dot{\eta} \, dx + \tfrac{1}{2} \int_{\mathbb{R}} \rho_1 \begin{pmatrix} -\dot{\eta} \\ \dot{\eta}_1 \end{pmatrix}^T \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\eta} \\ \dot{\eta}_1 \end{pmatrix} dx \\ & - \tfrac{1}{2} \int_{\mathbb{R}} g(\rho - \rho_1) \eta^2(x) \, dx - \tfrac{1}{2} \int_{\mathbb{R}} g \rho_1 (h_1 + \eta_1)^2(x) \, dx \, . \end{aligned} \tag{2.23}$$

In these terms we are able to deduce from 'first principles' the appropriate canonically conjugate variables for the problem, namely

$$\begin{aligned} \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix} &= \begin{pmatrix} \delta_{\dot{\eta}} L \\ \delta_{\dot{\eta}_1} L \end{pmatrix} = \rho \begin{pmatrix} G^{-1}(\eta) \dot{\eta} \\ 0 \end{pmatrix} + \rho_1 \begin{pmatrix} G_{11} & -G_{12} \\ -G_{21} & G_{22} \end{pmatrix}^{-1} \begin{pmatrix} \dot{\eta} \\ \dot{\eta}_1 \end{pmatrix} \\ &= \begin{pmatrix} \rho\Phi - \rho_1 \Phi_1 \\ \rho_1 \Phi_2 \end{pmatrix} . \end{aligned} \tag{2.24}$$

The expression (2.24) also appears in [1]. Using (2.24), the kinetic energy (2.21) has the form

$$\begin{aligned} K &= \tfrac{1}{2} \int_{\mathbb{R}} \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix}^T \begin{pmatrix} \dot{\eta} \\ \dot{\eta}_1 \end{pmatrix} dx \\ &= \tfrac{1}{2} \int_{\mathbb{R}} \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix}^T \begin{pmatrix} -G_{11} & -G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} dx \, . \end{aligned} \tag{2.25}$$

Solving (2.14)(2.24) for (Φ,Φ_1,Φ_2) in terms of (ξ,ξ_1), and defining $\rho G_{11}+\rho_1 G(\eta) = B$, we have

$$\Phi = B^{-1}(G_{11}\xi - G_{12}\xi_1) \tag{2.26}$$

$$\Phi_1 = B^{-1}\left(-G(\eta)\xi - \frac{\rho}{\rho_1}G_{12}\xi_1\right) \tag{2.27}$$

$$\rho_1\Phi_2 = \xi_1 \ , \tag{2.28}$$

and (2.25) can be written as

$$K = \tfrac{1}{2}\int_{\mathbb{R}} \begin{pmatrix}\xi\\ \xi_1\end{pmatrix}^T \begin{pmatrix} G_{11}B^{-1}G(\eta) & -G(\eta)B^{-1}G_{12} \\ -G_{21}B^{-1}G(\eta) & \frac{1}{\rho_1}G_{22} - \frac{\rho}{\rho_1}G_{21}B^{-1}G_{12}\end{pmatrix} \begin{pmatrix}\xi\\ \xi_1\end{pmatrix} dx \ . \tag{2.29}$$

The Hamiltonian for the free surface and free interface problem is $H = K+V$ where $K = K(\eta,\eta_1,\xi,\xi_1)$ is given by (2.29) and the potential energy $V = V(\eta,\eta_1)$ is simply (2.19). This expression corrects [1] in giving the full coupling in the kinetic energy between the variables ξ and ξ_1. Hamilton's equations of motion take the form

$$\partial_t\eta = \delta_\xi H \ , \qquad \partial_t\xi = -\delta_\eta H \ , \tag{2.30}$$

and

$$\partial_t\eta_1 = \delta_{\xi_1} H \ , \qquad \partial_t\xi_1 = -\delta_{\eta_1} H \ , \tag{2.31}$$

for the interface and free surface respectively.

3 Hamiltonian perturbation theory

Our approach to the systematic derivations of the long wave limiting equations is from the point of view of Hamiltonian perturbation theory, in which the Hamiltonian is a function of a small parameter ε. The approximating equations are also Hamiltonian systems, obtained by retaining a finite number of terms in the Taylor expansion in ε of the Hamiltonian. Namely, we are considering systems of differential equations which appear in the Hamiltonian form

$$\partial_t v = J\delta_v H \ , \tag{3.1}$$

where $H : X \to \mathbb{R}$ is the Hamiltonian defined on a phase space X of functions, and $J\delta_v H$ is the Hamiltonian vector field on X. For the problem of a free interface we will introduce the Hamiltonian $H = H(v,\varepsilon)$ depending on the variables $v = (\eta,\xi)$. For the problem of a free surface and a free interface, the phase space variables will be $v = (\eta,\eta_1,\xi,\xi_1)$. The topology of the function space X will not be specified precisely in the present work, because of the relatively formal nature of the task at hand. The small parameter ε will be introduced through choices of scaling of the independent variables x and the dependent variables v, corresponding to the scaling regimes of interest in these long wave problems. We will consider a variety of scaling regimes, corresponding firstly to the shallow water limits (and their thin layer analogues), and secondly to the Boussinesq and KdV scaling regimes, in which dispersive and nonlinear effects are both brought into play. The parameter ε enters in different ways in the different regimes, however the systematic point of view is retained throughout the asymptotic procedure.

The Taylor expansion of H in ε is denoted

$$H(v,\varepsilon) = H^{(0)}(v) + \varepsilon H^{(1)}(v) + \cdots = \sum_{j=0}^{\infty} \varepsilon^j H^{(j)}(v) \ . \tag{3.2}$$

All of our candidate systems of equations for long wave approximations will be Hamiltonian systems in their own right, in the form (3.1), with a Hamiltonian $H_m(v)$ obtained from systematically truncating the Taylor series of $H(v,\varepsilon)$,

$$\partial_t v = J\delta_v H_m \ , \quad H_m = H^{(0)}(v) + \varepsilon H^{(1)}(v) + \cdots + \varepsilon^m H^{(m)}(v) \ . \tag{3.3}$$

The operator J will be different in different settings, but in the present work it will be independent of v and always homogeneous in ε, unlike in certain cases considered by Olver [25] in which the operator J is nontrivially v and ε dependent.

3.1 The calculus of transformations

In the broad picture, our phase space X is a function space endowed with a *symplectic two-form* $\omega : T(X) \times T(X) \to \mathbb{R}$. Given a Hamiltonian function $H : X \to \mathbb{R}$, the Hamiltonian vector field X_H is given through the classical relationship

$$dH(V) = \omega(V, X_H) \ , \qquad \text{for all} \quad V \in T(X) \ . \tag{3.4}$$

In case X has a metric given by an inner product $(\cdot,\cdot)$, such as when it is a Hilbert space, the symplectic form can be represented as

$$\omega(V_1,V_2) = (V_1, J^{-1}V_2) \ , \tag{3.5}$$

where J is called the *structure map* or the *(co-)symplectic operator*. We are assuming that J is skew-symmetric and nondegenerate, although sometimes in practice it will be the case that it possesses a small dimensional null space associated with the constant functions. The inner product is also used to define gradients of functions, namely

$$dH(V) = (\delta H, V) \ , \tag{3.6}$$

where we denote $\mathrm{grad} H = \delta H$. In this setting the Hamiltonian vector field X_H is given by the expression

$$X_H = J\delta_v H \ , \tag{3.7}$$

as can be seen from (3.4)(3.5) and (3.6). Denote by $\Phi_H(v,t)$ the *flow* of the resulting Hamiltonian system

$$\partial_t v = J\delta_v H \ , \quad v(0) = v_0 \ . \tag{3.8}$$

Now consider two phase spaces X_1, X_2, with a symplectic form on X_1 given by J_1. Suppose that $H_1 : X_1 \to \mathbb{R}$ is a Hamiltonian on X_1. Given a transformation $f : X_1 \to X_2$, which we denote by $w = f(v)$, with $v \in X_1$, $w \in X_2$, define a Hamiltonian on X_2 by $H_2(w) = H_2(f(v)) = H_1(v)$. The Hamiltonian vector field $\delta_v H_1$ on X_1 is transformed to a vector field on X_2 which is expressed by

$$\partial_t w = \partial_v f J_1 (\partial_v f)^T \delta_w H_2(w) \; . \tag{3.9}$$

That is, a transformation f will induce a symplectic structure on X_2, given by the structure map $J = \partial_v f J_1 (\partial_v f)^T$, and the transformed vector field $J\delta_w H_2(w)$ is Hamiltonian in the phase space X_2.

When the phase space X_2 already has a symplectic structure J_2, and the transformation f is such that

$$\partial_v f J_1 (\partial_v f)^T = J_2 \tag{3.10}$$

then it is called a *canonical* transformation of X_1 to X_2. This is the case in particular when $X_1 = X_2$, and this class of canonical transformation plays a special rôle in the subject.

3.2 Examples of transformations

The elementary transformations that we will use repeatedly in this paper consist of spatial scalings, scalings of the dependent variables (amplitude scaling), translations to a moving frame, and changing coordinates in the description of surface and interface wave motion from elevation-potential variables to elevation-velocity variables. Our phase space will be $v \in L^2(\mathbb{R})$ (or an appropriate linear subspace consisting of sufficiently smooth functions), with a metric given by the usual inner product; namely in the case of free interface motion we have $v_j = (\eta_j, \xi_j) \in T(L^2(\mathbb{R}))$, $j = 1, 2$ for which

$$(v_1, v_2) = \int_{\mathbb{R}} \eta_1 \eta_2 + \xi_1 \xi_2 \, dx \; .$$

The problem (2.17) is given in Darboux coordinates, which is to say that the symplectic form is represented by the matrix operator

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \; .$$

The case of the coupled free surface and free interface is similar.

Amplitude scaling: Given $v = (\eta, \xi)$, consider the scaling $w = (\eta', \xi') = (\alpha\eta, \beta\xi) = f(\eta, \xi)$ for $\alpha, \beta \in \mathbb{R}$, which we view as a particularly simple coordinate transformation. The Jacobian of the transformation is

$$\partial_v f = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix}$$

and therefore the new symplectic form induced by the transformation is given by

$$J_2 = \partial_v f J (\partial_v f)^T = \alpha\beta J \; .$$

The scaling transformation is canonical only when $\alpha = \beta^{-1}$. However for a general choice of scalars α and β, the resulting modification from J to J_2 can be reversed by a simple time change $t' = \alpha\beta t$.

These amplitude scaling transformations introduce the small parameter ε into the Hamiltonian principally through their effect on the various Dirichlet-Neumann operators found in the problem. In fact it is known that the Dirichlet-Neumann operator for $S(\eta)$ is analytic in its dependence on $\eta \in \mathrm{Lip}(\mathbb{R})$ (see Coifman and Meyer [8] for the result in two dimensions, and Craig, Schanz and Sulem [14] for the higher dimensional case). In practice these facts imply that the operators $G(\eta)$, $G_1(\eta)$ and $G_{j\ell}(\eta,\eta_1)$, which appear in the expressions for the Dirichlet integral in the various Hamiltonians of this paper, can be written in terms of convergent Taylor series expansions in η;

$$G(\eta)\xi = \sum_{j=0}^{\infty} G^{(j)}(\eta)\xi \ . \tag{3.11}$$

and similarly for (η,η_1). Recursion relations for the Taylor polynomials $G^{(j)}(\eta)$ of the various Dirichlet-Neumann operators that appear in this paper are derived in the appendix. These polynomials $G^{(j)}(\eta)$ are homogeneous of degree j in η, so that the scaling transformation $w = f(\eta,\xi) = (\alpha\eta,\beta\xi) = (\eta',\xi')$ has the effect

$$G(\eta')\xi' = \sum_{j=0}^{\infty} \beta\alpha^j G^{(j)}(\eta)\xi \ . \tag{3.12}$$

Typically α and β are taken to be powers of the scaling parameter ε, introducing this parameter into the transformed Hamiltonian.

Surface elevation - velocity coordinates: It is common to write the equations of motion in the fluid dynamics of free boundaries in terms of the variables (η,u), where $\eta(x)$ is the elevation of the free surface or free interface, and $u = \partial_x\xi$ is proportional to the velocity of the fluid tangential to the interface. As a transformation $v = (\eta,\xi) \mapsto w = (\eta,u) = f(v)$, the Jacobian is given by

$$\partial_v f = \begin{pmatrix} I & 0 \\ 0 & \partial_x \end{pmatrix}$$

whereupon the induced symplectic form is represented by

$$J_2 = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix} \ . \tag{3.13}$$

This representation of a non-classical symplectic form commonly occurs when describing the Boussinesq system, for example.

Spatial scaling: Long wave theory is based on asymptotic expansions in which the small parameter is introduced through scaling of the spatial variables, namely $x \to \varepsilon x$. The resulting transformation of phase space X is that $v(x) \to w(x) = v(x/\varepsilon) = f(v)(x)$, with the Jacobian $\partial_v f(v)$ described best by its action on a vector field $V(x) \in T(X)$;

$$(\partial_v f(v))V(x) = \frac{d}{d\tau}\Big|_{\tau=0} f(v+\tau V) = \frac{d}{d\tau}\Big|_{\tau=0} \big(v(x/\varepsilon) + \tau V(x/\varepsilon)\big) = V(x/\varepsilon) \ .$$

The transpose $\partial_v f^T$ is expressed via the following computation

$$(V_1, \partial_v f V_2) = \int_{\mathbb{R}} V_1(x) V_2(x/\varepsilon)\, dx = \int_{\mathbb{R}} V_1(\varepsilon x) V_2(x)\, \varepsilon dx = (\partial_v f^T V_1, V_2) \;, \tag{3.14}$$

therefore $(\partial_v f)^T V(x) = \varepsilon V(\varepsilon x)$. The resulting induced symplectic form is represented by

$$J_2 = \partial_v f J \partial_v f^T = \varepsilon J \;. \tag{3.15}$$

This again recovers the original symplectic form, modulo a time change $\tau = \varepsilon t$.

In practice we find that the principal way in which a spatial scaling transformation introduces the parameter ε into the Hamiltonian is through its effect on Fourier multiplier operators. Indeed Fourier multipliers form an important component of our description of the Dirichlet-Neumann operator, and it is important to express the effect of spatial scaling in convenient form.

Lemma 3.1. *Let $f(v(x)) = v(x/\varepsilon) = w(x)$ be the transformation on X given by scaling of the spatial variables. Let $m(D)$ be a Fourier multiplication operator, defined by*

$$(m(D)v)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ik(x-x')} m(k) v(x')\, dx' dk \;. \tag{3.16}$$

Then the transformed Fourier multiplication operator is

$$f(m(D)v)(x) = (m(\varepsilon D) f(v))(x) \;. \tag{3.17}$$

Proof. Using the expression (3.16) for the Fourier multiplier, one has

$$\begin{aligned}
f(m(D)v)(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ik(x/\varepsilon - X')} m(k) v(X')\, dX' dk \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ik(x-x')/\varepsilon} m(k) v(x'/\varepsilon)\, \frac{dx' dk}{\varepsilon} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{iK(x-x')} m(\varepsilon K) v(x'/\varepsilon)\, dx' dK = m(\varepsilon D) f(v)(x) \qquad (3.18)
\end{aligned}$$

□

Moving reference frame: A transformation that is commonly employed in studying long wave limits in the fluid dynamics of free surfaces, is to change to a moving coordinate frame. In particular when the longest wavelength linear solutions have speed c_0, one introduces new variables $v'(x,t) = v(x - c_0 t, t)$, and transforms the governing partial differential equation accordingly. However the time variable t is distinguished in our point of view of systems of partial differential equations as Hamiltonian systems, so at first consideration this transformation, which mixes time and spatial variables, is not accommodated in the present picture. The substitute is to add a multiple of the momentum integral to the Hamiltonian. That is, the momentum for the free interface problem is

$$I(\eta, \xi) = \int_{\mathbb{R}} \xi \partial_x \eta\, dx \;, \tag{3.19}$$

whose Hamiltonian flow is simple constant speed translation,

$$\partial_t \eta = \partial_x \eta \ , \ \ \partial_t \xi = \partial_x \xi \ , \quad \Phi_I(t,\eta,\xi)(x) = (\eta(x+t), \xi(x+t)) \ . \tag{3.20}$$

Furthermore, the Hamiltonian equations that we consider are all of constant coefficients, implying that the flow conserves momentum, in other words the Hamiltonian H and the momentum integral I are Poisson commuting quantities;

$$\{H,I\} = \int_{\mathbb{R}} \delta_v H J \delta_v I \, dx = 0 \ .$$

Therefore their flows commute, $\Phi_I \circ \Phi_H = \Phi_H \circ \Phi_I$. Since the solution $v(x,t) = \Phi_H(t,v)(x)$ for fixed x represents observations at a point which is stationary in space, the quantity $\Phi_H \circ \Phi_{-c_0 I}(t,v)(x)$ represents observations in a reference frame moving with speed c_0. Because the flows for the momentum and the Hamiltonian are commuting, $\Phi_H \circ \Phi_{-c_0 I}(t,v)(x) = \Phi_{H-c_0 I}(t,v)$, which is to say that the flow of the Hamiltonian vector field for $H - c_0 I$ corresponds to the flow of the Hamiltonian vector field of H alone, observed in a reference frame traveling at velocity c_0.

