Chapter 4

Laplace's equation

4.1. Dirichlet, Poisson and Neumann boundary value problems

The most commonly occurring form of problem that is associated with Laplace's equation is a boundary value problem, normally posed on a domain $\Omega \subseteq \mathbb{R}^n$. That is, Ω is an open set of $\mathbb{R}^n$ whose boundary is smooth enough so that integrations by parts may be performed, thus at the very least rectifiable. The most common boundary value problem is the *Dirichlet problem*:

$$\begin{aligned} \Delta u(x) = 0 \ , \qquad & x \in \Omega \\ u(x) = f(x) \ , \qquad & x \in \partial\Omega \ . \end{aligned} \tag{4.1}$$

The function $f(x)$ is known as the Dirichlet data; physically it corresponds to a density of charge dipoles fixed on the boundary $\partial\Omega$, whereupon the solution $u(x)$ corresponds to the resulting electrostatic potential. A function satisfying $\Delta u = 0$ is called *harmonic*, as we have stated in Chapter 1.

Perhaps the second most common problem is called the *Poisson problem*;

$$\begin{aligned} \Delta u(x) = h(x) \ , \qquad & x \in \Omega \\ u(x) = 0 \ , \qquad & x \in \partial\Omega \ , \end{aligned} \tag{4.2}$$

for which the function $h(x)$ represents a distribution of fixed charges in the domain Ω , while the boundary $\partial\Omega$ is a perfect conductor. Again the solution $u(x)$ represents the resulting electrostatic potential. There are several other quite common boundary value problems that are similar in character to (4.1), for example the *Neumann problem*

$$\begin{aligned} \Delta u(x) &= 0\ , \qquad x \in \Omega \\ \partial_N u(x) &= g(x)\ , \qquad x \in \partial\Omega\ , \end{aligned} \tag{4.3}$$

where $N(x)$ is the outward unit normal vector to Ω and $\partial_N u(x) = \nabla u(x) \cdot N$. The solution corresponds to the electrostatic potential in Ω due to a charge density distribution on $\partial\Omega$. The *Robin problem*, or boundary value problem of the third kind, asks to find $u(x)$ such that

$$\begin{aligned} \Delta u(x) &= 0\ , \qquad x \in \Omega \\ \partial_N u(x) - \beta u(x) &= g(x)\ , \qquad x \in \partial\Omega\ , \end{aligned} \tag{4.4}$$

where β is a real constant, or possibly a real function of $x \in \Omega$. Often this problem is associated with an imposed impedence on the boundary.

4.2. Green's identities

Consider two function $u(x)$ and $v(x)$ defined on a domain $\Omega \subseteq \mathbb{R}^n$. Calculus identities for integrations by parts give the following formulae, which is known as *Green's first identity*

$$\int_\Omega v \Delta u\, dx = -\int_\Omega \nabla v \cdot \nabla u\, dx + \int_{\partial\Omega} v \partial_N u\, dS_x\ . \tag{4.5}$$

The notation is that the differential of surface area in the integral over the boundary is dS_x. To ensure that the manipulations in this formula are valid we ask that $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$, that is, all derivatives of u and v up to second order are continuous in the interior of Ω and at least their first derivatives have continuous limits on the boundary $\partial\Omega$.

Integrating again by parts (or alternatively using (4.5) in a symmetric way with the roles of u and v reversed) we obtain *Green's second identity*

$$\int_\Omega v \Delta u\, dx - \int_\Omega \Delta v\, u\, dx = \int_{\partial\Omega} \big(v \partial_N u - \partial_N v u \big)\, dS_x\ . \tag{4.6}$$

for $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$. The integral over the boundary $\partial\Omega$ is the analog of the Wronskian in ODEs.

Considering the function v as a test function and substituting several astute choices for it into Green's identities, we obtain information about solutions u of Laplace's equation. First of all, let $v(x) = 1$, then (4.5) gives

$$\int_\Omega \Delta u\, dx = \int_{\partial\Omega} \partial_N u\, dS_x\ .$$

In case u is harmonic, then $\Delta u = 0$ and the LHS vanishes. This is a compatibility condition for boundary data $g(x) = \partial_N u$ for the Neumann problem.

