Chapter 2

Wave equations

We will start the topic of PDEs and their solutions with a discussion of a class of wave equations, initially with several transport equations and then for the standard second order wave equation (1.9) in one space dimension. We will use this mission as motivation to introduce the Fourier transform.

2.1. Transport equations - the Fourier transform

The transport equation is a first order equation that describes the displacement of a quantity u in a background that possesses a (possibly) variable wave velocity $c(t,x)$, namely

$$\partial_t u + c(t,x) \cdot \partial_x u = 0 \ . \tag{2.1}$$

This typically gives rise to an *initial value problem*, where one asks that

$$u(0,x) = f(x) \ .$$

The function $f(x)$ is called the *initial data*. The problem is to describe how the solution $u(t,x)$ evolves, given its initial 'shape' $f(x)$ at $t = 0$. Often these equations are known as first order wave equations for the unknown function $u(t,x)$. The coefficient $c(t,x)$ is a vector, known as the 'speed of propagation', we will see the reason for this when we describe the solution process.

A first example is given by the case of c, the speed of propagation, is a constant; we will solve this using the Fourier transform. For convenience restrict ourselves to one space dimension. The process is elementary, but nevertheless it serves to introduce a surprisingly powerful technique for the study of PDEs.

Definition 2.1. The Fourier transform of a function $f(x)$ is given by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi y} f(y)\, dy := \mathcal{F} f(\xi) \ . \tag{2.2}$$

One aspect of the power of the Fourier transform is its invertibility.

Theorem 2.2 (Fourier inversion formula)**.** *Under suitable hypotheses on $f(x)$, there is an inversion formula for the Fourier transform*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi x} \hat{f}(\xi)\, d\xi \ .$$

At a later point we will discuss the hypotheses under which this theorem holds in much more detail; for now a perfectly reasonable condition would be that $f \in L^2(\mathbb{R})$, that is

$$\Big(\int_{-\infty}^{+\infty} |f(x)|^2 \, dx \Big)^{\frac{1}{2}} := \|f\|_{L^2(\mathbb{R})} < +\infty \ .$$

The Fouier transform is well adapted for representing derivatives.

Proposition 2.3. Under further suitable hypotheses,

$$\begin{aligned} \partial_x f(x) &= \frac{1}{\sqrt{2\pi}} \partial_x \int_{-\infty}^{+\infty} e^{-i\xi x} \hat{f}(\xi)\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \partial_x (e^{-i\xi x}) \hat{f}(\xi)\, d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi x}\, i\xi \hat{f}(\xi)\, d\xi \ . \end{aligned}$$

That is

$$\widehat{\partial_x f}(\xi) = i\xi \hat{f}(\xi) \ . \tag{2.3}$$

The analytic hypotheses will be needed in order to justify the Fourier inversion formula and the exchange of the derivative ∂_x with the integral in the above calculation.

Let us assume that the propagation speed c in (2.1) is constant, and that the solution $u(t, x)$ satisfies the required hypotheses of Proposition 2.3 at each time t (to ultimately have a rigorous argument this assumption must be verified, but for the present we will not do so). Then in order to be a solution of (2.1) the function

$$u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\xi x} \hat{u}(t, \xi)\, d\xi$$

must satisfy

$$(\partial_t + c\partial_x) u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\xi x} (\partial_t \hat{u} + ci\xi \hat{u})(t, \xi)\, d\xi = 0 \ .$$

Figure 1. This is an example of the evolution of a solution of a transport equation.

Since the Fourier transform is invertible, this means that for all $\xi \in \mathbb{R}^1$,

$$\partial_t \hat{u}(t,\xi) + ic\xi\hat{u}(t,\xi) = 0 \ ,$$

and this is a straightforward ordinary differential equation (ODE) in time for each value of the Fourier transform variable ξ;

$$\partial_t v(t) + ic\xi v(t) = 0 \ ,$$

with solution

$$v(t) = e^{-ic\xi t}v(0) = e^{-ic\xi t}\hat{f}(\xi) \ .$$

Therefore an expression for our solution is

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\xi x}\hat{u}(t,\xi)\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\xi x}\big(e^{-ic\xi t}\hat{f}(\xi)\big)\,d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\xi(x-ct)}\big(\hat{f}(\xi)\big)\,d\xi = f(x-ct) \ . \end{aligned}$$

The Fourier inversi on formula was used in the last line. A picture of the evolution of a solution is given in Figure 1.

You could ask whether the identical procedure works for the case of a variable speed of propagation $c(t,x)$; it does not in general, nor for a time independent speed $c(x)$. However when $c(t)$ is independent of x the answer is 'yes' and a similar solution representation is possible. This is given as an exercise at the end of the section.