Characteristic coordinates: It is common in the long-wave scaling regime for a Hamiltonian PDE to have the quadratic part of its Hamiltonian in the form

$$H^{(2)} = \tfrac{1}{2} \int_{\mathbb{R}} Au^2 + B\eta^2 \, dx \ , \tag{3.21}$$

when given in elevation - velocity coordinates. Both A and B are positive constants. The resulting Hamilton's equations for $H^{(2)}$ are linear wave equations

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} = \begin{pmatrix} 0 & -A \\ -B & 0 \end{pmatrix} \begin{pmatrix} \partial_x \eta \\ \partial_x u \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix} \delta H^{(2)} \ . \tag{3.22}$$

A transformation to characteristic coordinates

$$\begin{pmatrix} r \\ s \end{pmatrix} = F \begin{pmatrix} \eta \\ u \end{pmatrix} \tag{3.23}$$

is designed to accomplish three things. The first is to transform the hyperbolic system of equations (3.22) to the characteristic form

$$\partial_t \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} -C & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} \partial_x r \\ \partial_x s \end{pmatrix} \tag{3.24}$$

where $C = \sqrt{AB}$. The second is to transform the symplectic form so that the original structure map (3.13) becomes

$$J = \begin{pmatrix} -\partial_x & 0 \\ 0 & \partial_x \end{pmatrix} = F \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix} F^T \ . \tag{3.25}$$

The third desired property is to transform the Hamiltonian to the normal form

$$H^{(2)}(r,s) = \tfrac{1}{2} \int_{\mathbb{R}} \sqrt{AB}(r^2 + s^2) \, dx \ . \tag{3.26}$$

Clearly property three is the result of the first two. All three are accomplished by the transformation given by

$$F = \begin{pmatrix} \sqrt[4]{\frac{B}{4A}} & \sqrt[4]{\frac{A}{4B}} \\ \sqrt[4]{\frac{B}{4A}} & -\sqrt[4]{\frac{A}{4B}} \end{pmatrix} . \tag{3.27}$$

Normal mode decomposition: A basic theorem in mechanics states that a harmonic oscillator in n degrees of freedom can be transformed to a set of n decoupled oscillators, the normal modes of the system. In the long-wave regime for the free surface-free interface problem, the system is coupled at principal order in the quadratic part of the Hamiltonian. It is thus natural to use a normal mode decomposition. Consider the quadratic Hamiltonian form

$$H^{(2)} = \tfrac{1}{2}\int_{\mathbb{R}} \begin{pmatrix} \eta \\ \eta_1 \end{pmatrix}^T \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \eta \\ \eta_1 \end{pmatrix} + \begin{pmatrix} u \\ u_1 \end{pmatrix}^T \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} u \\ u_1 \end{pmatrix} dx , \tag{3.28}$$

where A, B and C are positive constants. The corresponding Hamilton's equations can be written as

$$\partial_t \begin{pmatrix} \eta \\ \eta_1 \\ u \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\partial_x & 0 \\ 0 & 0 & 0 & -\partial_x \\ -\partial_x & 0 & 0 & 0 \\ 0 & -\partial_x & 0 & 0 \end{pmatrix} \delta H^{(2)} . \tag{3.29}$$

Since the quadratic form in (u, u_1) in (3.28) is symmetric, the transformation to normal modes

$$\begin{pmatrix} \mu \\ \mu_1 \end{pmatrix} = R \begin{pmatrix} \eta \\ \eta_1 \end{pmatrix} , \quad \begin{pmatrix} v \\ v_1 \end{pmatrix} = R \begin{pmatrix} u \\ u_1 \end{pmatrix} , \tag{3.30}$$

where

$$R = \begin{pmatrix} a^- & b^- \\ a^+ & b^+ \end{pmatrix} = \left(R^T\right)^{-1} , \tag{3.31}$$

is a rotation. Setting

$$a^{\pm} = \left(2 + \frac{\theta^2}{2} \pm \frac{\theta}{2}\sqrt{1+\theta^2}\right)^{-1/2} ,$$

$$b^{\pm} = \frac{1}{2}\left(\theta \pm \sqrt{4+\theta^2}\right)\left(2 + \frac{\theta^2}{2} \pm \frac{\theta}{2}\sqrt{1+\theta^2}\right)^{-1/2} , \qquad \theta = \frac{C-A}{B} , \tag{3.32}$$

the result is the following new form for the principal quadratic part of the Hamiltonian

$$H^{(2)}(\mu, \mu_1, v, v_1) = \tfrac{1}{2}\int_{\mathbb{R}} \mu^2 + (c_0^-)^2 v^2 + \mu_1^2 + (c_0^+)^2 v_1^2 \, dx , \tag{3.33}$$

with

$$(c_0^\pm)^2 = \tfrac{1}{2}\left(A + C \pm \sqrt{(A-C)^2 + 4B^2}\right) . \tag{3.34}$$

Of course, the higher order terms of the Hamiltonian are transformed as well. The structure map is invariant under this transformation and the evolution equations for the normal modes are simply given by

$$\partial_t \begin{pmatrix} \mu \\ \mu_1 \\ \nu \\ \nu_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\partial_x & 0 \\ 0 & 0 & 0 & -\partial_x \\ -\partial_x & 0 & 0 & 0 \\ 0 & -\partial_x & 0 & 0 \end{pmatrix} \delta H^{(2)}(\mu, \mu_1, \nu, \nu_1) . \tag{3.35}$$

4 The linearized equations

A thorough understanding of the evolution of waves in a nonlinear system initially entails studying the equations linearized about an equilibrium solution. In our cases at hand, the equilibrium solution is simply the fluid at rest, thus $\delta_v H(0) = 0$. An elegant way to derive the linearized equations at a stationary point of a Hamiltonian system is to truncate the Taylor expansion of the Hamiltonian function at its quadratic term. We obtain the linearized free interface equations and the linearized system of free surface and free interface equations in precisely this manner, using the expressions for their respective Hamiltonians that were obtained in Section 2.

4.1 Linear free interfaces

Restricting to the quadratic part of the Hamiltonian (2.16), one obtains

$$H = \frac{1}{2}\int_{\mathbb{R}} \xi \frac{D\tanh(hD)\tanh(h_1 D)}{\rho\tanh(h_1 D) + \rho_1 \tanh(hD)} \xi + g(\rho - \rho_1)\eta^2 \, dx . \tag{4.1}$$

The linearized form of (2.17) then reads

$$\begin{aligned} \partial_t \eta &= \delta_\xi H = \frac{D\tanh(hD)\tanh(h_1 D)}{\rho\tanh(h_1 D) + \rho_1\tanh(hD)} \xi , \\ \partial_t \xi &= -\delta_\eta H = -g(\rho - \rho_1)\eta . \end{aligned} \tag{4.2}$$

The corresponding dispersion relation giving the wave frequency $\omega(k)$ as a function of the wavenumber k is

$$\omega^2 = \frac{g(\rho-\rho_1)k\tanh(kh)\tanh(kh_1)}{\rho\tanh(kh_1) + \rho_1\tanh(kh)} . \tag{4.3}$$

Equivalently, it can be stated in terms of the phase velocity of a single Fourier mode

$$c = \frac{\omega}{k} = \sqrt{\frac{g(\rho-\rho_1)\tanh(kh)\tanh(kh_1)}{k(\rho\tanh(kh_1) + \rho_1\tanh(kh))}} . \tag{4.4}$$

In the long-wave regime, we can distinguish three different situations giving rise to characteristic asymptotics for the phase speed (4.4); the first being where both $kh \to 0$ and $kh_1 \to 0$ (two finite layers), with the ratio h_1/h fixed,

$$c^2 \simeq c_0^2 = \frac{g(\rho - \rho_1)}{\rho/h + \rho_1/h_1} \ . \tag{4.5}$$

The second is where $kh > O(1)$ (deep lower layer) while $kh_1 \to 0$ (finite upper layer) (or the reverse situation in which $kh \to 0$ while $kh_1 > O(1)$). Then

$$c^2 \simeq c_0^2 = g\frac{\rho - \rho_1}{\rho_1/h_1} \tag{4.6}$$

(respectively, $c_0^2 = g(\rho-\rho_1)/(\rho/h)$). The third situation occurs for two deep layers separated by the free interface. Letting $k \to 0$ while both kh and $kh_1 > O(1)$, one finds

$$\omega_0^2 = \frac{g(\rho - \rho_1)}{\rho + \rho_1} k \ . \tag{4.7}$$

In the opposite regime, one lets $k \to +\infty$ while fixing the fluid domain geometry. The resulting asymptotic behavior of the dispersion relation is that

$$\omega_\infty^2 = \frac{g(\rho - \rho_1)}{\rho + \rho_1} k \ , \tag{4.8}$$

which coincides with the scaling invariant third situation above. These expressions are to be compared with the case of a free surface lying over a free interface in a two fluid system.

4.2 Linear free surfaces and interfaces

Using (2.19) and (2.29), the quadratic part of the Hamiltonian for the problem of a free interface underlying a free surface is given by

$$\begin{aligned} H \ = \ & \frac{1}{2}\int_{\mathbb{R}} \xi \frac{D\tanh(hD)\coth(h_1D)}{\rho\coth(h_1D) + \rho_1\tanh(hD)}\xi + 2\xi\frac{D\tanh(hD)\mathrm{csch}(h_1D)}{\rho\coth(h_1D) + \rho_1\tanh(hD)}\xi_1 \\ & + \xi_1\frac{D(\coth(h_1D)\tanh(hD) + (\rho/\rho_1))}{\rho\coth(h_1D) + \rho_1\tanh(hD)}\xi_1 + g(\rho-\rho_1)\eta^2 + g\rho_1\eta_1^2\, dx \ . \end{aligned} \tag{4.9}$$

The linearized equations of motion are

$$\begin{aligned} \partial_t\eta \ &= \ \delta_\xi H = \frac{D\tanh(hD)\coth(h_1D)}{\rho\coth(h_1D) + \rho_1\tanh(hD)}\xi + \frac{D\tanh(hD)\mathrm{csch}(h_1D)}{\rho\coth(h_1D) + \rho_1\tanh(hD)}\xi_1 \\ \partial_t\xi \ &= \ -\delta_\eta H = -g(\rho - \rho_1)\eta \ , \end{aligned}$$

and

$$\partial_t \eta_1 = \delta_{\xi_1} H = \frac{D \tanh(hD) \operatorname{csch}(h_1 D)}{\rho \coth(h_1 D) + \rho_1 \tanh(hD)} \xi + \frac{D(\coth(h_1 D) \tanh(hD) + (\rho/\rho_1))}{\rho \coth(h_1 D) + \rho_1 \tanh(hD)} \xi_1$$

$$\partial_t \xi_1 = -\delta_{\eta_1} H = -g\rho_1 \eta_1 \ . \tag{4.10}$$

The corresponding dispersion relation for ω^2 is determined by the quadratic equation

$$\omega^4 - g\rho k \frac{1 + \tanh(kh)\coth(kh_1)}{\rho \coth(kh_1) + \rho_1 \tanh(kh)} \omega^2 + g^2(\rho - \rho_1)k^2 \frac{\tanh(kh)}{\rho \coth(kh_1) + \rho_1 \tanh(kh)} = 0 \ . \tag{4.11}$$

The two solutions $\omega^\pm(k)$ of (4.11) are associated with two different modes of wave motion, namely surface and interface displacements. They are given by

$$\begin{aligned}(\omega^\pm)^2 = {} & \frac{1}{2} g\rho k \frac{1 + \tanh(hk)\coth(h_1 k)}{\rho \coth(h_1 k) + \rho_1 \tanh(hk)} \\ & \pm \frac{1}{2} gk \big[\rho^2 (1 - \tanh(hk)\coth(h_1 k))^2 \\ & \qquad + 4\rho\rho_1 \tanh(hk)(\coth(h_1 k) - \tanh(hk)) \\ & \qquad + 4\rho_1^2 \tanh(hk)^2\big]^{1/2} / (\rho \coth(h_1 k) + \rho_1 \tanh(hk)) \ . \end{aligned} \tag{4.12}$$

The radicand is always positive, as can be assured by the fact that for all wavenumbers $k > 0$, $\tanh(hk) < 1 < \coth(h_1 k)$. The branch ω^+ is associated with free surface wave motion, while the linear behavior of the interface is governed by ω^- (at least in the limit of large k). This expression also appears in [28].

Comparison of c_0 with $c_0^\pm$: It is important to compare the dispersion relation ω^- for the interfacial mode with the dispersion relation ω for the case with a rigid lid (4.3). In the regime where $k \to +\infty$, fixing other aspects of the fluid domain, one finds that

$$(\omega_\infty^+)^2 = gk \ , \qquad (\omega_\infty^-)^2 = \frac{g(\rho - \rho_1)}{\rho + \rho_1} k \ . \tag{4.13}$$

The latter agrees with the asymptotics as $k \to +\infty$ of the dispersion relation (4.8) of the case with a rigid lid. The expression for $(\omega_\infty^+)^2 = gk$ agrees with the dynamics of the free surface with no free interface present.

However the behavior of the dispersion relations for long wave regimes are very different when considering the case of a free surface lying over a free interface and the case of rigid lid upper boundary conditions. Letting kh and $kh_1 \to 0$ while fixing the ratio h/h_1 to be finite, one finds that the two phase speeds associated with the two branches of the dispersion curve $\omega^{\pm}$ are asymptotic to

$$(c_0^{\pm})^2 = \tfrac{1}{2} g \left(h + h_1 \pm \sqrt{(h-h_1)^2 + 4(\rho_1/\rho) h h_1} \right) . \tag{4.14}$$

We only consider $\rho_1 < \rho$, so the 'faster' free surface phase velocity c_0^+ is somewhat slower than if there were no interface present. Note that the phase velocity $(c_0^-)^2$ associated with the free interface (the 'slower' dispersion curve) is positive for $\rho > \rho_1$ (stable stratification). Examining c_0^- we conclude that it can behave completely differently than the case of the rigid lid, given in (4.5). There is also a significant difference between the dispersive behavior in this long wave regime, in the case of a free surface and a free interface, as compared to the case of a rigid lid.

In other situations, such as when $kh \to \infty$ (infinitely deep lower layer) and $kh_1 \to 0$ (finite upper layer),

$$(c_0^+)^2 = \frac{g}{k} \qquad \text{and} \qquad (c_0^-)^2 = g h_1 \left(1 - \frac{\rho_1}{\rho} \right) . \tag{4.15}$$

This differs from the regime of two finite layers where both $(c_0^{\pm})^2$ are of the same order of magnitude, as shown in (4.14).

In figure 4.1, we plot the linear phase speeds for the different configurations as functions of the wavenumber. The linear phase speed $c = \omega/k$ for the interface in the rigid lid case is given by (4.4), while those of the coupled system are given by (4.12) ($c^{\pm} = \omega^{\pm}/k$). We show the comparison between c and $c^{\pm}$ for two different values of the density ratio $\rho_1/\rho = 0.2, 0.8$ and for three different values of the depth ratio $h_1/h = 10, 1, 0.1$. As expected, c^- coincides with c at large k and their graphs always lie below that of c^+. The differences between c and c^- are most significant for small values of ρ_1/ρ. Also, the values of c and c^- are slightly larger for small ρ_1/ρ than large ρ_1/ρ. This is the fact that interfacial waves propagate more rapidly beneath a less dense fluid. For a given value of ρ_1/ρ, the differences between c and c^- are most important when the ratio h_1/h is small. When h_1/h is large, their graphs match perfectly since in this case the effects of a rigid lid or a free surface are negligible.

5 Long wave expansions for free interfaces

5.1 The Korteweg-deVries (KdV) regime

The first case of interest is the situation in which the fluid domain consists of two layers, each of finite depth; $0 < h, h_1 < +\infty$. We will start our study with the classical scaling regime of small amplitude long waves, for which we fix the asymptotic depths h, h_1 of the layers. More precisely, we derive an asymptotic description of waves in a regime in which wave amplitudes a/h, a/h_1 and typical wavelengths h/λ, h_1/λ are in balance; namely $a/h \simeq a/h_1 \simeq (h/\lambda)^2 \simeq (h_1/\lambda)^2 \simeq \varepsilon^2$, and we take $\varepsilon^2 \ll 1$ to be a small parameter. This regime was studied by Benjamin in [3].

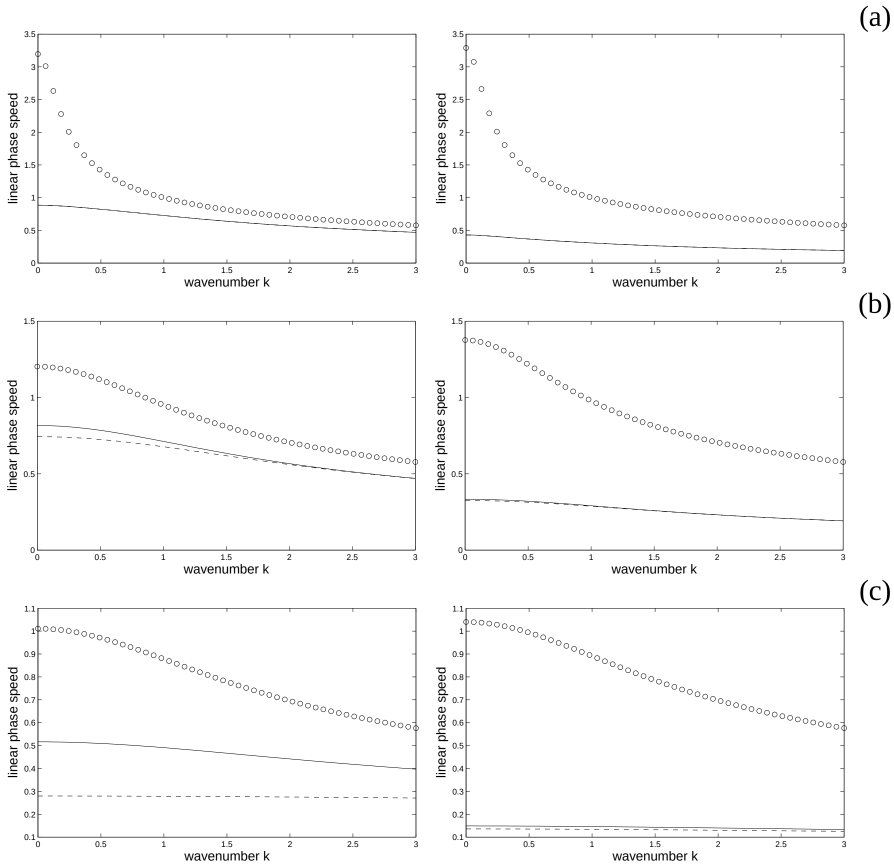

FIGURE 4.1. Linear phase speed c vs. wavenumber k for (left column) $\rho_1/\rho = 0.2$ and (right column) $\rho_1/\rho = 0.8$: (a) $h_1/h = 10$, (b) $h_1/h = 1$, (c) $h_1/h = 0.1$. The linear phase speed for the interface in the rigid lid case is represented in solid line. The linear phase speeds c^- and c^+ in the coupled system are represented in dashed line and circles respectively.