Proposition 4.1. In order for the Neumann problem (4.3) to have a solution, the Neumann data $g(x)$ must satisfy

$$\int_{\partial\Omega} g(x)\, dS_x = \int_{\partial\Omega} \partial_N u(x)\, dS_x = 0 \ .$$

For a second choice, let $v(x) = u(x)$ itself. Then Green's first identity (4.5) is an 'energy' identity

$$\int_\Omega |\nabla u(x)|^2\, dx + \int_\Omega u\Delta u\, dx = \int_{\partial\Omega} u\partial_N u\, dS_x \ . \tag{4.7}$$

One consequence of this is a uniqueness theorem.

Theorem 4.2. *Suppose that $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfies the Dirichlet problem* (4.1) *with $f(x) = 0$, or the Poisson problem* (4.2) *with $h(x) = 0$, or the Neumann problem* (4.3) *with $g(x) = 0$. Then*

$$\int_\Omega |\nabla u(x)|^2\, dx = \int_{\partial\Omega} u\partial_N u\, dS_x = 0 \ .$$

Therefore $u(x) = 0$ in the case of the Dirichlet problem (4.3) *and of the Poisson problem* (4.2)*. In the case of the Neumann problem, the conclusion is that $u(x)$ is constant.*

Proof. The identity (4.7) implies that

$$\int_\Omega |\nabla u(x)|^2\, dx = 0$$

in each of the three cases, which in turn implies that $\nabla u = 0$ almost everywhere in Ω. Therefore $u(x)$ must be constant as we have assumed that $u \in C^2(\Omega)$. In cases (4.1) and (4.2) this constant must vanish in order to satisfy the boundary conditions. In the case of the Neumann problem (4.3) we only conclude that $u(x)$ is constant. In this situation we conclude that the constant functions $u(x) \equiv C$ span the null space of the Laplace operator with Neumann boundary conditions. □

4.3. Poisson kernel

This component of the course is dedicated to techniques based on the Fourier transform, For Laplace's equation we can do this directly only in particular cases, the most straightforward being that the domain $\Omega = \mathbb{R}^n_+ := \{x = (x_1, \dots x_n) \in \mathbb{R}^n \ : \ x_n > 0\}$, the half-space, and this is the situation that we will discuss. It would also be possible to directly use Fourier series to solve Laplace's equation on the disk $\mathbb{D}^2 := \{x \in \mathbb{R}^2 \ : \ |x| < 1\}$, or the polydisc $D^{2n} := \{z = (x_1, \dots x_n, y_1, \dots y_n) \in \mathbb{R}^{2n} \ : \ (x_j^2 + y_j^2) < 1 \ , j = 1, \dots n\}$. We will however stick with the half space $\mathbb{R}^n_+$.

The Dirichlet problem (4.1) on $\mathbb{R}^n_+$ is to solve

$$\begin{aligned} \Delta u(x) &= 0 \;, \qquad x = (x_1, \dots x_n) \in \mathbb{R}^n_+ \\ u(x) &= f(x') \;, \qquad x = (x', 0)\,, \quad x' \in \mathbb{R}^{n-1} = \partial\mathbb{R}^n_+ \;. \end{aligned} \tag{4.8}$$

It is implied that $u(x)$ and $\nabla u(x)$ tend to zero as $x_n \to +\infty$. For the special boundary data $f(x') = e^{i\xi'\cdot x'}$, with $\xi' \in \mathbb{R}^{n-1}$ there are explicit solutions

$$u(x', x_n) = e^{i\xi'\cdot x'} e^{-|\xi'|x_n} \;, \tag{4.9}$$

since

$$\begin{aligned} \Delta u(x', x_n) &= \sum_{j=1}^{n-1} \partial^2_{x_j}\big(e^{i\xi'\cdot x'} e^{-|\xi'|x_n}\big) + \partial^2_{x_n}\big(e^{i\xi'\cdot x'} e^{-|\xi'|x_n}\big) \\ &= \big(\sum_{j=1}^{n-1} -\xi'^2_j + |\xi'|^2\big)\big(e^{i\xi'\cdot x'} e^{-|\xi'|x_n}\big) = 0 \;. \end{aligned}$$