2.2. Transport equations - the method of characteristics

There is a second method for solving equations of the form (2.1) which doesn't involve the Fourier transform. Again for simplicity we will take the case that $x \in \mathbb{R}^1$. Let $f \in C^1(\mathbb{R}^1)$, the space of once continuously differentiable functions, and examine the equation

$$(\partial_t + c\partial_x)u = 0 \ . \tag{2.4}$$

This is the statement that the directional derivative of $u(t,x)$ in the $(1,c)$ direction in the (t,x) plane vanishes;

$$(1,c)\cdot(\partial_t,\partial_x)u = 0 \ .$$

That is to say, $u(t,x)$ is constant along lines with tangent direction $(1,c)$,

Figure 2. Geometric interpretation of a transport equation.

Figure 3. Characteristic curves of the vector field (2.6).

and therefore for all x, x' such that $(x - x') = ct$ we have $u(t,x) = u(0,x') = f(x - ct)$. Verifying this for any initial data $f \in C^1(\mathbb{R}^1)$

$$\partial_t u(t,x) = \partial_t f(x - ct) = -cf'(x - ct) \ ,$$
$$c\partial_x u(t,x) = c\partial_x f(x - ct) = cf'(x - ct) \ ,$$

and the sum of the two terms cancels.

This observation extends to give a solution method for the general transport equation, which uses ordinary differential equations and which is known as the method of characteristics. Consider the equation

$$\begin{aligned} \partial_t u + c(t,x)\partial_x u &= 0 \ , \\ u(0,x) &= f(x) \ . \end{aligned} \tag{2.5}$$

Interpret the statement (2.5) in terms of directional derivatives

$$(1, c(t,x)) \cdot (\partial_t, \partial_x) u = 0 \ ,$$

which in turn is the statement that $u(t,x)$ is constant along integral curves of the vector field $X = (1, c(t,x))$ in the (t,x) plane;

$$X\, u(t,x) = (\partial_t + c(t,x)\partial_x) u = 0 \ . \tag{2.6}$$

The *characteristic equations* for the transport equation (2.5) are

$$\frac{dt}{ds} = 1 \ , \qquad \frac{dx}{ds} = c(t,x) \ , \tag{2.7}$$

whose solutions are the integral curves for X, called the *characteristic curves* of (2.5). This family of curves $(t(s) = s, x(s,\beta))$ is parametrized by the points $\beta \in \mathbb{R}^1$ at which the curves intersect the initial surface $\{t = 0\}$. Now define the solution by $u(t,x) = f(\beta)$, where $\beta = \beta(t,x)$ is the point of intersection with the initial surface of the characteristic curve which passes through (t,x). To verify that our construction satisfies the partial differential equation (2.5), evaluate $u = u(t(s), x(s))$ along a characteristic curve and use the chain rule

$$\begin{aligned} 0 &= \frac{d}{ds} u(t(s), x(s)) \\ &= \partial_t u \frac{dt}{ds} + \partial_x u \frac{dx}{ds} = \partial_t u + c(t, x)\, \partial_x u \ . \end{aligned}$$

Two examples of transport equations with differing behavior are given by the following:

$$\partial_t u + x \partial_x u = 0 \ , \tag{2.8}$$

and

$$\partial_t u + x^2 \partial_x u = 0 \ . \tag{2.9}$$

Exercise 2.3 asks to solve these two equations, giving a solution representation for the initial value problem $u(0, x) = f(x)$, using explicit expressions for the respective functions $\beta(t, x)$.

2.3. The d'Alembert formula

We now take up the study of the second order wave equation, starting with the one space dimensional case. In one dimension there is a particularly simple and explicit formula for solutions of

$$\begin{aligned} &\partial_t^2 u - \partial_x^2 u = 0 \ , \\ &u(0, x) = f(x) \ , \quad \partial_t u(0, x) = g(x) \ . \end{aligned} \tag{2.10}$$

The initial data comprises the two functions f and g that are to be specified. Introduce characteristic coordinates

$$x + t := r \ , \quad x - t := s \ ,$$

so that the differential operators are transformed as follows:

$$\partial_t - \partial_x = 2\partial_r \ , \quad \partial_t + \partial_x = -2\partial_s \ ,$$

then equation (2.10) is transformed to

$$(\partial_t - \partial_x)(\partial_t + \partial_x) u = -4 \partial_r \partial_s u = 0 \ . \tag{2.11}$$

The meaning of equation (2.10) is now interpreted to be that $\partial_s(\partial_r u) = 0$, which is to say that $\partial_r u$ is independent of the variable s. That is,

$$\partial_r u = v(r)$$

for some function $v(r)$. Therefore, using $W = W(s)$ as an integration constant in the r-integral,

$$u(r, s) = \int^r v(r')\, dr' + W(s) := V(r) + W(s) \ .$$

Figure 4. Solution of (2.10) using the d'Alembert formula (2.13), exhibiting the dependence on the initial data.