To implement our scheme of Hamiltonian perturbation theory in this regime, we introduce amplitude scaling and spatial scaling as follows

$$x' = \varepsilon x\,, \qquad \varepsilon^2\eta' = \eta\,, \qquad \varepsilon\xi' = \xi\,, \tag{5.1}$$

which has the effect that η and $u = \partial_x\xi$ are considered of the same order of magnitude $O(\varepsilon^2)$. This introduces the small parameter into the Hamiltonian (2.16) for the interface problem. To make the parametric dependence explicit, we use the description of the Taylor expansion for the Dirichlet-Neumann operators that is given in the appendix A.1,

$$\begin{aligned}
G(\eta) &= D\tanh(hD) + (D\eta D - D\tanh(hD)\eta D\tanh(hD)) + O(|\eta|^2|D|^3) \\
G_1(\eta) &= D\tanh(h_1D) - (D\eta D - D\tanh(h_1D)\eta D\tanh(h_1D)) + O(|\eta|^2|D|^3) \\
B &= \rho_1 G(\eta) + \rho G_1(\eta) \ .
\end{aligned}$$

Under the transformation given by the scaling (5.1), the Dirichlet-Neumann operator $G(\eta)$ for the lower fluid domain becomes

$$\begin{aligned}
G(\eta') &= \varepsilon D'\tanh(\varepsilon hD') \\
&\quad + \varepsilon^4(D'\eta'D' - D'\tanh(\varepsilon hD')\eta'D'\tanh(\varepsilon hD')) + O(\varepsilon^8) \\
&= \varepsilon^2 hD'^2 + \varepsilon^4\big(-\tfrac{1}{3}h^3D'^4 + D'\eta'D'\big) \\
&\quad + \varepsilon^6\big(\tfrac{2}{15}h^5D'^6 - h^2D'^2\eta'D'^2\big) + O(\varepsilon^8) \ ,
\end{aligned} \tag{5.2}$$

and the Dirichlet-Neumann operator $G_1(\eta)$ for the upper fluid domain is

$$\begin{aligned}
G_1(\eta') &= \varepsilon D'\tanh(\varepsilon h_1D') \\
&\quad - \varepsilon^4(D'\eta'D' - D'\tanh(\varepsilon h_1D')\eta'D'\tanh(\varepsilon h_1D')) + O(\varepsilon^8) \\
&= \varepsilon^2 h_1D'^2 + \varepsilon^4\big(-\tfrac{1}{3}h_1^3D'^4 - D'\eta'D'\big) \\
&\quad + \varepsilon^6\big(\tfrac{2}{15}h_1^5D'^6 + h_1^2D'^2\eta'D'^2\big) + O(\varepsilon^8)
\end{aligned} \tag{5.3}$$

where we have used that for $j \geq 2$, the quantities $G^{(j)}(\eta')$ and $G_1^{(j)}(\eta')$ are of order $O(\varepsilon^8)$ or higher in this scaling regime. Combining these expressions for the asymptotic description of the operator B in this regime gives

$$\begin{aligned}
B &= \rho_1 G(\eta') + \rho G_1(\eta') \\
&= \varepsilon^2(\rho_1 h + \rho h_1)D'^2 + \varepsilon^4\big((\rho_1 - \rho)D'\eta'D' - \tfrac{1}{3}(\rho_1 h^3 + \rho h_1^3)D'^4\big) \\
&\quad + \varepsilon^6\big(\tfrac{2}{15}(\rho_1 h^5 + \rho h_1^5)D'^6 - (\rho_1 h^2 - \rho h_1^2)D'^2\eta'D'^2\big) + O(\varepsilon^8) \ .
\end{aligned} \tag{5.4}$$

Therefore with respect to this expansion, the inverse operator is

$$\begin{aligned}
B^{-1} &= \frac{1}{\varepsilon^2(\rho_1 h + \rho h_1)} D'^{-1} \\
&\quad \times \Bigg[1 + \varepsilon^2\left(\frac{1}{3}\frac{\rho_1 h^3 + \rho h_1^3}{\rho_1 h + \rho h_1}D'^2 - \frac{\rho_1 - \rho}{\rho_1 h + \rho h_1}\eta'\right) \\
&\quad + \varepsilon^4\Bigg(-\frac{2}{15}\frac{\rho_1 h^5 + \rho h_1^5}{\rho_1 h + \rho h_1}D'^4 + \frac{\rho_1 h^2 - \rho h_1^2}{\rho_1 h + \rho h_1}D'\eta'D' \\
&\quad + \left[-\frac{1}{3}\frac{\rho_1 h^3 + \rho h_1^3}{\rho_1 h + \rho h_1}D'^2 + \frac{\rho_1 - \rho}{\rho_1 h + \rho h_1}\eta'\right]^2\Bigg) + O(\varepsilon^6)\Bigg] D'^{-1} \ .
\end{aligned} \tag{5.5}$$

Boussinesq system: Using this information, the Boussinesq system for interfacial wave evolution can be derived from the appropriately scaled Hamiltonian (2.16) for the dynamics of the interface. The integrand of (2.16) is given in terms of the rational function of Dirichlet-Neumann operators $G_1(\eta)B^{-1}G(\eta)$. In this scaling regime, using the expressions (5.2)(5.3)(5.5) and retaining terms of up to order $O(\varepsilon^6)$, this takes the form

$$
\begin{aligned}
H(\eta,\xi) = {} & \frac{\varepsilon^4}{2}\int_{\mathbb{R}} \xi \frac{hh_1}{\rho_1 h+\rho h_1} D^2\xi + g(\rho-\rho_1)\eta^2 \,\frac{dx}{\varepsilon} \\
& + \frac{\varepsilon^6}{2}\int_{\mathbb{R}} \xi \left(-\frac{1}{3}\left(\frac{hh_1}{\rho_1 h+\rho h_1}\right)^2 (\rho_1 h_1+\rho h) D^4 \right. \\
& \left. + \frac{\rho h_1^2-\rho_1 h^2}{(\rho_1 h+\rho h_1)^2} D\eta D \right) \xi \,\frac{dx}{\varepsilon} + O(\varepsilon^7) \ ,
\end{aligned} \tag{5.6}
$$

where and hereafter the primes are dropped for convenience. According to the transformation laws for the structure map J, this Hamiltonian is accompanied by the transformed structure map $J_2 = \varepsilon^{-3}J$, so the large powers of ε that enter in (5.6) should not be alarming. Retaining terms in the Hamiltonian of order $O(\varepsilon^5)$ or lower, the resulting approximate system of equations of motion (3.3) describing the long wave small amplitude regime is the following Boussinesq system for η and ξ

$$
\begin{aligned}
\partial_t \eta &= \varepsilon^{-3}\delta_\xi H \\
&= -\frac{hh_1}{\rho_1 h+\rho h_1}\partial_x^2\xi \\
&\quad -\varepsilon^2\left(\frac{1}{3}\frac{(hh_1)^2(\rho_1 h_1+\rho h)}{(\rho_1 h+\rho h_1)^2}\partial_x^4\xi + \frac{\rho h_1^2-\rho_1 h^2}{(\rho_1 h+\rho h_1)^2}\partial_x(\eta\partial_x\xi)\right) \\
\partial_t \xi &= -\varepsilon^{-3}\delta_\eta H \\
&= -g(\rho-\rho_1)\eta - \frac{\varepsilon^2}{2}\frac{\rho h_1^2-\rho_1 h^2}{(\rho_1 h+\rho h_1)^2}(\partial_x\xi)^2 \ .
\end{aligned} \tag{5.7}
$$

We note that the coefficient of the nonlinear terms $(\rho h_1^2-\rho_1 h^2)/(\rho_1 h+\rho h_1)^2$ changes sign at the parameter values $\rho/h^2 = \rho_1/h_1^2$, corresponding to the transition between the regime of soliton-like solutions of elevation above the mean level $\eta = 0$ when $\rho/h^2 > \rho_1/h_1^2$, to ones of depression [3].

Writing (5.6) in terms of the interface elevation - velocity coordinates, the Hamiltonian is

$$
\begin{aligned}
H(\eta,u) = {} & \frac{\varepsilon^4}{2}\int_{\mathbb{R}} \frac{hh_1}{\rho_1 h+\rho h_1}u^2 + g(\rho-\rho_1)\eta^2\,\frac{dx}{\varepsilon} \\
& + \frac{\varepsilon^6}{2}\int_{\mathbb{R}} -\frac{1}{3}\left(\frac{hh_1}{\rho_1 h+\rho h_1}\right)^2(\rho_1 h_1+\rho h)(\partial_x u)^2 + \frac{\rho h_1^2-\rho_1 h^2}{(\rho_1 h+\rho h_1)^2}\eta u^2\,\frac{dx}{\varepsilon} \ ,
\end{aligned} \tag{5.8}
$$

where $u = \partial_x \xi$. The resulting system of equations is

$$\begin{aligned}
\partial_t \eta &= -\partial_x \Bigg(\frac{hh_1}{\rho_1 h + \rho h_1} u \\
&\qquad + \varepsilon^2 \left(\frac{1}{3} \frac{(hh_1)^2(\rho_1 h_1 + \rho h)}{(\rho_1 h + \rho h_1)^2} \partial_x^2 u + \frac{\rho h_1^2 - \rho_1 h^2}{(\rho_1 h + \rho h_1)^2} (\eta u) \right) \Bigg) \\
\partial_t u &= -\partial_x \left(g(\rho - \rho_1)\eta + \frac{\varepsilon^2}{2} \frac{\rho h_1^2 - \rho_1 h^2}{(\rho_1 h + \rho h_1)^2} u^2 \right) ,
\end{aligned} \tag{5.9}$$

which is in the general form of the original Boussinesq system for surface water waves, however making explicit the parametric dependence of the coefficients. This system was studied by Kaup [20]. As described in Section 3, the structure map J in these coordinates is given in non-classical form (3.13).

KdV equations: Framing the Boussinesq system (5.9) in characteristic coordinates as in Section 3.2, we make a transformation as in (3.23)

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \sqrt[4]{\frac{g(\rho-\rho_1)(\rho_1 h + \rho h_1)}{4hh_1}} & \sqrt[4]{\frac{hh_1}{4g(\rho-\rho_1)(\rho_1 h+\rho h_1)}} \\ \sqrt[4]{\frac{g(\rho-\rho_1)(\rho_1 h + \rho h_1)}{4hh_1}} & -\sqrt[4]{\frac{hh_1}{4g(\rho-\rho_1)(\rho_1 h+\rho h_1)}} \end{pmatrix} \begin{pmatrix} \eta \\ u \end{pmatrix} . \tag{5.10}$$

The result is that the Hamiltonian is transformed to

$$\begin{aligned}
H(r,s) &= \frac{\varepsilon^4}{2} \int_{\mathbb{R}} \sqrt{\frac{g(\rho-\rho_1)hh_1}{\rho_1 h + \rho h_1}} \left(r^2 + s^2\right) \frac{dx}{\varepsilon} \\
&\quad + \frac{\varepsilon^6}{2} \int_{\mathbb{R}} -\frac{1}{6} \left(\frac{hh_1}{\rho_1 h + \rho h_1} \right)^{3/2} \sqrt{g(\rho - \rho_1)} \\
&\qquad \times (\rho_1 h_1 + \rho h) \left[(\partial_x r)^2 - 2(\partial_x r)(\partial_x s) + (\partial_x s)^2 \right] \\
&\qquad + \frac{1}{2\sqrt{2}} \frac{\rho h_1^2 - \rho_1 h^2}{(\rho_1 h + \rho h_1)^{7/4}} \sqrt[4]{\frac{g(\rho-\rho_1)}{hh_1}} \left(r^3 - r^2 s - r s^2 + s^3 \right) \frac{dx}{\varepsilon} ,
\end{aligned} \tag{5.11}$$

and the Boussinesq system (5.9) can be viewed as a system of two coupled equations of KdV type, namely

$$\begin{aligned}
\partial_t r = &-\sqrt{\frac{g(\rho-\rho_1)hh_1}{\rho_1 h+\rho h_1}}\partial_x r \\
&-\frac{\varepsilon^2}{6}\left(\frac{hh_1}{\rho_1 h+\rho h_1}\right)^{3/2}\sqrt{g(\rho-\rho_1)}(\rho_1 h_1+\rho h)\left(\partial_x^3 r-\partial_x^3 s\right) \\
&-\frac{\varepsilon^2}{4\sqrt{2}}\frac{\rho h_1^2-\rho_1 h^2}{(\rho_1 h+\rho h_1)^{7/4}}\sqrt[4]{\frac{g(\rho-\rho_1)}{hh_1}}\partial_x\left(3r^2-2rs-s^2\right)
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\partial_t s = &\sqrt{\frac{g(\rho-\rho_1)hh_1}{\rho_1 h+\rho h_1}}\partial_x s \\
&-\frac{\varepsilon^2}{6}\left(\frac{hh_1}{\rho_1 h+\rho h_1}\right)^{3/2}\sqrt{g(\rho-\rho_1)}(\rho_1 h_1+\rho h)\left(\partial_x^3 r-\partial_x^3 s\right) \\
&-\frac{\varepsilon^2}{4\sqrt{2}}\frac{\rho h_1^2-\rho_1 h^2}{(\rho_1 h+\rho h_1)^{7/4}}\sqrt[4]{\frac{g(\rho-\rho_1)}{hh_1}}\partial_x\left(r^2+2rs-3s^2\right) .
\end{aligned} \tag{5.13}$$

The component of the solution $r(x,t)$ corresponds to elements of the solution which are principally right-moving, while $s(x,t)$ are principally left-moving.

The KdV regime consists in restricting one's attention to the region of phase space in which s is itself of order $O(\varepsilon^2)$. More precisely, we will examine orbits of the system of equations (5.12) along which $\|s\|_{H^m} \le O(\varepsilon^2)$, for a Sobolev index $m \ge 3$. Taking this into account, the first equation (5.12) can be rewritten

$$\begin{aligned}
\partial_t r = &-\sqrt{\frac{g(\rho-\rho_1)hh_1}{\rho_1 h+\rho h_1}}\partial_x r-\frac{\varepsilon^2}{6}\left(\frac{hh_1}{\rho_1 h+\rho h_1}\right)^{3/2}\sqrt{g(\rho-\rho_1)}(\rho_1 h_1+\rho h)\partial_x^3 r \\
&-\frac{3\varepsilon^2}{2\sqrt{2}}\frac{\rho h_1^2-\rho_1 h^2}{(\rho_1 h+\rho h_1)^{7/4}}\sqrt[4]{\frac{g(\rho-\rho_1)}{hh_1}}r\partial_x r+O(\varepsilon^4) .
\end{aligned} \tag{5.14}$$

Eliminating terms of orders $O(\varepsilon^4)$ and higher, the resulting equation gives the KdV description for uni-directional long waves in the interface, as in [3]. It is in the form of a Hamiltonian system, with the symplectic structure given by the structure map $J = \partial_x$. It is useful to transform the system (5.14) to a coordinate frame moving with the characteristic velocity $c_0 = \sqrt{g(\rho-\rho_1)hh_1/(\rho_1 h+\rho h_1)}$ of the highest order component of the Hamiltonian, which is effected by subtracting a term proportional to the momentum integral $I(r,s) = (\varepsilon^3/2)\int r^2 - s^2 dx$. In the KdV regime, in which

$s \simeq O(\varepsilon^2)$, we have

$$
\begin{aligned}
&\frac{1}{\varepsilon^5}\left(H(r) - \sqrt{\frac{g(\rho-\rho_1)hh_1}{\rho_1 h + \rho h_1}} I\right) \\
&= \frac{1}{2}\int_{\mathbb{R}} -\frac{1}{6}\left(\frac{hh_1}{\rho_1 h + \rho h_1}\right)^{3/2} \sqrt{g(\rho-\rho_1)}(\rho_1 h_1 + \rho h)(\partial_x r)^2\, dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}} \frac{1}{2\sqrt{2}} \frac{\rho h_1^2 - \rho_1 h^2}{(\rho_1 h + \rho h_1)^{7/4}} \sqrt[4]{\frac{g(\rho-\rho_1)}{hh_1}} r^3\, dx\ .
\end{aligned} \tag{5.15}
$$

The equations of motion have been transformed to

$$
\partial_\tau r = -\partial_x \delta_r (H - c_0 I)/\varepsilon^2 = c_1 \partial_x^3 r + c_2 r \partial_x r\ , \tag{5.16}
$$

which is written with respect to a rescaled time $\tau = \varepsilon^2 t$, and with the constants defined by

$$
\begin{aligned}
c_1 &= -\frac{1}{6}\left(\frac{hh_1}{\rho_1 h + \rho h_1}\right)^{3/2} \sqrt{g(\rho-\rho_1)}(\rho_1 h_1 + \rho h)\ , \\
c_2 &= -\frac{3}{2\sqrt{2}} \frac{\rho h_1^2 - \rho_1 h^2}{(\rho_1 h + \rho h_1)^{7/4}} \sqrt[4]{\frac{g(\rho-\rho_1)}{hh_1}}\ .
\end{aligned} \tag{5.17}
$$