The other possible solution is $e^{i\xi'\cdot x'} e^{+|\xi'|x_n}$ but this is ruled out by its growth as $x_n \to +\infty$. The Fourier transform allows us to decompose a general function $f(x')$ on the boundary (in $L^2(\mathbb{R}^{n-1})$ or perhaps in $L^1(\mathbb{R}^{n-1})$) into a composite of complex exponentials

$$f(x') = \frac{1}{\sqrt{2\pi}^{\,n-1}} \int e^{i\xi'\cdot x'} \hat{f}(\xi')\, d\xi' \;.$$

By using (4.9) the solution $u(x)$ is expressed as a superposition

$$\begin{aligned} u(x) &= \frac{1}{\sqrt{2\pi}^{\,n-1}} \int e^{i\xi'\cdot x'} e^{-|\xi'|x_n} \hat{f}(\xi')\, d\xi' \\ &= \frac{1}{(2\pi)^{n-1}} \int \Big(\int e^{i\xi'\cdot(x'-y')} e^{-|\xi'|x_n}\, d\xi' \Big) f(y')\, dy' \\ &= \int D(x'-y', x_n) f(y')\, dy' \;. \end{aligned} \tag{4.10}$$

The function $D(x', x_n)$ is called the Poisson kernel for $\mathbb{R}^n_+$ or the double layer potential, and the solution $u(x)$ is evidently given by convolution with $D(x', x_n)$. Evaluating the above Fourier integral expression (4.10) we get an explicit expression,

$$D(x', x_n) = \frac{1}{(2\pi)^{n-1}} \int e^{i\xi'\cdot x'} e^{-|\xi'|x_n}\, d\xi' \tag{4.11}$$

$$= \frac{2}{\omega_n} \Big(\frac{x_n}{(|x'|^2 + x_n^2)^{n/2}} \Big) \;. \tag{4.12}$$

The second line of (4.11) will be verified later; the constant ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

Theorem 4.3. *The solution of the Dirichlet problem on $\mathbb{R}^n_+$ with data $f(x') \in L^2(\mathbb{R}^{n-1})$ is given by*

$$u(x) = \int_{\mathbb{R}^{n-1}} D(x'-y, x_n) f(y')\, dy' \tag{4.13}$$
$$= \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot(x'-y')} e^{-|\xi'_n| x_n}\, d\xi' \Big) f(y')\, dy'$$

For $x_n > 0$ this function is C^∞ (differentiable an arbitrary number of times).

Proof. For $x_n > 0$ both of the above integrals in (4.13) converge absolutely, as indeed we have

$$\int_{\mathbb{R}^{n-1}} \big| e^{i\xi'\cdot x'} e^{-|\xi'|x_n} \hat{f}(\xi') \big|\, d\xi' \le \|\hat{f}\|_{L^2(\mathbb{R}^{n-1})} \Big(\int_{\mathbb{R}^{n-1}} e^{-2|\xi'|x_n}\, d\xi' \Big)^{1/2} ,$$

(where we have used the Cauchy-Schwarz inequality and the Plancherel identity) and hence we also learn that the solution $u(x)$ has an upper bound in $\mathbb{R}^n_+$ which quantifies its decay rate in $x_n \to +\infty$. Namely

$$|u(x', x_n)| \le \Big(\int_{\mathbb{R}^{n-1}} e^{-2|\xi'|x_n}\, d\xi' \Big)^{1/2} \|\hat{f}\|_{L^2(\mathbb{R}^{n-1})} \le \frac{C}{|x_n|^{(n-1)/2}} \|f\|_{L^2(\mathbb{R}^{n-1})} .$$

Further derivatives of the expression (4.13) don't change the properties of absolute convergence of the integral for $x_n > 0$, and we verify that the function we have produced is indeed harmonic;

$$\Delta u(x) = \frac{1}{\sqrt{2\pi}^{n-1}} \int_{\mathbb{R}^{n-1}} \Delta \big(e^{i\xi'\cdot x'} e^{-|\xi'_n|x_n} \big) \hat{f}(\xi')\, d\xi' = 0 .$$

The issue is whether the harmonic function $u(x', x_n)$ that we have produced converges to $f(x')$ as $x_n \to 0$, and in what sense. In this paragraph we will show that for every $x_n > 0$, $u(x', x_n)$ is an $L^2(\mathbb{R}^{n-1})$ function of the horizontal variables $x' \in \mathbb{R}^{n-1}$, and that $u(x', x_n) \to f(x')$ in the $L^2(\mathbb{R}^{n-1})$ sense as $x_n \to 0$. By Plancherel,

$$\|u(x', x_n) - f(x')\|_{L^2(\mathbb{R}^{n-1})} = \|e^{-|\xi'|x_n} \hat{f}(\xi') - \hat{f}(\xi')\|_{L^2(\mathbb{R}^{n-1})} .$$