Returning to the original variables (t, x) the solution takes the form

$$u(t,x) = V(x+t) + W(x-t) \ , \tag{2.12}$$

which has the interpretation of a decomposition of the solution $u(t, x)$ into a left-moving component V and a right-moving component W. That is, any solution of the one-dimensional wave equation (2.10) can be decomposed into the sum of a left-moving wave $V(x + t)$ and a right-moving wave $W(x - t)$.

The components V and W are to be expressed in terms of the initial data $u(0, x) = f(x)$ and $\partial_t u(0, x) = g(x)$. In order to do this, evaluate the expression (2.12) of the decomposition on the surface $x = 0$;

$$\begin{aligned} u(0,x) &= V(x) + W(x) = f(x) \ , \\ \partial_t u(0,x) &= V'(x) - W'(x) = g(x) \ . \end{aligned}$$

Differentiating the first line with respect to x and solving for V' and W', we find

$$V'(x) = \tfrac{1}{2} f'(x) + \tfrac{1}{2} g(x) \ , \quad W'(x) = \tfrac{1}{2} f'(x) - \tfrac{1}{2} g(x) \ ;$$

thus

$$\begin{aligned} V(x) &= \tfrac{1}{2} f(x) + \tfrac{1}{2} \int_0^x g(x')\, dx' + c_1 \\ W(x) &= \tfrac{1}{2} f(x) - \tfrac{1}{2} \int_0^x g(x')\, dx' + c_2 \ , \end{aligned}$$

where necessarily $c_1 + c_2 = 0$. This is to say that

$$\begin{aligned} u(t,x) &= V(x+t) + W(x-t) \\ &= \tfrac{1}{2}\big(f(x+t) + f(x-t)\big) + \tfrac{1}{2} \int_{x-t}^{x+t} g(x')\, dx' \ . \end{aligned} \tag{2.13}$$

This is known as *d'Alembert's formula* for solutions of the one dimensional wave equation. We note that the wave equation is a second order partial differential equation, however (2.13) is well defined for data $f, g \in C(\mathbb{R}^1)$ (continuous functions) and gives a certain sense of a solution even though the resulting expression for $u(t, x)$ is not necessarily C^2. It gives a sense of *weak solution* to the wave equation in this case, a topic that will be explored more deeply at a later point.

Figure 5. The d'Alembert formula (2.13) exhibits the principle of finite propagation speed.

The solution formula (2.13) shows how the solution u at a space-time point depends upon the initial data. Indeed at the point (t, x) the solution $u(t, x)$ depends only upon data in the backward *light cone*, and in particular it depends upon f at the two points $x \pm t$, and on $g(x')$ only over the interval $x - t \le x' \le x + t$. This is an instance of solutions satisfying the principle of *finite propagation speed*. Defining the support of a function to be $\mathrm{supp}(h) := \overline{\{x \in \mathbb{R}^1 : h(x) \neq 0\}}$, and taking both (f, g) to have compact support, then if $\mathrm{supp}(f), \mathrm{supp}(g) \subseteq [-R, +R]$, then the support of the solution $u(t, x)$ satisfies $\mathrm{supp}(u(t, x)) \subseteq [-R - |t|, R + |t|]$ as can be seen in Figure 5. At time t the solution vanishes outside of the region $\{-R - |t| \le x \le R + |t|\}$, this property of finite propagation speed is called *Huygens' principle*, and is a common feature of solutions of wave equations. Furthermore, it is evident from the d'Alembert formula and Figure 5 that the solution $u(t, x)$ is constant *inside* the union of light cones emanating from the support of the initial data. Indeed, for $t > R$ and then for $|x| < t - R$,

$$u(t, x) = \tfrac{1}{2} \int_{x-t}^{x+t} g(x')\, dx' = \tfrac{1}{2} \int_{-R}^{+R} g(x')\, dx'$$

which is a constant, independent of both (t, x). This property is called the *strong Huygens' principle*; it holds for solutions of the wave equation (1.9) in odd space dimensions, but not in even space dimensions nor for most other wave equations.