Higher order Boussinesq and KdV equations: For many reasons it is desirable to extend the long-wave expansion of the Hamiltonian to orders higher than $O(\varepsilon^6)$ as appearing in (5.6). In particular, the Boussinesq system as it appears in (5.7) and (5.9) is badly ill-posed, and solutions of the initial value problem for the most part instantly leave the regime of slowly varying functions of x,t which characterize the hypotheses underlying the derivation of the long wave equations in the Boussinesq scaling regime. It is natural to stabilize this phenomenon by including the next higher order in the equations of motion. Secondly, there are values of the basic parameters for which the coefficients of the nonlinear term in the Boussinesq and KdV regimes are effectively of smaller order, namely when

$$
\frac{\rho}{h^2} - \frac{\rho_1}{h_1^2} \simeq O(\varepsilon^2)\ . \tag{5.18}
$$

In this situation, a valid asymptotic description of the interface motion is only available in the context of a higher order expansion. From our present point of view of Hamiltonian perturbation theory, the expressions (5.2)(5.3)(5.5) are used in the Hamiltonian, and terms of orders up to $O(\varepsilon^8)$ retained in the approximate equations. The resulting Hamiltonian is

$$
\begin{aligned}
H(\eta,\xi) = \ & \frac{\varepsilon^4}{2}\int_{\mathbb{R}} \xi \frac{hh_1}{\rho_1 h+\rho h_1} D^2\xi + g(\rho-\rho_1)\eta^2 \, \frac{dx}{\varepsilon} \\
& + \frac{\varepsilon^6}{2}\int_{\mathbb{R}} \xi \left(-\frac{1}{3}\left(\frac{hh_1}{\rho_1 h+\rho h_1}\right)^2 (\rho_1 h_1+\rho h) D^4 \right. \\
& \qquad\qquad \left. + \frac{\rho h_1^2-\rho_1 h^2}{(\rho_1 h+\rho h_1)^2} D\eta D \right) \xi \, \frac{dx}{\varepsilon} \\
& + \frac{\varepsilon^8}{2}\int_{\mathbb{R}} \xi \left(\frac{2}{15}\left(\frac{hh_1}{\rho_1 h+\rho h_1}\right)^2 \left(\rho_1 h_1^3+\rho h^3\right) \right. \\
& \qquad\qquad \left. -\frac{1}{9}\frac{\rho\rho_1 h^2 h_1^2}{(\rho_1 h+\rho h_1)^3}\left(h^2-h_1^2\right)^2 \right) D^6\xi \\
& \qquad -\xi \frac{(\rho-\rho_1)h^2h_1^2}{(\rho_1 h+\rho h_1)^2} D^2\eta D^2\xi - \xi \frac{\rho\rho_1(h+h_1)^2}{(\rho_1 h+\rho h_1)^3} D\eta^2 D\xi \\
& \qquad +\xi \frac{1}{3}\frac{\rho\rho_1 hh_1}{(\rho_1 h+\rho h_1)^3}\left(h^3+h^2h_1-hh_1^2-h_1^3\right)\left(D^3\eta D + D\eta D^3\right)\xi \, \frac{dx}{\varepsilon} \\
& + O(\varepsilon^9)\ .
\end{aligned}
\tag{5.19}
$$

Using this Hamiltonian, the higher order Boussinesq system takes the form

$$
\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon^{-3}\partial_x \\ -\varepsilon^{-3}\partial_x & 0 \end{pmatrix} \delta H(\eta,\xi)\ . \tag{5.20}
$$

Since the well posedness of the resulting system of equations is dependent on the term with highest order derivatives having positive coefficient, it is of interest to note the following identity

$$
\begin{aligned}
& \frac{2}{15}\left(\frac{hh_1}{\rho_1 h+\rho h_1}\right)^2 \left(\rho_1 h_1^3+\rho h^3\right) - \frac{1}{9}\frac{\rho\rho_1 h^2h_1^2}{(\rho_1 h+\rho h_1)^3}\left(h^2-h_1^2\right)^2 \\
& = \frac{(hh_1)^2}{(\rho h_1+\rho_1 h)^3}\left(\frac{2}{15}\left(\rho_1^2 h_1^2+\rho^2 h^2\right)hh_1 + \frac{2}{9}\rho\rho_1 h^2h_1^2 + \frac{1}{45}\rho\rho_1\left(h^4+h_1^4\right) \right)\ .
\end{aligned}
$$

The LHS is the coefficient of the D^6 term in the Hamiltonian, while the RHS is clearly a positive quantity for any choices of values of the basic parameters ρ,ρ_1,h,h_1.

The KdV regime at higher order of approximation results when examining solutions which are principally right-moving, which is made clear in characteristic coordinates. Transforming with (5.10), and considering a region of phase space in which $s \leq O(\varepsilon^4)$ (in an appropriate norm), the Hamiltonian takes the form

$$
\begin{aligned}
H = {} & \frac{\varepsilon^4}{2}\int_{\mathbb{R}} \sqrt{\frac{g(\rho-\rho_1)hh_1}{\rho_1 h+\rho h_1}}\, r^2\, \frac{dx}{\varepsilon} \\
& +\frac{\varepsilon^6}{2}\int_{\mathbb{R}} -\frac{1}{6}\left(\frac{hh_1}{\rho_1 h+\rho h_1}\right)^{3/2} \sqrt{g(\rho-\rho_1)}(\rho_1 h_1+\rho h)(\partial_x r)^2 \\
& \qquad +\frac{1}{2\sqrt{2}}\frac{\rho h_1^2-\rho_1 h^2}{(\rho_1 h+\rho h_1)^{7/4}}\sqrt[4]{\frac{g(\rho-\rho_1)}{hh_1}}\, r^3\, \frac{dx}{\varepsilon} \\
& +\frac{\varepsilon^8}{2}\int_{\mathbb{R}} \sqrt{\frac{g(\rho-\rho_1)(hh_1)^3}{(\rho_1 h+\rho h_1)^5}} \\
& \qquad \times\left(\frac{1}{15}(\rho^2 h^2+\rho_1^2 h_1^2)hh_1+\frac{1}{9}\rho\rho_1 h^2 h_1^2+\frac{1}{90}\rho\rho_1(h^4+h_1^4)\right)(\partial_x^2 r)^2 \\
& \qquad +\left(\frac{2}{3\sqrt{2}}\sqrt[4]{\frac{g(\rho-\rho_1)(hh_1)^3}{(\rho_1 h+\rho h_1)^{11}}}\rho\rho_1\big(h^3+h^2 h_1-hh_1^2-h_1^3\big)\right. \\
& \qquad\qquad \left. -\frac{1}{2\sqrt{2}}\sqrt[4]{\frac{g(\rho-\rho_1)^5(hh_1)^7}{(\rho_1 h+\rho h_1)^7}}\right) r(\partial_x r)^2-\frac{\rho\rho_1(h+h_1)^2}{4(\rho_1 h+\rho h_1)^3}\, r^4\, \frac{dx}{\varepsilon}\ .
\end{aligned}
$$

The equations of motion appear in the form $\partial_\tau r = -\partial_x \delta_r (H-c_0 I)/\varepsilon^2$ which is the following fifth order dispersive evolution equation (or Kawahara equation [21])

$$
\partial_\tau r = c_1\partial_x^3 r + c_2 r\partial_x r + \varepsilon^2\Big(c_3\partial_x^5 r + c_4 r\partial_x^3 r + 2c_4(\partial_x r)(\partial_x^2 r) + c_5 r^2\partial_x r\Big)\ . \tag{5.21}
$$

We have again scaled the temporal variable $\tau = \varepsilon^2 t$, the coefficients c_1 and c_2 are given in (5.17), and the higher order coefficients are

$$
\begin{aligned}
c_3 &= -\sqrt{\frac{g(\rho-\rho_1)(hh_1)^3}{(\rho_1 h+\rho h_1)^5}}\left(\frac{1}{15}(\rho^2 h^2+\rho_1^2 h_1^2)hh_1+\frac{1}{9}\rho\rho_1 h^2 h_1^2+\frac{1}{90}\rho\rho_1(h^4+h_1^4)\right) \\
c_4 &= \frac{2}{3\sqrt{2}}\sqrt[4]{\frac{g(\rho-\rho_1)(hh_1)^3}{(\rho_1 h+\rho h_1)^{11}}}\rho\rho_1\big(h^3+h^2 h_1-hh_1^2-h_1^3\big)-\frac{1}{2\sqrt{2}}\sqrt[4]{\frac{g(\rho-\rho_1)^5(hh_1)^7}{(\rho_1 h+\rho h_1)^7}} \\
c_5 &= \frac{3\rho\rho_1(h+h_1)^2}{2(\rho_1 h+\rho h_1)^3}\ .
\end{aligned}
$$

Note that when $c_2 = 0$ in (5.17) (that is, when $\rho h_1^2 - \rho_1 h^2 = 0$), then c_3 does not vanish.

5.2 Regime of small steepness for two finite layers

We change our focus to the regime in which the typical wavelength λ of the internal waves is long compared to the depths h and h_1 of the two layers. However the typical wave amplitude a is not assumed to be small compared to h or h_1 unlike the classical Boussinesq regime. This situation is particularly relevant to the study of internal waves, as in realistic conditions their amplitude a/h_1 is often significant, while exhibiting small steepness. We take the small parameter to be $\varepsilon^2 \simeq (h/\lambda)^2 \simeq (h_1/\lambda)^2 \simeq (a/\lambda)^2 \ll 1$ characterizing steepness, and we introduce the following scaling

$$x' = \varepsilon x, \qquad \eta' = \eta, \qquad \xi' = \varepsilon\xi \,. \tag{5.22}$$

As before, expanding $G^{(0)} = D\tanh(hD) = \varepsilon D'\tanh(\varepsilon hD') = \varepsilon^2 hD'^2 - \frac{1}{3}\varepsilon^4 h^3 D'^4 + O(\varepsilon^6)$ and $G_1^{(0)}$, together with higher order contributions which come from $G^{(j)}$, $G_1^{(j)}$ $(j = 1,2,3)$, and collecting terms in powers of ε in the Hamiltonian, one finds up to order $O(1/\varepsilon)$

$$H = \frac{1}{2\varepsilon}\int_{\mathbb{R}} R_0(\eta)u^2 + g(\rho-\rho_1)\eta^2\,dx + O(\varepsilon)\,, \tag{5.23}$$

where $u = \partial_x\xi$ and

$$R_0(\eta) = \frac{(h+\eta)(h_1-\eta)}{\rho_1(h+\eta)+\rho(h_1-\eta)}\,. \tag{5.24}$$

For convenience, we have dropped the primes in (5.23). The corresponding approximate equations of motion are given by

$$\begin{aligned} \partial_t\eta &= -\partial_x(R_0 u)\,,\\ \partial_t u &= -\partial_x\Big[\tfrac{1}{2}(\partial_\eta R_0)u^2 + g(\rho-\rho_1)\eta\Big]\,. \end{aligned} \tag{5.25}$$

Note that the factor $R_0(\eta)$ is nonsingular in the whole domain $-h < \eta < h_1$, vanishing at both endpoints $\eta = -h$ and $\eta = h_1$. In the case $\rho_1 = 0$, the canonical variables are $\eta(x)$ and $\xi(x) = \rho\Phi(x)$, and the equations of motion (5.25) reduce to

$$\partial_t\eta = -\frac{1}{\rho}\partial_x\left((h+\eta)u\right)\,, \quad \partial_t u = -\frac{1}{\rho}u\partial_x u - g\rho\partial_x\eta\,, \tag{5.26}$$

which are the classical shallow water equations for surface water waves.

The next approximation can be derived in a straightforward manner. Retaining terms of up to order $O(\varepsilon)$, one gets

$$\begin{aligned} H = {} & \frac{1}{2\varepsilon}\int_{\mathbb{R}} R_0(\eta)u^2 + g(\rho-\rho_1)\eta^2\,dx \\ & + \frac{\varepsilon}{2}\int_{\mathbb{R}} R_1(\eta)(\partial_x u)^2 + (\partial_x R_2(\eta))\partial_x\left(u^2\right) + R_3(\eta)(\partial_x\eta)^2 u^2\,dx + O(\varepsilon^3)\,. \end{aligned}$$

The corresponding equations of motion read

$$\begin{aligned} \partial_t\eta &= -\partial_x(R_0 u) - \varepsilon^2\partial_x\Big[-\partial_x(R_1\partial_x u) - \partial_x^2(R_2)u + R_3(\partial_x\eta)^2 u\Big]\,,\\ \partial_t u &= -\partial_x\Big[\tfrac{1}{2}(\partial_\eta R_0)u^2 + g(\rho-\rho_1)\eta\Big] - \varepsilon^2\partial_x\Big[\tfrac{1}{2}(\partial_\eta R_1)(\partial_x u)^2 \\ & \quad -\tfrac{1}{2}(\partial_\eta R_2)\partial_x^2\left(u^2\right) + \tfrac{1}{2}(\partial_\eta R_3)(\partial_x\eta)^2 u^2 - \partial_x\left(R_3(\partial_x\eta)u^2\right)\Big]\,, \end{aligned} \tag{5.27}$$

where

$$R_1(\eta) = -\frac{1}{3}\frac{(h+\eta)^2(h_1-\eta)^2(\rho_1(h_1-\eta)+\rho(h+\eta))}{(\rho_1(h+\eta)+\rho(h_1-\eta))^2},$$

$$\partial_\eta R_2(\eta) = -\frac{1}{3}\rho\rho_1(h+h_1)(h+\eta)(h_1-\eta)\frac{(h_1-\eta)^2-(h+\eta)^2}{(\rho_1(h+\eta)+\rho(h_1-\eta))^3},$$

$$R_3(\eta) = -\frac{1}{3}\rho\rho_1(h+h_1)^2\frac{\rho_1(h+\eta)^3+\rho(h_1-\eta)^3}{(\rho_1(h+\eta)+\rho(h_1-\eta))^4}.$$

These are novel evolution equations, not unrelated to the rational dispersive system obtained by Choi and Camassa [7], which exhibit nonlinear variations in wave speed and in their coefficients of dispersion.

5.3 The Benjamin-Ono (BO) regime

In this section, the series expansion of the Hamiltonian is used to derive the model equation for the wave motion at the interface between the fluids in the case when the lower layer has infinite depth, and the upper layer has a depth h_1. The significant quantities are the height a and the wavelength λ of a typical wave. The section has two parts. First, we assume that a/h_1 and h_1/λ are small and approximately of the same magnitude. This is the situation in which the BO equation was originally derived [4, 26], and when restricted to one-way propagation, our method will indeed yield the BO equation. In the second, we only assume that h_1/λ is small; it is a scaling regime analogous to that of Section 5.2, encompassing small steepness but with no *a priori* assumptions on the amplitude.

Boussinesq-like system: Since the typical wavelength is assumed to be large when compared to the depth of the upper layer, and the amplitude of a typical wave is assumed to be small when compared to h_1, the following scaling is used

$$x' = \varepsilon x, \qquad \varepsilon\eta' = \eta, \qquad \xi' = \xi\,, \tag{5.28}$$

where $\varepsilon^2 \simeq (h_1/\lambda)^2 \simeq (a/h_1)^2 \ll 1$. The operator for an infinite lower layer ($h = \infty$) is $G^{(0)} = |D|$. Inserting the expansions for the various operators into (2.16) (and then dropping the prime notation) yields the following expression for the Hamiltonian up to order $O(\varepsilon^2)$

$$\begin{aligned} H = {} & \frac{\varepsilon}{2}\frac{h_1}{\rho_1}\int_{\mathbb{R}} u^2\,dx + \frac{\varepsilon}{2}g(\rho-\rho_1)\int_{\mathbb{R}}\eta^2\,dx \\ & -\frac{\varepsilon^2}{2}\frac{\rho h_1^2}{\rho_1^2}\int_{\mathbb{R}} u|\partial_x|u\,dx - \frac{\varepsilon^2}{2\rho_1}\int_{\mathbb{R}}\eta u^2\,dx + O(\varepsilon^3)\,, \end{aligned} \tag{5.29}$$

which is expressed in the (η,u)-variables. The operator $|\partial_x|$ has Fourier symbol $|k|$ and is the composition of ∂_x with the Hilbert transform. The resulting Boussinesq system of equations of motion is given by

$$\begin{aligned} \partial_t \eta &= -\frac{h_1}{\rho_1}\partial_x u + \varepsilon \frac{\rho h_1^2}{\rho_1^2}|\partial_x|\partial_x u + \frac{\varepsilon}{\rho_1}\partial_x(\eta u)\,, \\ \partial_t u &= -g(\rho-\rho_1)\partial_x \eta + \frac{\varepsilon}{\rho_1} u\partial_x u\,, \end{aligned} \tag{5.30}$$

using the structure map J_2 of (3.13). On comparison with equations (4.17) and (4.18) in Choi and Camassa [6], the constants seem to be reversed in sign. This can be explained by the fact that we write our equations with respect to the quantity $\partial_x \xi = \rho \partial_x \Phi - \rho_1 \partial_x \Phi_1$ instead of the velocity $\partial_x \Phi_1$. With this relation, the linear hyperbolic terms in our equations are transformed directly into the equations obtained by Choi & Camassa.