Because $\|\hat{f}\|_{L^2} < +\infty$, for any $\delta > 0$ there is a (possibly large) $R > 0$ such that the integral

$$\int_{|\xi'|>R} |\hat{f}(\xi')|^2\, d\xi' < \delta .$$

We now estimate

$$\|u(x', x_n) - f(x')\|^2_{L^2(\mathbb{R}^{n-1})} = \| \big(e^{-|\xi'|x_n} - 1 \big) \hat{f}(\xi') \|^2_{L^2(\mathbb{R}^{n-1})}$$
$$= \int_{|\xi'|\le R} \big| \big(e^{-|\xi'|x_n} - 1 \big) \hat{f}(\xi') \big|^2\, d\xi' + \int_{|\xi'|>R} \big| \big(e^{-|\xi'|x_n} - 1 \big) \hat{f}(\xi') \big|^2\, d\xi'$$
$$\le \big| e^{-Rx_n} - 1 \big| \int_{|\xi'|\le R} |\hat{f}(\xi')|^2\, d\xi' + \delta .$$

The first term of the RHS vanishes in the limit as $\delta \to 0$. Since δ is arbitrary, we conclude that $u(x', x_n)$ converges to $f(x')$ as $x_n \to 0$ in the $L^2(\mathbb{R}^{n-1})$ sense, which is that the L^2 norm of their difference tends to zero. □

4.4. Maximum principle

A property that is evident of the Poisson kernel is that for $x_n > 0$,

$$D(x' - y', x_n) = \frac{2}{\omega_n}\Big(\frac{x_n}{(|x'|^2 + x_n^2)^{n/2}}\Big) > 0 \ .$$

Therefore whenever a solution of the Dirichlet problem (4.8) has the property that $f(x') \geq 0$, then

$$u(x) = \int_{\mathbb{R}^{n-1}} D(x' - y', x_n) f(y')\, dy' > 0 \ ;$$

this follows from an argument that is very similar to the one we used for Theorem 3.5 on the heat equation.

Theorem 4.4. *Suppose that $f(x') \geq 0$, then for $x \in \mathbb{R}^n_+$ we have $u(x) \geq 0$, and in fact if at any point $x \in \mathbb{R}^n_+$ (meaning, with $x_n > 0$) it happens that $u(x) = 0$, then we conclude that $u(x) \equiv 0$ and $f(x') = 0$.*

From this result we have a comparison between solutions.

Corollary 4.5. Suppose that $f_1(x') \leq f_2(x')$ for all $x' \in \mathbb{R}^{n-1}$. Then either

$$u_1(x) = (D * f_1)(x) < u_2(x) = (D * f_2)(x)$$

on all of $\mathbb{R}^n_+$, or else $u_i(x) \equiv u_2(x)$ if equality holds at any point $x \in \mathbb{R}^n_+$. In particular if $f_1 = f_2$ then both $f_1 \leq f_2$ and $f_1 \geq f_2$, so that $u_1 \equiv u_2$.

This is our second encounter with the recurring theme in elliptic and parabolic PDEs of comparison and maximum principles. In the case of the Laplace's equation and other elliptic equations, closely analog results hold on essentially arbitrary domains as well, although the Poisson kernel is not in general so explicit and the proof is different. In our present setting, the form of the Poisson kernel gives us a lower bound on the decay rates of solutions $u(x', x_n)$ for large x_n.

Corollary 4.6. Suppose that $f(x') \geq 0$ and that $A \subseteq \{x' : f(x') \geq \delta\}$ is a bounded set of positive measure $\mathrm{meas}(A) > 0$. Then

$$u(x', x_n) \geq \frac{2}{\omega_n}\Big(\frac{x_n}{\sup_{y' \in A}(|x' - y'|^2 + x_n^2)^{n/2}}\Big)\delta\, \mathrm{meas}(A) \ ,$$

and in particular solutions $u(x', x_n)$ that are positive cannot decay too rapidly as $x_n \to +\infty$.