2.4. The method of images

We have been discussing the initial value problem, or the *Cauchy problem* for the one dimensional wave equation, with initial data given for $x \in \mathbb{R}^1$. However in many situations the spatial domain has boundaries, on which the solution is normally required to satisfy additional boundary conditions. Such problems are called *initial - boundary value problems*. For the one dimensional problem let us take the case of the spatial domain $\mathbb{R}^1_+ := \{x > 0\}$, with the boundary at $x = 0$. Two typical boundary conditions are *Dirichlet conditions*

$$u(t, 0) = 0 \ ,$$

Figure 6. The method of images for Dirichlet boundary conditions.

and alternatively *Neumann conditions*

$$\partial_x u(t, 0) = 0 \ .$$

Of course if the support of the initial data does not include $x = 0$, by the property of finite propagation speed the solution will satisfy the boundary conditions for a certain time interval. However even in this case the situation will not in general last for all time $t \in \mathbb{R}^1$. The method of images is a classical technique used to solve these two cases, and as well it works for some others.

The method as applied to the problem of Dirichlet boundary conditions is based on the simple fact that a function that is odd, $h(-x) = -h(x)$ and continuous at zero must necessarily satisfy $h(0) = 0$. Since the spatial domain is $\mathbb{R}^1_+$, the initial data $f(x), g(x)$ are only defined for $x \geq 0$. Assume for simplicity that both $f, g \in C(\overline{\mathbb{R}^1_+})$. We require that both $f(0), g(0) = 0$ as compatibility conditions between the initial and the boundary values of the solution. The method of images consists in defining $f(x) = -f(-x)$ and $g(x) = -g(-x)$ for $x < 0$, the odd reflections of $f(x)$ and $g(x)$ through the origin; these are defined and continuous on all of $x \in \mathbb{R}^1$. The d'Alembert formula gives a solution to the Cauchy problem, namely

$$u(t, x) = \tfrac{1}{2}\big(f(x+t) + f(x-t)\big) + \tfrac{1}{2}\int_{x-t}^{x+t} g(x')\, dx' \ .$$

For any $x > t$ this expression only involves the arguments of f, g for positive x, however for $x < t$ we use the fact that f, g are odd to rewrite the expression as

$$u(t, x) = \tfrac{1}{2}\big(f(t+x) - f(t-x)\big) + \tfrac{1}{2}\int_{t-x}^{x+t} g(x')\, dx' \ .$$

The effect is that the solution satisfies Dirichlet boundary conditions as the expression for u is odd in x for each t; it is as though the solution is reflected off of the boundary with a negative mirror image of the incident signal, as in the Figure 6

In the case of Neumann boundary conditions at $x = 0$, ask that the boundary data $f, g \in C^1(\overline{\mathbb{R}^1_+})$ and that it satisfy the compatibility conditions that

$$\partial_x f(0) = 0 = \partial_x g(0) \ .$$

Figure 7. The method of images for Neumann boundary conditions.

Then reflect f, g to be even functions of $x \in \mathbb{R}^1$. We use the fact that an even function $h(x)$ whose first derivative is continuous at $x = 0$ must satisfy $\partial_x h(0) = 0$. Using the even property in the d'Alembert formula, for $x < t$

$$\begin{aligned} u(t,x) &= \tfrac{1}{2}\big(f(x+t) + f(x-t)\big) + \tfrac{1}{2}\int_{x-t}^{x+t} g(x')\,dx' \\ &= \tfrac{1}{2}\big(f(t+x) + f(t-x)\big) + \tfrac{1}{2}\int_{t-x}^{t+x} g(x')\,dx' + \int_{0}^{t-x} g(x')\,dx' \ . \end{aligned}$$

The reflection of the solution off of the boundary is now a positive mirror image of the incident signal. There is also a component of the solution coming from the continuing reflection off the boundary of the contribution of g.

Exercises: Chapter 2

Exercise 2.1. For the wave speed $c(t)$ which is a general (locally integrable) function of time, give a Fourier integral representation of the solution of the transport equation (2.1).

Exercise 2.2. Find an explicit expression for the solution of the equation

$$\partial_t u + c\partial_x u - ru = 0 \ , \qquad u(0,x) = f(x) \ ,$$

using either (or both) of the above solution methods. When $r > 0$ what happens to the solution $u(t,x)$ asymptotically for $t \to +\infty$.

Exercise 2.3. Give the solution of both equations (2.8) and (2.9) using the method of characteristics. Then differentiate to check your answer. Explain the differences between the two cases. Are the solutions unique in both cases? What are the asymptotic states

$$v(x) = \lim_{t\to+\infty} u(t,x) \ ,$$

if the limit exists, that is.