BO equation: Introducing the transformation to characteristic coordinates

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{4h_1}} & \sqrt[4]{\frac{h_1}{4g\rho_1(\rho-\rho_1)}} \\ \sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{4h_1}} & -\sqrt[4]{\frac{h_1}{4g\rho_1(\rho-\rho_1)}} \end{pmatrix} \begin{pmatrix} \eta \\ u \end{pmatrix}\,, \tag{5.31}$$

and assuming $s \simeq O(\varepsilon^2)$ so that we are studying solutions which are principally right-moving, the Hamiltonian (5.29) becomes

$$\begin{aligned} H = \varepsilon \int_{\mathbb{R}} \sqrt{\frac{gh_1(\rho-\rho_1)}{4\rho_1}} r^2\,dx - \frac{\varepsilon^2}{2}\int_{\mathbb{R}} \frac{\rho h_1^2}{\rho_1^2}\sqrt{\frac{g\rho_1(\rho-\rho_1)}{4h_1}} r|\partial_x| r\,dx \\ -\frac{\varepsilon^2}{2}\int_{\mathbb{R}} \frac{1}{2\rho_1}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{4h_1}} r^3\,dx\,. \end{aligned} \tag{5.32}$$

Thus Hamilton's equation for r is

$$\begin{aligned} \partial_t r = -\sqrt{\frac{gh_1(\rho-\rho_1)}{\rho_1}}\partial_x r + \frac{\varepsilon}{2}\frac{\rho h_1^2}{\rho_1^2}\sqrt{\frac{g\rho_1(\rho-\rho_1)}{h_1}}|\partial_x|\partial_x r \\ +\frac{3\sqrt{2}}{4\rho_1}\varepsilon\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}} r\partial_x r\,, \end{aligned} \tag{5.33}$$

which is the usual BO equation as derived in [4]. As in Section 3, we change to coordinates moving with velocity $c_0 = \sqrt{gh_1(\rho-\rho_1)/\rho_1}$. This is equivalent to using the Hamiltonian $H - c_0 I$, where I is the momentum functional. Rescaling time to $\tau = \varepsilon t$, equation (5.33) then becomes

$$\partial_\tau r = \frac{\rho h_1^2}{2\rho_1^2}\sqrt{\frac{g\rho_1(\rho-\rho_1)}{h_1}}|\partial_x|\partial_x r + \frac{3\sqrt{2}}{4\rho_1}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}} r\partial_x r\,. \tag{5.34}$$

Higher order Boussinesq-like and BO equations: Retaining the terms of order $O(\varepsilon^3)$ in the Hamiltonian, we obtain the next highest order correction of the previous equations. The Hamiltonian is

$$\begin{aligned}H = {} & \frac{\varepsilon}{2}\frac{h_1}{\rho_1}\int_{\mathbb{R}} u^2\,dx + \frac{\varepsilon}{2}g(\rho-\rho_1)\int_{\mathbb{R}}\eta^2\,dx - \frac{\varepsilon^2}{2}\frac{\rho h_1^2}{\rho_1^2}\int_{\mathbb{R}} u|\partial_x|u\,dx \\ & -\frac{\varepsilon^2}{2\rho_1}\int_{\mathbb{R}}\eta u^2\,dx + \frac{\varepsilon^3}{2}\frac{h_1}{\rho_1}\int_{\mathbb{R}}\left(\frac{\rho^2 h_1^2}{\rho_1^2} - \frac{h_1^2}{3}\right)(\partial_x u)^2\,dx \\ & +\varepsilon^3\frac{\rho h_1}{\rho_1^2}\int_{\mathbb{R}}\eta u|\partial_x|u\,dx + O(\varepsilon^4)\,,\end{aligned} \tag{5.35}$$

and we note that no quartic terms appear at this order. This can be explained by the fact that they contain some dispersion, so that they actually only contribute at higher order. The corresponding Hamilton's equations are

$$\begin{aligned}\partial_t\eta = {} & -\frac{h_1}{\rho_1}\partial_x u + \varepsilon\frac{\rho h_1^2}{\rho_1^2}|\partial_x|\partial_x u + \frac{\varepsilon}{\rho_1}\partial_x(\eta u) + \varepsilon^2\frac{h_1}{\rho_1}\left(\frac{\rho^2 h_1^2}{\rho_1^2} - \frac{h_1^2}{3}\right)\partial_x^3 u \\ & -\varepsilon^2\frac{\rho h_1}{\rho_1^2}|\partial_x|\partial_x(\eta u) - \varepsilon^2\frac{\rho h_1}{\rho_1^2}\partial_x(\eta|\partial_x|u)\,, \\ \partial_t u = {} & -g(\rho-\rho_1)\partial_x\eta + \frac{\varepsilon}{\rho_1}u\partial_x u - \varepsilon^2\frac{\rho h_1}{\rho_1^2}\partial_x(u|\partial_x|u)\,.\end{aligned} \tag{5.36}$$

Making the further change of variables (5.31) and restricting our attention to principally right-moving solutions, the Hamiltonian (5.35) takes the form

$$\begin{aligned}H = {} & \varepsilon\int_{\mathbb{R}}\sqrt{\frac{gh_1(\rho-\rho_1)}{4\rho_1}}r^2\,dx - \frac{\varepsilon^2}{2}\int_{\mathbb{R}}\frac{\rho h_1^2}{\rho_1^2}\sqrt{\frac{g\rho_1(\rho-\rho_1)}{4h_1}}r|\partial_x|r\,dx \\ & -\frac{\varepsilon^2}{2}\int_{\mathbb{R}}\frac{1}{2\rho_1}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{4h_1}}r^3\,dx + \frac{\varepsilon^3}{2}\int_{\mathbb{R}}\frac{\rho h_1}{\rho_1^2}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{4h_1}}r^2|\partial_x|r\,dx \\ & +\frac{\varepsilon^3}{2}\int_{\mathbb{R}}\left(\frac{\rho^2 h_1^2}{\rho_1^2} - \frac{h_1^2}{3}\right)\sqrt{\frac{gh_1(\rho-\rho_1)}{4\rho_1}}(\partial_x r)^2\,dx\,,\end{aligned}$$

and we obtain the following higher order equation for r

$$\begin{aligned}\partial_\tau r = {} & \frac{\rho h_1^2}{2\rho_1^2}\sqrt{\frac{g\rho_1(\rho-\rho_1)}{h_1}}|\partial_x|\partial_x r + \frac{3\sqrt{2}}{4\rho_1}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}}r\partial_x r \\ & -\frac{\sqrt{2}}{2}\varepsilon\frac{\rho h_1}{\rho_1^2}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}}\Big[\partial_x(r|\partial_x|r) + |\partial_x|(r\partial_x r)\Big] \\ & +\frac{\varepsilon}{2}\left(\frac{\rho^2 h_1^2}{\rho_1^2} - \frac{h_1^2}{3}\right)\sqrt{\frac{gh_1(\rho-\rho_1)}{\rho_1}}\partial_x^3 r\,.\end{aligned} \tag{5.37}$$

This represents the higher order corrections to the BO equation (5.34).

5.4 Regime of small steepness for an infinite lower layer

Shallow water-like system: In this regime, the interface elevation is not assumed to be small compared to h_1, but only that it is of small slope. The new variables are defined as

$$x' = \varepsilon x\,, \qquad \eta' = \eta\,, \qquad \xi' = \varepsilon\,\xi\,, \tag{5.38}$$

where $\varepsilon^2 \simeq (h_1/\lambda)^2 \simeq (a/\lambda)^2 \ll 1$ characterizes steepness. Taking into account all terms of up to order $O(1)$ in the Hamiltonian, we have

$$\begin{aligned} H = {} & \frac{h_1}{2\varepsilon\rho_1}\int_{\mathbb{R}} u^2\,dx + \frac{1}{2\varepsilon} g(\rho-\rho_1)\int_{\mathbb{R}} \eta^2\,dx - \frac{1}{2\varepsilon\rho_1}\int_{\mathbb{R}} \eta u^2\,dx \\ & - \frac{\rho h_1^2}{2\rho_1^2}\int_{\mathbb{R}} u|\partial_x|u\,dx + \frac{\rho h_1}{\rho_1^2}\int_{\mathbb{R}} u|\partial_x|(\eta u)\,dx \\ & - \frac{\rho}{2\rho_1^2}\int_{\mathbb{R}} \eta u|\partial_x|(\eta u)\,dx + O(\varepsilon). \end{aligned} \tag{5.39}$$

If we only consider terms of order $O(1/\varepsilon)$ in (5.39), the corresponding system of Hamilton's equations is given by

$$\begin{aligned} \partial_t \eta &= -\frac{h_1}{\rho_1}\partial_x u + \frac{1}{\rho_1}\partial_x(\eta u)\,, \\ \partial_t u &= -g(\rho-\rho_1)\partial_x\eta + \frac{1}{\rho_1} u\partial_x u\,, \end{aligned} \tag{5.40}$$

which are the usual shallow water equations. Note that the nonlinear terms are not small corrections of the linear hyperbolic system, but are of the same order as the linear terms. Including terms of order $O(1)$ in (5.39), the following equations for η and u are obtained

$$\begin{aligned} \partial_t \eta = {} & -\frac{h_1}{\rho_1}\partial_x u + \frac{1}{\rho_1}\partial_x(\eta u) + \varepsilon\,\frac{\rho h_1^2}{\rho_1^2}|\partial_x|\partial_x u \\ & -\varepsilon\,\frac{\rho h_1}{\rho_1^2}|\partial_x|\partial_x(\eta u) - \varepsilon\,\frac{\rho h_1}{\rho_1^2}\partial_x(\eta|\partial_x|u) + \varepsilon\,\frac{\rho}{\rho_1^2}\partial_x(\eta|\partial_x|(\eta u))\,, \\ \partial_t u = {} & -g(\rho-\rho_1)\partial_x\eta + \frac{1}{\rho_1}u\partial_x u - \varepsilon\,\frac{\rho h_1}{\rho_1^2}\partial_x(u|\partial_x|u) + \varepsilon\,\frac{\rho}{\rho_1^2}\partial_x(u|\partial_x|(\eta u))\,. \end{aligned}$$

These equations are fully nonlinear in the sense that the dispersive terms containing the highest spatial derivatives are nonlinear in the variables (η, u).

Burgers-like equations: We derive equations for principally right-moving solutions using the same procedure as before. Using the new variables r and s defined in (5.31) with $s \simeq O(\varepsilon^2)$, the Hamiltonian (5.39) can be expressed as

$$
\begin{aligned}
H = & \frac{1}{\varepsilon} \int_{\mathbb{R}} \sqrt{\frac{gh_1(\rho-\rho_1)}{4\rho_1}} r^2\, dx - \frac{1}{2\varepsilon} \int_{\mathbb{R}} \frac{1}{2\rho_1} \sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{4h_1}} r^3\, dx \\
& - \frac{1}{2} \int_{\mathbb{R}} \frac{\rho h_1^2}{\rho_1^2} \sqrt{\frac{g\rho_1(\rho-\rho_1)}{4h_1}} r|\partial_x| r\, dx - \frac{1}{2} \int_{\mathbb{R}} \frac{\rho}{4\rho_1^2} r^2 |\partial_x| r^2\, dx \\
& + \frac{1}{2} \int_{\mathbb{R}} \frac{\rho h_1}{\rho_1^2} \sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{4h_1}} r^2 |\partial_x| r\, dx\, .
\end{aligned} \tag{5.41}
$$

Retaining only terms of order $O(1/\varepsilon)$ in (5.41), the evolution of r is governed by

$$
\partial_t r = -\sqrt{\frac{gh_1(\rho-\rho_1)}{\rho_1}} \partial_x r + \frac{3\sqrt{2}}{4\rho_1} \sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}} r\partial_x r\, . \tag{5.42}
$$

As expected, this is the inviscid Burgers equation. The next order terms in (5.41) introduce some nonlinear dispersion in the equation, yielding

$$
\begin{aligned}
\partial_t r = & -\sqrt{\frac{gh_1(\rho-\rho_1)}{\rho_1}} \partial_x r + \frac{3\sqrt{2}}{4\rho_1} \sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}} r\partial_x r \\
& + \frac{\varepsilon}{2} \frac{\rho h_1^2}{\rho_1^2} \sqrt{\frac{g\rho_1(\rho-\rho_1)}{h_1}} |\partial_x| \partial_x r + \frac{\varepsilon}{2} \frac{\rho}{\rho_1^2} \partial_x (r|\partial_x| r^2) \\
& - \frac{\sqrt{2}}{2} \varepsilon \frac{\rho h_1}{\rho_1^2} \sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}} \Big[\partial_x (r|\partial_x| r) + |\partial_x| (r\partial_x r) \Big]\, .
\end{aligned} \tag{5.43}
$$

5.5 The intermediate long wave (ILW) regime

Shifting attention to a different situation, we now derive several model equations for waves at the interface between two layers of fluid, where the top fluid has finite depth, while the depth of the lower layer is taken to be comparable to the wavelength λ of a typical wave. However, the lower layer is now taken to be finite, with a depth assumed to be comparable to the wavelength of a typical wave. As before, we investigate two situations. First, it is assumed that the amplitude of a wave is small when compared to the height h_1 of the upper layer. In this case, one of the model equations that appears is the well-known ILW equation, as derived by Joseph [19] and Kubota, Ko and Dobbs [22]. In the second case, the amplitude of a typical wave is not small when compared to depth of the upper layer. This situation can be called the shallow-water regime, and as was mentioned already, its importance in the the present context stems from the observation that internal waves may have amplitudes which are comparable to the depth of the upper layer. As the calculations are nearly identical to the previous configuration (Sections 5.3 and 5.4), only the resulting equations are given here.

ILW equations: We use the same scaling as in (5.28)

$$x' = \varepsilon x, \qquad \varepsilon \eta' = \eta, \qquad \xi' = \xi, \tag{5.44}$$

where $\varepsilon^2 \simeq (h_1/\lambda)^2 \simeq (a/h_1)^2 \ll 1$. However we will additionally assume $\varepsilon h \simeq O(1)$. Retaining terms of up to order $O(\varepsilon^3)$, the Hamiltonian can be expressed (after dropping the primes) as

$$\begin{aligned} H = {} & \frac{\varepsilon}{2}\frac{h_1}{\rho_1}\int_{\mathbb{R}} u^2\,dx + \frac{\varepsilon}{2} g(\rho-\rho_1)\int_{\mathbb{R}} \eta^2\,dx \\ & - \frac{\varepsilon^2}{2}\frac{\rho h_1^2}{\rho_1^2}\int_{\mathbb{R}} u\mathscr{T}_h\partial_x u\,dx - \frac{\varepsilon^2}{2\rho_1}\int_{\mathbb{R}} \eta u^2\,dx \\ & - \frac{\varepsilon^3}{2}\frac{h_1}{\rho_1}\int_{\mathbb{R}} (\partial_x u)\left(\frac{\rho^2 h_1^2}{\rho_1^2}\mathscr{T}_h^2 + \frac{h_1^2}{3}\right)\partial_x u\,dx \\ & + \varepsilon^3 \frac{\rho h_1}{\rho_1^2}\int_{\mathbb{R}} u\mathscr{T}_h\partial_x(\eta u)\,dx \; + \; O(\varepsilon^4), \end{aligned} \tag{5.45}$$

where $\mathscr{T}_h$ denotes the Fourier multiplier $-i\coth(\varepsilon hD)$. This operator reduces to the Hilbert transform in the limit $h \to \infty$. Neglecting terms of order $O(\varepsilon^3)$ in (5.45) yields the following equations for η and u

$$\begin{aligned} \partial_t \eta &= -\frac{h_1}{\rho_1}\partial_x u + \varepsilon \frac{\rho h_1^2}{\rho_1^2}\mathscr{T}_h\partial_x^2 u + \frac{\varepsilon}{\rho_1}\partial_x(\eta u), \\ \partial_t u &= -g(\rho-\rho_1)\partial_x \eta + \frac{\varepsilon}{\rho_1} u\partial_x u, \end{aligned} \tag{5.46}$$

and, when terms of order $O(\varepsilon^3)$ are retained but higher order terms truncated,

$$\begin{aligned} \partial_t \eta = {} & -\frac{h_1}{\rho_1}\partial_x u + \varepsilon \frac{\rho h_1^2}{\rho_1^2}\mathscr{T}_h\partial_x^2 u + \frac{\varepsilon}{\rho_1}\partial_x(\eta u) - \varepsilon^2 \frac{h_1}{\rho_1}\left(\frac{\rho^2 h_1^2}{\rho_1^2}\mathscr{T}_h^2 + \frac{h_1^2}{3}\right)\partial_x^3 u \\ & - \varepsilon^2 \frac{\rho h_1}{\rho_1^2}\mathscr{T}_h\partial_x^2(\eta u) - \varepsilon^2 \frac{\rho h_1}{\rho_1^2}\partial_x(\eta\mathscr{T}_h\partial_x u), \\ \partial_t u = {} & -g(\rho-\rho_1)\partial_x \eta + \frac{\varepsilon}{\rho_1} u\partial_x u - \varepsilon^2 \frac{\rho h_1}{\rho_1^2}\partial_x(u\mathscr{T}_h\partial_x u). \end{aligned} \tag{5.47}$$

Disregarding the linear dispersive terms in (5.46) leads to the same linear hyperbolic system as in the previous section.