Proof. Express the solution in terms of the Poisson kernel

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^{n-1}} D(x'-y', x_n) f(y')\, dy' \\ &\geq \int_A D(x'-y', x_n) f(y')\, dy' \\ &\geq \inf_{y'\in A} \big(D(x'-y', x_n)\big) \int_A f(y')\, dy' \ , \end{aligned}$$

and of course

$$\int_A f(y')\, dy' \geq \delta \, \mathrm{meas}(A) \ ,$$

while

$$\inf_{y'\in A} \big(D(x'-y', x_n)\big) \geq \Big(\frac{x_n}{\sup_{y'\in A}(|x'-y'|^2 + x_n^2)^{n/2}} \Big) \ .$$

□

4.5. Oscillation and attenuation estimates

Another principle exhibited by solutions of Laplace's equation is the property of attenuation of oscillatory data. This is a feature that is related to the interior regularity of elliptic equations. It also is very relevant to applications, such as to imaging strategies in electrical impedence tomography, which is a medical imaging technique where the idea is to use electrostatic potentials to probe the interior of a patient's body in real time.

Theorem 4.7. *Suppose that $f(x') \in L^2(\mathbb{R}^{n-1})$ is the Dirichlet data for (4.8) on $\mathbb{R}^n_+$, and suppose in addition that*

$$\mathrm{dist}\big(\mathrm{supp}(\hat{f}(\xi')), 0\big) > \rho \ .$$

Then the solution $u(x', x_n)$ decays as $x_n \to +\infty$ with the upper bounds

$$|u(x', x_n)| \leq \frac{C}{|x_n|^{(n-1)/2}} e^{-\rho|x_n|} \ . \tag{4.14}$$

This estimate (4.14) gives an effective penetration depth of the solution $u(x)$ into the interior of the domain, in the situation in which the Dirichlet data $f(x')$ has no low frequency component.

Proof. Since $\mathrm{dist}\big(\mathrm{supp}(\hat{f}(\xi')), 0\big) > \rho$ there is a $\delta > 0$ such that $\inf_{\xi'\in \mathrm{supp}(\hat{f})} > \rho(1+\delta)$. Using the Fourier representation for $u(x)$,

$$u(x) = \frac{1}{\sqrt{2\pi}^{n-1}} \int_{\mathrm{supp}(\hat{f})} e^{i\xi'\cdot x'} e^{-|\xi'|x_n} \hat{f}(\xi')\, d\xi' \ .$$

Therefore

$$
\begin{aligned}
|u(x', x_n)| &\le \frac{1}{\sqrt{2\pi}^{n-1}} \int_{\mathrm{supp}(\hat f)} \left| e^{-|\xi'| x_n} \hat f(\xi') \right| d\xi' \\
&\le \frac{1}{\sqrt{2\pi}^{n-1}} \int_{\mathrm{supp}(\hat f)} e^{-|\xi'| x_n (\frac{1}{1+\delta})} e^{-|\xi'| x_n (\frac{\delta}{1+\delta})} |\hat f(\xi')| \, d\xi' \\
&\le e^{-\inf_{\xi' \in \mathrm{supp}(\hat f)} (\frac{|\xi'| x_n}{1+\delta})} \frac{1}{\sqrt{2\pi}^{n-1}} \int_{\mathrm{supp}(\hat f)} e^{-|\xi'| x_n (\frac{\delta}{1+\delta})} |\hat f(\xi')| \, d\xi' \\
&\le e^{-\rho |x_n|} \Big(\int |\hat f|^2 \, d\xi' \Big)^{1/2} \Big(\int e^{-|\xi'| x_n (\frac{2\delta}{1+\delta})} \, d\xi' \Big)^{1/2} ,
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality on the last line. We thus have the estimate on the decay of $u(x', x_n)$ for large x_n, namely

$$
|u(x)| \le \frac{C(\delta)}{|x_n|^{(n-1)/2}} e^{-\rho |x_n|} \|f\|_{L^2} .
$$

□

A similar bound holds for derivatives of $u(x)$ using the same lines of argument as in the proof above. This is again related to the interior regularity of solutions of elliptic equations. We will give a bound on multiple derivatives of $u(x)$. Using multiindex notation

$$
\partial_x^\alpha u(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u(x) ,
$$