The corresponding one-way equations for r are, respectively

$$\partial_\tau r = \frac{\rho h_1^2}{2\rho_1^2}\sqrt{\frac{g\rho_1(\rho-\rho_1)}{h_1}}\mathscr{T}_h\partial_x^2 r + \frac{3\sqrt{2}}{4\rho_1}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}} r\partial_x r\,, \tag{5.48}$$

and

$$
\begin{aligned}
\partial_\tau r = {} & \frac{\rho h_1^2}{2\rho_1^2}\sqrt{\frac{g\rho_1(\rho-\rho_1)}{h_1}}\mathscr{T}_h\partial_x^2 r + \frac{3\sqrt{2}}{4\rho_1}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}}\, r\partial_x r \\
& -\frac{\varepsilon}{2}\left(\frac{\rho^2 h_1^2}{\rho_1^2}\mathscr{T}_h^2 + \frac{h_1^2}{3}\right)\sqrt{\frac{gh_1(\rho-\rho_1)}{\rho_1}}\partial_x^3 r \\
& -\frac{\sqrt{2}}{2}\varepsilon\frac{\rho h_1}{\rho_1^2}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}}\Big[\partial_x(r\mathscr{T}_h\partial_x r) + \mathscr{T}_h\partial_x(r\partial_x r)\Big] ,
\end{aligned}
\tag{5.49}
$$

for which we have used the same transformation (5.31) as in the BO regime. Equation (5.48) is the ILW equation as derived in [19], while (5.49) gives the corrections to it at the next order of approximation.

Burgers-like equations: If the wave amplitude is not assumed to be small, then the present analysis is very similar to the case of infinite depth. At first order, the shallow-water equations (5.40) are obtained. If higher order terms are included, the Hamiltonian can be written as

$$
\begin{aligned}
H = {} & \frac{h_1}{2\varepsilon\rho_1}\int_{\mathbb{R}} u^2\,dx + \frac{1}{2\varepsilon}g(\rho-\rho_1)\int_{\mathbb{R}}\eta^2\,dx - \frac{1}{2\varepsilon\rho_1}\int_{\mathbb{R}}\eta u^2\,dx \\
& -\frac{\rho h_1^2}{2\rho_1^2}\int_{\mathbb{R}} u\mathscr{T}_h\partial_x u\,dx + \frac{\rho h_1}{\rho_1^2}\int_{\mathbb{R}} u\mathscr{T}_h\partial_x(\eta u)\,dx \\
& -\frac{\rho}{2\rho_1^2}\int_{\mathbb{R}}\eta u\mathscr{T}_h\partial_x(\eta u)\,dx + O(\varepsilon) ,
\end{aligned}
\tag{5.50}
$$

and the resulting approximate equations of motion read

$$
\begin{aligned}
\partial_t\eta = {} & -\frac{h_1}{\rho_1}\partial_x u + \frac{1}{\rho_1}\partial_x(\eta u) + \varepsilon\frac{\rho h_1^2}{\rho_1^2}\mathscr{T}_h\partial_x^2 u - \varepsilon\frac{\rho h_1}{\rho_1^2}\mathscr{T}_h\partial_x^2(\eta u) \\
& -\varepsilon\frac{\rho h_1}{\rho_1^2}\partial_x(\eta\mathscr{T}_h\partial_x u) + \varepsilon\frac{\rho}{\rho_1^2}\partial_x\Big(\eta\mathscr{T}_h\partial_x(\eta u)\Big) , \\
\partial_t u = {} & -g(\rho-\rho_1)\partial_x\eta + \frac{1}{\rho_1}u\partial_x u \\
& -\varepsilon\frac{\rho h_1}{\rho_1^2}\partial_x(u\mathscr{T}_h\partial_x u) + \varepsilon\frac{\rho}{\rho_1^2}\partial_x\Big(u\mathscr{T}_h\partial_x(\eta u)\Big) .
\end{aligned}
\tag{5.51}
$$

The corresponding equation for the right-moving component r is

$$
\begin{aligned}
\partial_t r = &-\sqrt{\frac{gh_1(\rho-\rho_1)}{\rho_1}}\partial_x r + \frac{3\sqrt{2}}{4\rho_1}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}}\, r\partial_x r \\
&+\frac{\varepsilon}{2}\frac{\rho h_1^2}{\rho_1^2}\sqrt{\frac{g\rho_1(\rho-\rho_1)}{h_1}}\mathscr{T}_h\partial_x^2 r + \frac{\varepsilon}{2}\frac{\rho}{\rho_1^2}\partial_x(r\mathscr{T}_h\partial_x r^2) \\
&-\frac{\sqrt{2}}{2}\varepsilon\frac{\rho h_1}{\rho_1^2}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}}\Big[\partial_x(r\mathscr{T}_h\partial_x r) + \mathscr{T}_h\partial_x(r\partial_x r)\Big]\ .
\end{aligned}
\tag{5.52}
$$

6 Long wave expansions for free surfaces and interfaces

In this section, we consider the more general situation in which the upper fluid layer is bounded on top by a free surface. We restrict our analysis to the case of two finite layers. The case of an infinite lower layer involves multiple space and time scales as suggested by (4.15), and so it should be described by a modulational analysis as has been done in the context of the surface water wave problem. This interesting regime is beyond the scope of the present paper. Here similarly to Section 5.1, we assume that both interfacial and surface waves are of small amplitude, and they are long (of comparable wavelengths) compared to the layer depths. Our goal is to quantify the differences between the rigid lid and free surface configurations.

6.1 Regime of two finite layers

The general expression of the Hamiltonian in the configuration with one free surface and one free interface is

$$
\begin{aligned}
H = &\frac{1}{2}\int_{\mathbb{R}} \xi GB^{-1}G_{11}\,\xi - \xi GB^{-1}G_{12}\,\xi_1 - \xi_1 G_{21}B^{-1}G\xi + \frac{1}{\rho_1}\xi_1 G_{22}\,\xi_1 \\
&-\frac{\rho}{\rho_1}\xi_1 G_{21}B^{-1}G_{12}\,\xi_1 + g(\rho-\rho_1)\eta^2 + g\rho_1\eta_1^2\, dx\ .
\end{aligned}
\tag{6.1}
$$

Let us consider first the case where both the internal and surface waves are long and of small amplitude according to the scaling

$$
x' = \varepsilon x\ , \qquad \varepsilon^2\eta' = \eta\ , \qquad \varepsilon\xi' = \xi\ , \qquad \varepsilon^2\eta_1' = \eta_1\ , \qquad \varepsilon\xi_1' = \xi_1\ . \tag{6.2}
$$

The Hamiltonian up to order $O(\varepsilon^5)$ can be written (after dropping the primes) as

$$
\begin{aligned}
H = \frac{\varepsilon^3}{2}\int_{\mathbb{R}} & g(\rho-\rho_1)\eta^2 + g\rho_1\eta_1^2 + \frac{h}{\rho}u^2 + \frac{2h}{\rho}uu_1 + \frac{1}{\rho\rho_1}(\rho_1 h+\rho h_1)u_1^2 \\
& - \frac{\varepsilon^2}{3}\frac{h^2}{\rho^2}(\rho h + 3\rho_1 h_1)(\partial_x u)^2 - \frac{\varepsilon^2}{3}\frac{h}{\rho^2}\Big(2\rho h^2 + 6\rho_1 hh_1 + 3\rho h_1^2\Big)(\partial_x u)(\partial_x u_1) \\
& - \frac{\varepsilon^2}{3\rho^2\rho_1}\Big(\rho^2 h_1^3 + \rho\rho_1 h^3 + 3\rho\rho_1 hh_1^2 + 3\rho_1^2 h^2 h_1\Big)(\partial_x u_1)^2 + \frac{\varepsilon^2}{\rho}\eta u^2 \\
& + \frac{2\varepsilon^2}{\rho}\eta uu_1 - \frac{\varepsilon^2}{\rho\rho_1}(\rho-\rho_1)\eta u_1^2 + \frac{\varepsilon^2}{\rho_1}\eta_1 u_1^2\, dx + O(\varepsilon^7)\ ,
\end{aligned}
\tag{6.3}
$$

in terms of $u = \partial_x\xi$ and $u_1 = \partial_x\xi_1$. It turns out that some contributions from $G_{j\ell}^{(2,0)}$, $G_{j\ell}^{(0,2)}$ and $G_{j\ell}^{(1,1)}$ also come out at order $O(\varepsilon^5)$ in the Hamiltonian but they cancel in the systematic treatment. The equations of motion for the interface and surface are therefore approximated by

$$
\begin{aligned}
\partial_t\eta &= -\frac{h}{\rho}\partial_x u - \frac{h}{\rho}\partial_x u_1 - \frac{\varepsilon^2}{\rho}\partial_x(\eta u) - \frac{\varepsilon^2}{\rho}\partial_x(\eta u_1) \\
&\quad - \frac{\varepsilon^2}{3\rho^2}\Big(\rho h^3 + 3\rho_1 h^2 h_1\Big)\partial_x^3 u - \frac{\varepsilon^2}{6\rho^2}\Big(2\rho h^3 + 6\rho_1 h^2 h_1 + 3\rho hh_1^2\Big)\partial_x^3 u_1\ , \\
\partial_t u &= -g(\rho-\rho_1)\partial_x\eta - \frac{\varepsilon^2}{\rho}u\partial_x u - \frac{\varepsilon^2}{\rho}\partial_x(uu_1) + \frac{\varepsilon^2}{\rho\rho_1}(\rho-\rho_1)u_1\partial_x u_1\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_t\eta_1 &= -\frac{h}{\rho}\partial_x u - \frac{1}{\rho\rho_1}(\rho_1 h + \rho h_1)\partial_x u_1 - \frac{\varepsilon^2}{\rho}\partial_x(\eta u) + \frac{\varepsilon^2}{\rho\rho_1}(\rho-\rho_1)\partial_x(\eta u_1) \\
&\quad - \frac{\varepsilon^2}{\rho_1}\partial_x(\eta_1 u_1) - \frac{\varepsilon^2}{6\rho^2}\Big(2\rho h^3 + 6\rho_1 h^2 h_1 + 3\rho hh_1^2\Big)\partial_x^3 u \\
&\quad - \frac{\varepsilon^2}{3\rho^2\rho_1}\Big(\rho^2 h_1^3 + \rho\rho_1 h^3 + 3\rho\rho_1 hh_1^2 + 3\rho_1^2 h^2 h_1\Big)\partial_x^3 u_1\ , \\
\partial_t u_1 &= -g\rho_1\partial_x\eta_1 - \frac{\varepsilon^2}{\rho_1}u_1\partial_x u_1\ .
\end{aligned}
$$

This set of equations represents the fully coupled Boussinesq system of the free surface/free interface problem.

6.2 The KdV regime for the interface

Because the interface and the free surface are coupled at first order in the Hamiltonian (6.3), we perform a normal mode decomposition of the system (see Section

3.2) by applying successively the canonical transformations

$$\begin{pmatrix}\eta'\\ \eta_1'\\ u'\\ u_1'\end{pmatrix} = \begin{pmatrix}\sqrt{g(\rho-\rho_1)} & 0 & 0 & 0\\ 0 & \sqrt{g\rho_1} & 0 & 0\\ 0 & 0 & \frac{1}{\sqrt{g(\rho-\rho_1)}} & 0\\ 0 & 0 & 0 & \frac{1}{\sqrt{g\rho_1}}\end{pmatrix} \begin{pmatrix}\eta\\ \eta_1\\ u\\ u_1\end{pmatrix} , \tag{6.4}$$

and

$$\begin{pmatrix}\mu\\ \mu_1\\ v\\ v_1\end{pmatrix} = \begin{pmatrix}a^- & b^- & 0 & 0\\ a^+ & b^+ & 0 & 0\\ 0 & 0 & a^- & b^-\\ 0 & 0 & a^+ & b^+\end{pmatrix} \begin{pmatrix}\eta\\ \eta_1\\ u\\ u_1\end{pmatrix} , \tag{6.5}$$

where $(a^\pm, b^\pm)^T$ are the eigenvectors corresponding to $(c_0^\pm)^2$ defined in (4.14). They are given by

$$a^\pm = \frac{1}{\sqrt{1+(d^\pm)^2}} , \qquad b^\pm = \frac{d^\pm}{\sqrt{1+(d^\pm)^2}} , \tag{6.6}$$

with

$$d^\pm = \frac{1}{h\sqrt{\rho_1(\rho-\rho_1)}} \left(\rho_1 h + \frac{1}{2}\rho h_1 - \frac{1}{2}\rho h \pm \frac{1}{2}\rho\sqrt{(h-h_1)^2 + 4\frac{\rho_1}{\rho} h h_1}\right) . \tag{6.7}$$

For simplicity, we will still refer to (μ, v) as the interfacial modes and to (μ_1, v_1) as the surface modes. However the reader should keep in mind that the new variables are linear combinations of both (η, u) and (η_1, u_1) according to (6.5).

Boussinesq system: Assuming additionally that the free surface is of smaller amplitude than the interface, with the scaling

$$\varepsilon\mu_1' = \mu_1 , \qquad \varepsilon v_1' = v_1 , \tag{6.8}$$

the resulting Hamiltonian can be expressed (after dropping the primes) as

$$H = \frac{\varepsilon^3}{2}\int_{\mathbb{R}} \mu^2 + (c_0^-)^2 v^2 + \varepsilon^2\Big(\mu_1^2 + (c_0^+)^2 v_1^2 + \mathscr{D}(\partial_x v)^2 + \mathscr{N}\mu v^2\Big)\,dx + O(\varepsilon^7) , \tag{6.9}$$

where

$$\begin{aligned}\mathscr{D} = &-\frac{gh^2}{3\rho^2}(\rho-\rho_1)(\rho h + 3\rho_1 h_1)(a^-)^2\\ &-\frac{gh}{3\rho^2}\sqrt{\rho_1(\rho-\rho_1)}\left(2\rho h^2 + 6\rho_1 h h_1 + 3\rho h_1^2\right) a^- b^-\\ &-\frac{g}{3\rho^2}\left(\rho^2 h_1^3 + \rho\rho_1 h^3 + 3\rho\rho_1 h h_1^2 + 3\rho_1^2 h^2 h_1\right)(b^-)^2 ,\end{aligned} \tag{6.10}$$

and

$$\begin{aligned}\mathscr{N} \;=\; & \frac{\sqrt{g(\rho-\rho_1)}}{\rho}(a^-)^3 + 2\frac{\sqrt{g\rho_1}}{\rho}(a^-)^2 b^- \\ & - \frac{\sqrt{g(\rho-\rho_1)}}{\rho} a^-(b^-)^2 + \sqrt{\frac{g}{\rho_1}}(b^-)^3 \;.\end{aligned} \tag{6.11}$$

The corresponding system of equations of motion takes the form

$$\partial_t \begin{pmatrix}\mu \\ \mu_1 \\ v \\ v_1\end{pmatrix} = \begin{pmatrix} 0 & 0 & -\partial_x & 0 \\ 0 & 0 & 0 & -\varepsilon^{-2}\partial_x \\ -\partial_x & 0 & 0 & 0 \\ 0 & -\varepsilon^{-2}\partial_x & 0 & 0 \end{pmatrix} \delta H(\mu,\mu_1,v,v_1)\;. \tag{6.12}$$

More explicitly, we have

$$\begin{aligned}\partial_t\mu \;&=\; -\partial_x\Big((c_0^-)^2 v - \varepsilon^2(\mathscr{D}\partial_x^2 v - \mathscr{N}\mu v)\Big)\;, \\ \partial_t v \;&=\; -\partial_x\left(\mu + \tfrac{1}{2}\varepsilon^2\mathscr{N} v^2\right)\;,\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}\partial_t\mu_1 \;&=\; -\partial_x\Big((c_0^+)^2 v_1\Big)\;, \\ \partial_t v_1 \;&=\; -\partial_x\mu_1\;,\end{aligned} \tag{6.14}$$

for the interface and free surface respectively. At this order of approximation, their evolutions are decoupled. The evolution of the interface is governed by a Boussinesq-type system of equations, while the evolution of the free surface is purely linear.

KdV equation: If we make the further change of variables

$$\begin{pmatrix} r \\ s\end{pmatrix} = \begin{pmatrix} \sqrt[4]{\frac{1}{4(c_0^-)^2}} & \sqrt[4]{\frac{(c_0^-)^2}{4}} \\ \sqrt[4]{\frac{1}{4(c_0^-)^2}} & -\sqrt[4]{\frac{(c_0^-)^2}{4}} \end{pmatrix} \begin{pmatrix}\mu \\ v\end{pmatrix}\;, \tag{6.15}$$

while leaving (μ_1, v_1) unchanged, and restrict our attention to right-moving solutions for the interface by assuming $s \simeq O(\varepsilon^2)$, the Hamiltonian (6.9) becomes

$$H = \frac{\varepsilon^3}{2}\int_{\mathbb{R}} c_0^- r^2\,dx + \frac{\varepsilon^5}{2}\int_{\mathbb{R}} \mu_1^2 + (c_0^+)^2 v_1^2 + \frac{\mathscr{D}}{2c_0^-}(\partial_x r)^2 + \frac{\mathscr{N}}{2\sqrt{2c_0^-}} r^3\,dx\;. \tag{6.16}$$

The right-moving component r thus satisfies the KdV equation

$$\partial_t r = -c_0^-\partial_x r + \varepsilon^2\frac{\mathscr{D}}{2c_0^-}\partial_x^3 r - \varepsilon^2\frac{3\mathscr{N}}{2\sqrt{2c_0^-}} r\partial_x r\;, \tag{6.17}$$

which can be simplified into

$$\partial_\tau r = \frac{\mathscr{D}}{2c_0^-}\partial_x^3 r - \frac{3\mathscr{N}}{2\sqrt{2c_0^-}} r\partial_x r \ , \tag{6.18}$$

in the reference frame moving with velocity c_0^- and evolving over time scale $\tau = \varepsilon^2 t$.

Comparison with the rigid lid case: In order to quantify the differences between the rigid lid and coupled cases in the Boussinesq regime, we plot in figure 6.1 the ratios of nonlinearity to dispersion for both configurations as functions of the density ratio ρ_1/ρ. For the interface in the rigid lid configuration, the coefficients of nonlinearity and dispersion are given in (5.8), and the corresponding ratio reads

$$R_L = -\frac{3(\rho h_1^2 - \rho_1 h^2)}{(hh_1)^2(\rho_1 h_1 + \rho h)\sqrt{g(\rho-\rho_1)}} \ . \tag{6.19}$$

For the interface in the coupled configuration, the ratio of nonlinearity to dispersion is

$$R_S = \frac{\mathscr{N}}{\mathscr{D}} \ , \tag{6.20}$$

where $\mathscr{D}$ and $\mathscr{N}$ are given by (6.10) and (6.11). Note that there is an extra factor $\sqrt{g(\rho-\rho_1)}$ in (6.19) due to the renormalization of the term in η^2 in (5.8), as this should be consistent with the coupled Hamiltonian (6.9) in which the term in μ^2 is normalized. Figure 6.1 shows the comparison between R_L and R_S for eight different values of the depth ratio $h_1/h = 10, 1.5, 1.2, 1.1, 1.05, 1, 0.8, 0.4$. It is clear that there are significant differences between these two cases. First, one can see that R_L is always negative for $h_1/h > 1$, while R_S is always positive for $h_1/h < 1$. The ratio R_L changes sign only once in the range $\rho_1/\rho \in (0,1)$ for $h_1/h < 1$. On the contrary, R_S changes sign once and then twice as h_1/h increases from 1. This property has important implications since the sign of the ratio determines the polarity of solitary wave solutions (i.e. of elevation or depression). Benjamin [3] found that, in the rigid lid case, the sign of R_L changes for $\rho_1/\rho = (h_1/h)^2$. We note that there is a widely varying difference between the sign of R_S and that of R_L for many parameter choices. Regarding the relative importance of nonlinearity and dispersion, it is observed that, for $\rho_1/\rho \simeq 0.9$ (which is close to realistic conditions), both $R_L, R_S \simeq O(1)$ in magnitude when $h_1/h \simeq 1$ and larger. This observation also holds true for a smaller density ratio, say $\rho_1/\rho \simeq 0.2$. As expected, the nonlinear effects prevail over the dispersive effects when h_1/h is small. We can nevertheless conclude that the Boussinesq and KdV regimes for the interface, in which dispersive and nonlinear effects are balanced, remain valid over a significant range of parameters.

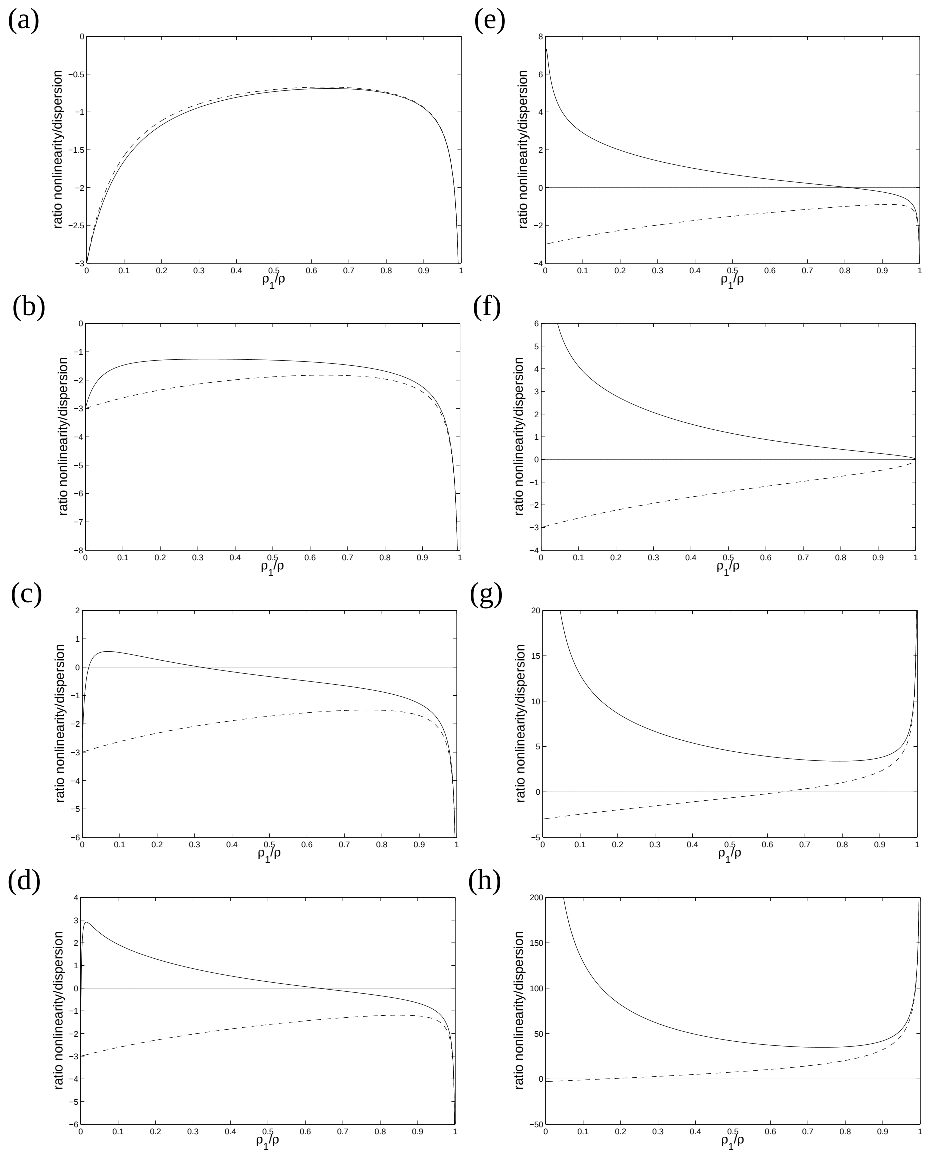

FIGURE 6.1. Ratio of nonlinearity to dispersion in the Boussinesq regime vs. density ratio ρ_1/ρ for (a) $h_1/h = 10$, (b) $h_1/h = 1.5$, (c) $h_1/h = 1.2$, (d) $h_1/h = 1.1$, (e) $h_1/h = 1.05$, (f) $h_1/h = 1$, (g) $h_1/h = 0.8$, (h) $h_1/h = 0.4$. The ratios for the interface in the rigid lid approximation and in the coupled system are represented in dashed and solid lines respectively.

7 Conclusions

In this paper we derive a Hamiltonian formulation for the problem of coupled free interface and free surface wave motion, in the spirit of the Hamiltonian given by Benjamin and Bridges [5] and Craig and Groves [10] for the case of one free interface with an upper rigid lid. Our Hamiltonian corrects the one proposed by Ambrosi [1]. We use the Hamiltonian for the free interface problem and the Hamiltonian for the free surface/free interface problem to develop a systematic long wave perturbation analysis, based on a perturbation theory for Hamiltonian PDE which we give in Section 2. In this section we take the opportunity to systematize a sense of canonical transformations for PDE, in particular in the context of a variety of scaling transformations that are employed in the long wave perturbation analysis.

Using the framework of Hamiltonian perturbation theory, we derive in a uniform and systematic way the principal nonlinear dispersive equations of the long wave, small amplitude scaling regimes. In case of the free interface problem bounded above by a rigid lid, and in the presence of a finite bottom to the fluid region, we derive in particular the Boussinesq system (5.9), and the classical KdV equation (5.16) given in Benjamin [3]. We extend the derivation to the higher order analogs of these equations, such as the Kawahara equation (5.21). We note that the extended Boussinesq system (5.20) that arises at this order of perturbation theory is a natural regularization of (5.9) at high wavenumbers.

The case in which the lower fluid layer is infinitely deep and the upper layer remains bounded by a rigid lid (or vice versa, with appropriate changes of sign) was studied by Benjamin [4] and Ono [26]. In this setting we derive the Benjamin-Ono equation (5.34) and its bi-directional Boussinesq-like variant (5.30) which has been studied by Choi and Camassa [6]. From our point of view, the perturbation analysis associates naturally a Hamiltonian function with the equations of motion. We also derive the higher order extensions of these two systems, (5.36) and (5.37). The extended Boussinesq-like system (5.36) represents a natural regularization of the system of Choi & Camassa (5.30).

In the third and intermediate regime of two finite layers, with one layer asymptotically thin, the result is the ILW equation (5.48). Again there is an analogous Boussinesq-like counterpart (5.46) and both of these equations can be carried to higher order in a straightforward manner, resulting in the extended Boussinesq-like system (5.47) and its uni-directional counterpart (5.49). The latter system is the extension of the ILW equation (5.48).

While the Benjamin-Ono regime above allows interface deformations which are an order of magnitude larger than the KdV regime, nonetheless amplitudes are assumed to be small when compared with the depth of the fluid layers themselves. This does not necessarily hold true in the ocean. We find that by systematically working within a regime of small slope, but making no assumptions whatever on the smallness of amplitudes, there is a well defined perturbation regime which accommodates deformations of the free interface which are of the same order of magnitude as the depth of each of the fluid layers. This small slope regime corresponds most closely to the observed scales in oceanic internal waves; the resulting Hamiltonian systems of equations are of novel form, involving coefficients of dispersion and nonlinearity which are themselves rational functions of the interface displacement. In the case of two finite fluid layers, the resulting equations (5.27) are not unrelated to those given in Choi and Camassa [7]; we have described them in a previous announcement [11]. Similar systems of equations occur in the infinitely deep setting, where we find a Boussinesq-like system of equations with nonlinear dispersive terms. A similar equation (5.52), with nonlinear dispersive terms and the finite depth Hilbert transform, is derived in the regime of the intermediate long wave scaling, using only the small steepness assumption throughout.

Turning to the situation in which a free surface bounds the upper fluid layer in addition to the free interface between the two fluids, we have focused on the setting of two finite layers. We have worked through the long-wave perturbation analysis for the Boussinesq and KdV scalings, and compared the resulting model equations with the case of rigid lid upper boundary conditions. We have found a number of significant differences between the two cases. Even at the level of the linear dispersion relation, the linear phase and group velocities can differ. We show that for small values of the density difference $\rho-\rho_1$, the differences are small between the rigid lid and the free surface cases. However there can be significant deviations when the difference in densities is large; this is illustrated in figure 1 with a number of choices of parameters. The deviations are most important when the ratio h_1/h is small, as one would expect. On the level of the nonlinearity and dispersion present in the problem, which are reflected in the coefficients of the KdV model equations, the differences between the two upper boundary conditions can be very important. In figure 2 we have plotted the ratio of the coefficients of nonlinearity to dispersion as a function of $\rho_1/\rho \in (0,1)$, for a number of choices of h_1/h. The sign of the ratio determines the geometry of solitary waves (whether positive or negative) and the size of the ratio indicates the relative importance of the two competing phenomena. From Benjamin [3] it is known that the rigid lid case changes sign once at most, at the critical value $\rho_1/\rho = (h_1/h)^2$ (when the latter quantity lies in the interval $(0,1)$). In contrast, the same ratio of nonlinearity to dispersion can behave completely differently in the case of the upper free surface. It can change sign once or twice as ρ_1/ρ varies over $(0,1)$, in situations in which rigid lid conditions predict no sign change. It also has different behavior at the singular limits $\rho_1/\rho = 0$ or 1 in many cases. As many models assume rigid lid conditions, we feel it important to understand the differences that are apparent in this behavior.

When one pursues a similar line of perturbation analysis with an upper free surface and in the presence of an infinite lower layer, there are multiple space and time scales present in the problem, and a modulational regime of analysis is called for. This interesting regime is beyond the scope of the present paper; one presumably encounters surface ripple effects due to the presence of large internal waves, which strike us as a possibly very realistic prediction.

There are a number of perspectives for future research that are put forward by this perturbation analysis. We would like to understand the free surface/free interface system more thoroughly, including the effects of the interface on the surface modes. The novel nonlinear dispersive systems which model large amplitude free interface motions are very interesting, and merit a thorough analysis, perhaps first with a numerical study of their solitary wave solutions. In addition there is the potential for numerical simulations of the initial value problem, based on the evaluation of the Dirichlet-Neumann operators as in [15], and the comparison with the data of Grue *et al.* [18] and Segur and Hammack [29] for counter- and co-propagating solitary wave solutions in the interface and on the free surface.

Our methods are not restricted to two-dimensional flows, and it would be worthwhile to extend the analysis to the full three-dimensional setting. The approach can also be applied in principle to systems with bottom topography (see [12]) and consisting of multiple layers of immiscible fluids separated by sharp free interfaces, with possibly one free surface lying over the region occupied by the fluid.

Appendix: The Dirichlet-Neumann operator

In this appendix we give an analysis of the Dirichlet-Neumann operators for the lower fluid region $S(\eta)$ and the upper region $S(\eta,\eta_1)$. Given the data $\Phi(x)$ posed on the interface $\{(x,y) : y = \eta(x)\}$, the operator $G(\eta)$ for the lower fluid region returns the (non-normalized) normal derivative of the velocity potential

$$G(\eta)\Phi(x) = \nabla\varphi \cdot N(1+|\partial_x\eta|^2)^{1/2} \; , \tag{A.1}$$

satisfying Neumann boundary conditions on the fixed bottom $\{y = -h\}$;

$$\begin{aligned} \Delta\varphi &= 0 \; , & \text{for } (x,y) \in S(\eta) \\ \varphi(x,\eta(x)) &= \Phi(x) \; , & -\partial_y\varphi(x,-h) = 0 \; . \end{aligned} \tag{A.2}$$

The Dirichlet-Neumann operator for the upper fluid domain $S_1(\eta,\eta_1)$ gives the (non-normalized) normal derivatives of φ_1 on the two boundaries, from the boundary values of the velocity potential $\varphi_1(x,y)$ on the two boundaries as data. Namely, let $\varphi_1(x,y)$ solve the equation

$$\begin{aligned} \Delta\varphi_1 &= 0 \; , & \text{for } (x,y) \in S_1(\eta,\eta_1) \\ \varphi_1(x,\eta(x)) &= \Phi_1(x) \; , & \varphi_1(x,h_1+\eta_1(x)) = \Phi_2(x) \; . \end{aligned} \tag{A.3}$$

Let $-N(x)$ be the exterior unit normal to $S_1(\eta,\eta_1)$ on its lower boundary (since $N(x)$ is the exterior unit normal to the lower domain $S(\eta)$), and let $N_1(x)$ be the exterior unit normal to the upper boundary of $S_1(\eta,\eta_1)$. The Dirichlet-Neumann operator is the following matrix operator

(A.4)
$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix} = \begin{pmatrix} -(\nabla\varphi_1 \cdot N)(x,\eta(x))(1+(\partial_x\eta(x))^2)^{1/2} \\ (\nabla\varphi_1 \cdot N_1)(x,h_1+\eta_1(x))(1+(\partial_x\eta_1(x))^2)^{1/2} \end{pmatrix} ,$$

which appears in (2.20). This matrix operator is analytic in its dependence on the domain, as parametrized locally by the two functions $\eta(x)$ and $\eta_1(x)$. Its Taylor expansion in (η,η_1) about zero plays a useful rôle in the systematic long wave expansions of this paper. We derive expressions for the Taylor expansion of the Dirichlet-Neumann operator (A.4) which are explicit in their dependence upon (η,η_1), and where the Taylor coefficients are in fact recursively defined (and can be, for example, calculated by a computer up to arbitrary order). This is a similar situation to the case of the Dirichlet-Neumann operator $G(\eta)$ for the domain $S(\eta)$, where only the top boundary is perturbed. The analogous Taylor expansion and recursion formula for the Taylor coefficients of $G(\eta)$ appears in [15, 13]. We review the computation in this appendix for the convenience of the reader, before giving the more complicated Taylor series for (A.4).

A.1 Lower fluid domain $S(\eta)$

Start with the case of the operator $G(\eta)$ for the lower fluid domain $S(\eta)$. A particular basis of harmonic functions is given by $\varphi_k(x,y) = a(k)e^{ky}e^{ikx} + b(k)e^{-ky}e^{ikx}$. Satisfying the bottom boundary conditions in (A.2) we find that $a(k) = e^{kh}/(e^{kh} + e^{-kh})$ and $b(k) = e^{-kh}/(e^{kh}+e^{-kh})$. Its boundary values on the free surface are

$$\Phi_k(x) = \varphi_k(x,\eta(x)) = \sum_{j\geq 0} \frac{1}{j!}\eta^j(x)k^j \left(\frac{e^{kh}}{e^{kh}+e^{-kh}} + (-1)^j \frac{e^{-kh}}{e^{kh}+e^{-kh}} \right) e^{ikx} \tag{A.5}$$

which has the normalization property that $\varphi_k(x,0) = e^{ikx}$. Relating the normal derivative of $\varphi_k(x,y)$ on the free surface,

$$\begin{aligned} &\nabla\varphi_k(x,y)\cdot N(1+|\partial_x\eta(x)|^2)^{1/2}|_{y=\eta(x)} \\ &= \sum_{j\geq 0} \frac{1}{j!}\eta^j(x)(-\partial_x\eta(x))(ik^{j+1}) \left(\frac{e^{kh}}{e^{kh}+e^{-kh}} + (-1)^j \frac{e^{-kh}}{e^{kh}+e^{-kh}} \right) e^{ikx} \\ &\quad + \sum_{j\geq 0} \frac{1}{j!}\eta^j(x)(k^{j+1}) \left(\frac{e^{kh}}{e^{kh}+e^{-kh}} + (-1)^{j+1} \frac{e^{-kh}}{e^{kh}+e^{-kh}} \right) e^{ikx} , \end{aligned} \tag{A.6}$$

to the Taylor series expansion of $G(\eta)\Phi_k$, the constant term is $G^{(0)}e^{ikx} = k\tanh(hk)e^{ikx}$. Writing this Fourier multiplication operator in terms of $D = -i\partial_x$, it reads

$$G^{(0)}e^{ikx} = D\tanh(hD)e^{ikx} . \tag{A.7}$$

Reading the higher terms of the Taylor expansion from (A.5)(A.6), we find

$$G^{(j)}(\eta)e^{ikx} = \frac{1}{j!}D\eta^j(x)D^j\left(\frac{e^{hD}}{e^{hD}+e^{-hD}}+(-1)^{j+1}\frac{e^{-hD}}{e^{hD}+e^{-hD}}\right)e^{ikx}$$

$$- \sum_{\ell=1}^{j} G^{(j-\ell)}(\eta)\frac{1}{\ell!}\eta^\ell(x)D^\ell\left(\frac{e^{hD}}{e^{hD}+e^{-hD}}+(-1)^{\ell}\frac{e^{-hD}}{e^{hD}+e^{-hD}}\right)e^{ikx} , \tag{A.8}$$

from which one can read in a recursive manner the expressions for the Taylor coefficients of $G(\eta)$ as a function of η. In particular one has the first and second order terms

$$G^{(1)}(\eta) = D\eta(x)D - G^{(0)}\eta(x)G^{(0)} , \tag{A.9}$$
$$G^{(2)}(\eta) = -\tfrac{1}{2}\left(D^2\eta^2(x)G^{(0)} + G^{(0)}\eta^2(x)D^2 - 2G^{(0)}\eta(x)G^{(0)}\eta(x)G^{(0)}\right) ,$$

which appear in [15]. In practice in numerical computations involving the numerical Fourier transform, it is more efficient in terms of computational time and memory to use the adjoint of the formula to (A.8), as this only requires vector operations; this has been pointed out in [13].