and assuming the hypotheses on the support of the Dirichlet data $f(x)$ as in Theorem 4.7, one follows the same line of argument as in the proof above to show that

$$
\begin{aligned}
|\partial_x^\alpha u(x', x_n)| &\le e^{-\rho} \int \big(|\xi_1^{\alpha_1} \dots \xi_{n-1}^{\alpha_{n-1}}| |\xi'|^{\alpha_n} |\hat f(\xi')| e^{-|\xi'| x_n (\frac{\delta}{1+\delta})} \big) \, d\xi' \\
&\le \frac{C(\delta, \alpha)}{|x_n|^{(n-1)/2 + |\alpha|}} e^{-\rho |x_n|} \|f\|_{L^2} .
\end{aligned}
$$

4.6. The fundamental solution

The Laplace operator is invariant under rotations, and one can imagine that solutions which are also rotationally invariant are of special interest. In polar coordinates (r, φ) in $\mathbb{R}^n$, where $r \in [0, +\infty)$ and $\varphi \in \mathbb{S}^{n-1}$ the Laplacian is expressed

$$
\Delta u = \partial_r^2 + \frac{n-1}{r} \partial_r u + \frac{1}{r^2} \Delta_\varphi u , \tag{4.15}
$$

where Δ_φ is the Laplace operator on the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. A rotationally invariant solution $\Gamma(r)$ of Laplace's equation must satisfy

$$
\partial_r^2 \Gamma + \frac{n-1}{r} \partial_r \Gamma = 0 ,
$$

Figure 1. Domain Ω with a ball $B_\rho(y)$ excised.

That is $\partial_r\Gamma = \frac{C}{r^{n-1}}$, which in turn implies that

$$\Gamma(r) = \frac{C}{2-n}\frac{1}{r^{n-2}}\ , \qquad n \geq 3$$
$$\Gamma(r) = C\log(r)\ , \qquad n = 2\ .$$

The function $\Gamma(r)$ is harmonic for $0 < r < +\infty$ but singular for $r = 0$. It is called a *fundamental solution*, which will be explained by the following computation that uses Green's identities. Suppose that $u \in C^2(\Omega) \cap C^1(\partial\Omega)$ and consider $y \in \Omega$ a point inside the domain under consideration. Take $\rho > 0$ sufficiently small so that the ball $B_\rho(y) \subseteq \Omega$, as in the Figure 1. Using Green's second identity (4.6) over the region $\Omega\backslash B_\rho(y)$ we have

$$\begin{aligned}(4.16)\qquad &\int_{\Omega\backslash B_\rho(y)} \Gamma(|x-y|)\Delta u\,dx \\ &= \int_{\partial\Omega} \big(\Gamma\partial_N u - \partial_N\Gamma u\big)\,dS_x + \int_{S_\rho(y)} \big(\Gamma\partial_N u - \partial_N\Gamma u\big)\,dS_x \\ &+ \int_{\Omega\backslash B_\rho(y)} \Delta\Gamma\, u\,dx\ .\end{aligned}$$

The fundamental solution Γ is harmonic in $\Omega\backslash B_\rho(y)$, the singularity at $x = y$ being inside $B_\rho(y)$, therefore that last term of the RHS is zero. The first term of the RHS is our usual boundary integral from (4.6). We are to calculate the second integral and its limit as $\rho \to 0$. On the sphere $S_\rho(y)$ we have $\Gamma(|x-y|) = \Gamma(\rho)$, while $\partial_N\Gamma(|x-y|) = -\partial_r\Gamma(\rho)$, the latter minus sign coming from the fact that the outward unit normal to $\Omega\backslash B_\rho(y)$ on $S_\rho(y)$ is pointing inwards towards y. Therefore

$$\int_{S_\rho(y)} \Gamma\partial_N u\,dS_x = \Gamma(\rho)\int_{S_\rho(y)} \partial_N u\,dS_x = -\Gamma(\rho)\int_{B_\rho(y)} \Delta u\,dx\ ,$$

and the limit of this quantity vanishes as $\rho \to 0$. Indeed, in case $n \geq 3$ then

$$(4.17)\qquad \lim_{\rho\to 0}\Big|\Gamma(\rho)\int_{B_\rho(y)} \Delta u\,dx\Big| \leq \lim_{\rho\to 0}\frac{C}{n-2}\frac{1}{\rho^{n-2}}\rho^n\|u\|_{C^2}$$