There is an analogous expression for the Dirichlet-Neumann operator $G_1(\eta)$ for the upper doman $S_1(\eta)$ in the case of the problem of a single free interface, with Neumann boundary conditions posed on the rigid lid $\{y = h_1\}$. It is obtained from (A.8) by substituting h_1 for h and $-\eta(x)$ for $\eta(x)$, and in particular the first three terms in the Taylor expansion are

$$G_1^{(0)} = D\tanh(h_1D) , \qquad G_1^{(1)}(\eta) = -D\eta(x)D + G_1^{(0)}\eta(x)G_1^{(0)} , \tag{A.10}$$
$$G_1^{(2)}(\eta) = -\tfrac{1}{2}\left(D^2\eta^2(x)G_1^{(0)} + G_1^{(0)}\eta^2(x)D^2 - 2G_1^{(0)}\eta(x)G_1^{(0)}\eta(x)G_1^{(0)}\right) .$$

A.2 Upper fluid domain $S_1(\eta,\eta_1)$

In the problem with a free surface coupled to a free interface, we need to address the Dirichlet-Neumann operator (A.4) for the upper domain $S_1(\eta,\eta_1)$. Consider the family of harmonic functions $\varphi_{1,k}(x,y) = (a(k)e^{ky} + b(k)e^{-ky})e^{ikx}$ which solve (A.3) with the boundary values

$$\Phi_{1,k}(x) = (a(k)e^{k\eta(x)} + b(k)e^{-k\eta(x)})e^{ikx} \qquad \text{on } y = \eta(x) \tag{A.11}$$

$$\Phi_{2,k}(x) = (a(k)e^{kh_1}e^{k\eta_1(x)} + b(k)e^{-kh_1}e^{-k\eta_1(x)})e^{ikx} \quad \text{on } y = h_1 + \eta_1(x) . \tag{A.12}$$

As in (A.5), these expressions have convergent Taylor expansions in η and in η_1 respectively;

$$\Phi_{1,k}(x) = \sum_{j\geq 0}\frac{1}{j!}\eta^j(x)k^j\left(a(k) + (-1)^j b(k)\right)e^{ikx} \tag{A.13}$$

$$\Phi_{2,k}(x) = \sum_{j\geq 0}\frac{1}{j!}\eta_1^j(x)k^j\left(a(k)e^{kh_1} + (-1)^j b(k)e^{-kh_1}\right)e^{ikx} . \tag{A.14}$$

The exterior normal derivatives of φ_1 on the two boundaries are given by

$$\begin{aligned} &-\nabla\varphi_{1,k}\cdot N(1+|\partial_x\eta(x)|^2)^{1/2}|_{y=\eta(x)} \\ &= \sum_{j\geq 0}\frac{1}{j!}\eta^j(x)(i\partial_x\eta(x))k^{j+1}(a(k)+(-1)^j b(k))e^{ikx} \\ &\quad - \sum_{j\geq 0}\frac{1}{j!}\eta^j(x)k^{j+1}(a(k)+(-1)^{j+1}b(k))e^{ikx} \end{aligned} \tag{A.15}$$

and

$$\begin{aligned} &\nabla\varphi_{1,k}\cdot N_1(1+|\partial_x\eta_1(x)|^2)^{1/2}|_{y=h_1+\eta_1(x)} \\ &= \sum_{j\geq 0}\frac{1}{j!}\eta_1^j(x)(-i\partial_x\eta_1(x))k^{j+1}(a(k)e^{h_1k}+(-1)^j b(k)e^{-h_1k})e^{ikx} \\ &\quad + \sum_{j\geq 0}\frac{1}{j!}\eta_1^j(x)k^{j+1}(a(k)e^{h_1k}+(-1)^{j+1}b(k)e^{-h_1k})e^{ikx}\ . \end{aligned} \tag{A.16}$$

Using (A.13), (A.14) (A.15), (A.16), the relation (A.4) can be solved for expressions for the Taylor coeffi cients of the Dirichlet-Neumann operator as a double power series in η and η_1. For this, one takes a basis of the harmonic functions (A.11)(A.12), by setting in turn $(a_1(k),b_1(k)) = (-e^{-h_1k}/(e^{h_1k}-e^{-h_1k}), e^{h_1k}/(e^{h_1k}-e^{-h_1k}))$, $(a_2(k),b_2(k)) = (1/(e^{h_1k}-e^{-h_1k}), -1/(e^{h_1k}-e^{-h_1k}))$. First of all, from direct comparison in the relation (A.4) one fi nds that the constant term in the Taylor expansion is

$$\begin{pmatrix} G_{11}^{(0)} & G_{12}^{(0)} \\ G_{21}^{(0)} & G_{22}^{(0)} \end{pmatrix} = \begin{pmatrix} D\coth(h_1D) & -D\mathrm{csch}(h_1D) \\ -D\mathrm{csch}(h_1D) & D\coth(h_1D) \end{pmatrix}\ . \tag{A.17}$$

We denote the general term in the Taylor expansion by $G_{j\ell}^{(m_0,m_1)}$, where $j,\ell = 1,2$, which is homogeneous of degree m_0 in η and of degree m_1 in η_1, so that the operator can be written

$$\begin{pmatrix} G_{11}(\eta,\eta_1) & G_{12}(\eta,\eta_1) \\ G_{21}(\eta,\eta_1) & G_{22}(\eta,\eta_1) \end{pmatrix} = \sum_{m_1,m_2=0}^{\infty} \begin{pmatrix} G_{11}^{(m_0,m_1)}(\eta,\eta_1) & G_{12}^{(m_0,m_1)}(\eta,\eta_1) \\ G_{21}^{(m_0,m_1)}(\eta,\eta_1) & G_{22}^{(m_0,m_1)}(\eta,\eta_1) \end{pmatrix}\ .$$

The fi rst order terms are of particular importance in the long wave expansions of this paper. From (A.13), (A.14), (A.15), (A.16) and the relation (A.4) we fi nd

$$\begin{pmatrix} G_{11}^{(10)}(\eta,\eta_1) & G_{12}^{(10)}(\eta,\eta_1) \\ G_{21}^{(10)}(\eta,\eta_1) & G_{22}^{(10)}(\eta,\eta_1) \end{pmatrix} =$$
$$\begin{pmatrix} D\coth(h_1D)\eta(x)D\coth(h_1D) - D\eta(x)D & -D\coth(h_1D)\eta(x)D\mathrm{csch}(h_1D) \\ -D\mathrm{csch}(h_1D)\eta(x)D\coth(h_1D) & D\mathrm{csch}(h_1D)\eta(x)D\mathrm{csch}(h_1D) \end{pmatrix}\ .$$

Similarly,

$$\begin{pmatrix} G_{11}^{(01)}(\eta,\eta_1) & G_{12}^{(01)}(\eta,\eta_1) \\ G_{21}^{(01)}(\eta,\eta_1) & G_{22}^{(01)}(\eta,\eta_1) \end{pmatrix} =$$
$$\begin{pmatrix} -D\operatorname{csch}(h_1D)\eta_1(x)D\operatorname{csch}(h_1D) & D\operatorname{csch}(h_1D)\eta_1(x)D\coth(h_1D) \\ D\coth(h_1D)\eta_1(x)D\operatorname{csch}(h_1D) & -D\coth(h_1D)\eta_1(x)D\coth(h_1D)+D\eta_1(x)D \end{pmatrix} .$$

There is a recursion formula for the higher order terms in the Taylor series expansion for $G_{j\ell}^{(m)}(\eta,\eta_1)$, analogous to the concise formula (A.8). We distinguish two cases. The first is the special case where $m=(m_0,0)$ or $(0,m_1)$, and the second is the more general case, where $m=(m_0,m_1)$ where neither $m_0,m_1=0$. In the first case let $m=(m_0,0)$. Then we can read from the matrix equation (A.4), using (A.13), (A.14), (A.15), and (A.16), the following expressions for the matrix coefficients: the (11)-coefficient is

$$\begin{aligned} G_{11}^{(m_0,0)}(\eta) &= \frac{1}{m_0!}D\eta^{m_0}(x)D^{m_0}\left(\frac{e^{-h_1D}}{e^{h_1D}-e^{-h_1D}}+\frac{(-1)^{m_0}e^{h_1D}}{e^{h_1D}-e^{-h_1D}}\right) \\ &\quad + \sum_{\substack{p_0\geq 1\\ q_0+p_0=m_0\\ q_1=0=p_1}} G_{11}^{(q_0,0)}(\eta)\frac{1}{p_0!}\eta^{p_0}(x)D^{p_0}\left(\frac{e^{-h_1D}}{e^{h_1D}-e^{-h_1D}}+\frac{(-1)^{p_0+1}e^{h_1D}}{e^{h_1D}-e^{-h_1D}}\right), \end{aligned} \tag{A.18}$$

the (21)-coefficient is

$$G_{21}^{(m_0,0)}(\eta) = \sum_{\substack{p_0\geq 1\\ q_0+p_0=m_0\\ q_1=0=p_1}} G_{21}^{(q_0,0)}(\eta)\frac{1}{p_0!}\eta^{p_0}(x)D^{p_0}\left(\frac{e^{-h_1D}}{e^{h_1D}-e^{-h_1D}}+\frac{(-1)^{p_0+1}e^{h_1D}}{e^{h_1D}-e^{-h_1D}}\right), \tag{A.19}$$

the (12)-coefficient is

$$\begin{aligned} G_{12}^{(m_0,0)}(\eta) &= -\frac{1}{m_0!}D\eta^{m_0}(x)D^{m_0}\left(\frac{1}{e^{h_1D}-e^{-h_1D}}+\frac{(-1)^{m_0}}{e^{h_1D}-e^{-h_1D}}\right) \\ &\quad - \sum_{\substack{p_0\geq 1\\ q_0+p_0=m_0\\ q_1=0=p_1}} G_{11}^{(q_0,0)}(\eta)\frac{1}{p_0!}\eta^{p_0}(x)D^{p_0}\left(\frac{1}{e^{h_1D}-e^{-h_1D}}+\frac{(-1)^{p_0+1}}{e^{h_1D}-e^{-h_1D}}\right), \end{aligned} \tag{A.20}$$

and the (22)-coefficient is

$$G_{22}^{(m_0,0)}(\eta) = -\sum_{\substack{p_0\geq 1\\ q_0+p_0=m_0\\ q_1=0=p_1}} G_{21}^{(q_0,0)}(\eta)\frac{1}{p_0!}\eta^{p_0}(x)D^{p_0}\left(\frac{1}{e^{h_1D}-e^{-h_1D}}+\frac{(-1)^{p_0+1}}{e^{h_1D}-e^{-h_1D}}\right). \tag{A.21}$$

A recursive computation of $G_{j\ell}^{(m_0,0)}(\eta)$ can be based upon formula (A.18) for $G_{11}^{(m_0,0)}(\eta)$, $m_0>0$ and formula (A.19) for $G_{21}^{(m_0,0)}(\eta)$, $m_0>0$. This is sufficient information in order to calculate $G_{12}^{(m_0,0)}(\eta)$ and $G_{22}^{(m_0,0)}(\eta)$ from respectively (A.20) and (A.21).

It is a general fact that

$$G^{(m_0,m_1)}_{j\ell}(\eta,\eta_1) = G^{(m_1,m_0)}_{\ell j}(-\eta_1,-\eta) \tag{A.22}$$

which allows us to deduce the form of $G^{(0,m_1)}_{j\ell}(\eta_1)$, with $j,\ell = 1,2$ from the above expressions. As well, each matrix operator $G^{(m)}_{j\ell}$ is self adjoint, which is not necessarily self evident from the above formulae. Thus in particular $(G^{(m)}_{12})^* = G^{(m)}_{21}$. Therefore the latter can be obtained from (A.20), which itself depends upon the recursion (A.18) alone.

The second case consists of those multi indices $m = (m_0,m_1)$ where neither m_0 nor m_1 vanish. The order m terms of the RHS of the relation (A.4) are zero, as is seen in (A.15)(A.16). Working as in the first case, we find expressions for the (11)-coefficient to be

$$\begin{aligned} G^{(m_0,m_1)}_{11}(\eta,\eta_1) = & \sum_{\substack{1\le p_0\le m_0\\ q_0+p_0=m_0\\ p_1=0}} G^{(q_0,m_1)}_{11}(\eta,\eta_1)\tfrac{1}{p_0!}\eta^{p_0}(x)D^{p_0}\left(\tfrac{e^{-h_1D}}{e^{h_1D}-e^{-h_1D}} + \tfrac{(-1)^{p_0+1}e^{h_1D}}{e^{h_1D}-e^{-h_1D}}\right) \\ & + \sum_{\substack{p_0=0\\ 1\le p_1\le m_1\\ q_1+p_1=m_1}} G^{(m_0,q_1)}_{12}(\eta,\eta_1)\tfrac{1}{p_1!}\eta_1^{p_1}(x)D^{p_1}\left(\tfrac{1}{e^{h_1D}-e^{-h_1D}} + \tfrac{(-1)^{p_1+1}}{e^{h_1D}-e^{-h_1D}}\right). \end{aligned}$$

The (21)-coefficient is

$$\begin{aligned} G^{(m_0,m_1)}_{21}(\eta,\eta_1) = & \sum_{\substack{1\le p_0\le m_0\\ q_0+p_0=m_0\\ p_1=0}} G^{(q_0,m_1)}_{21}(\eta,\eta_1)\tfrac{1}{p_0!}\eta^{p_0}(x)D^{p_0}\left(\tfrac{e^{-h_1D}}{e^{h_1D}-e^{-h_1D}} + \tfrac{(-1)^{p_0+1}e^{h_1D}}{e^{h_1D}-e^{-h_1D}}\right) \\ & + \sum_{\substack{p_0=0\\ 1\le p_1\le m_1\\ q_1+p_1=m_1}} G^{(m_0,q_1)}_{22}(\eta,\eta_1)\tfrac{1}{p_1!}\eta_1^{p_1}(x)D^{p_1}\left(\tfrac{1}{e^{h_1D}-e^{-h_1D}} + \tfrac{(-1)^{p_1+1}}{e^{h_1D}-e^{-h_1D}}\right). \end{aligned}$$

The (12)-matrix coefficient is the operator

$$\begin{aligned} G^{(m_0,m_1)}_{12}(\eta,\eta_1) = & -\sum_{\substack{1\le p_0\le m_0\\ q_0+p_0=m_0\\ p_1=0}} G^{(q_0,m_1)}_{11}(\eta,\eta_1)\tfrac{1}{p_0!}\eta^{p_0}(x)D^{p_0}\left(\tfrac{1}{e^{h_1D}-e^{-h_1D}} + \tfrac{(-1)^{p_0+1}}{e^{h_1D}-e^{-h_1D}}\right) \\ & - \sum_{\substack{p_0=0\\ 1\le p_1\le m_1\\ q_1+p_1=m_1}} G^{(m_0,q_1)}_{12}(\eta,\eta_1)\tfrac{1}{p_1!}\eta_1^{p_1}(x)D^{p_1}\left(\tfrac{e^{h_1D}}{e^{h_1D}-e^{-h_1D}} + \tfrac{(-1)^{p_1+1}e^{-h_1D}}{e^{h_1D}-e^{-h_1D}}\right). \end{aligned}$$

Finally, the (22)-matrix coefficient is

$$\begin{aligned} G^{(m_0,m_1)}_{22}(\eta,\eta_1) = & -\sum_{\substack{1\le p_0\le m_0\\ q_0+p_0=m_0\\ p_1=0}} G^{(q_0,m_1)}_{21}(\eta,\eta_1)\tfrac{1}{p_0!}\eta^{p_0}(x)D^{p_0}\left(\tfrac{1}{e^{h_1D}-e^{-h_1D}} + \tfrac{(-1)^{p_0+1}}{e^{h_1D}-e^{-h_1D}}\right) \\ & - \sum_{\substack{p_0=0\\ 1\le p_1\le m_1\\ q_1+p_1=m_1}} G^{(m_0,q_1)}_{22}(\eta,\eta_1)\tfrac{1}{p_1!}\eta_1^{p_1}(x)D^{p_1}\left(\tfrac{e^{h_1D}}{e^{h_1D}-e^{-h_1D}} + \tfrac{(-1)^{p_1+1}e^{-h_1D}}{e^{h_1D}-e^{-h_1D}}\right). \end{aligned}$$

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