which vanishes like ρ^2 as $\rho \to 0$. The case $n = 2$ also vanishes in the limit, which involves $\log(\rho)$ instead. On the other hand the second term of (4.16) is

$$-\int_{S_\rho(y)} \partial_N \Gamma \, u \, dS_x = \frac{C}{\rho^{n-1}} \int_{S_\rho(y)} u \, dS_x \ ,$$

and as ρ tends to zero, by the continuity of $u(x)$ at $x = y$ this has the limit

$$\lim_{\rho \to 0} \frac{C}{\rho^{n-1}} \int_{S_\rho(y)} u \, dS_x = \lim_{\rho \to 0} \frac{C}{\rho^{n-1}} \Big(\frac{\omega_n}{\rho^{n-1}} u(y) \Big) = C \omega_n u(y) \ , \tag{4.18}$$

where the quantity ω_n is the surface area of the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$.

Theorem 4.8. *For $u \in C^2(\Omega) \cap C^1(\partial\Omega)$ and for $y \in \Omega$ we have the identity*

$$\int_\Omega \Gamma(|x-y|) \Delta u(x) \, dx = C \omega_n u(y) + \int_{\partial\Omega} \big(\Gamma \partial_N u - \partial_N \Gamma \, u \big) \, dS_x \ . \tag{4.19}$$

Setting $C = \omega_n^{-1}$ we recover precisely the value of $u(y)$ with this identity. In fact this procedure need not be restricted to bounded domains Ω. If we specify that $u \in C^2(\mathbb{R}^n)$ is such that $\lim_{R \to +\infty} \int_{S_R(0)} (\Gamma \partial_N u - \partial_N \Gamma \, u) \, dS_x = 0$ for $y \in \mathbb{R}^n$ then the statement of Theorem 4.8 can be interpreted distributionaly in terms of the Dirac-δ function;

$$\Delta_x \Gamma(|x-y|) = \delta_y(x) \ .$$

We comment further that away from its pole the fundamental solution Γ is C^∞ smooth, in fact it is real analytic C^ω. The results of Theorem 4.8 can be used to deduce that a harmonic functions are C^∞, indeed even real analytic, in the interior of their domain of definition.

Theorem 4.9. *Let $u \in C^2(\Omega) \cap C^1(\partial\Omega)$ be a harmonic function in Ω, then in fact $u \in C^\omega$.*

Proof. For harmonic functions $u(x)$ the identity (4.19) reads

$$u(y) = \int_{\partial\Omega} \big(u \, \partial_N \Gamma - \partial_N u \, \Gamma \big) \, dS_x \ .$$

Since $y \in \Omega$ in this expression, while the boundary integral only involves $x \in \partial\Omega$, then $\mathrm{dist}(x, y)$ is bounded from below and the RHS clearly is a C^ω function of y, whence the result. □

We may add any harmonic function $w(x)$ to the fundamental solution $\Gamma(|x-y|)$ and retain the above properties, as

$$\Delta_x(\Gamma + w) = \Delta_x \Gamma + \Delta_x w = \delta_y(x) \ .$$

In particular consider any ball $B_\rho(y) \subseteq \Omega$ within the domain of definition of a harmonic function $u(x)$. Use the quantity $\Gamma(|x-y|)-\Gamma(\rho)$ as a fundamental solution in (4.19) to find that

$$u(y) = \int_{S_\rho(y)} \big(\Gamma(|x-y|) - \Gamma(\rho)\big)\partial_N u \, dS_x - \int_{S_\rho(y)} u \, \partial_r \Gamma\Big|_{r=\rho} dS_x$$
$$= \int_{S_\rho(y)} u(x) \frac{1}{\omega_n \rho^{n-1}} \, dS_x \ .$$

This has proved the following result.

Theorem 4.10 (Gauss' law of arithmetic mean)**.** *A harmonic function satisfies the integral identity*

$$u(y) = \frac{1}{\omega_n \rho^{n-1}} \int_{S_\rho(y)} u(x) \, dS_x = \frac{1}{\omega_n} \int_{v \in S_1(0)} u(y + \rho v) \, dS_v \ , \tag{4.20}$$

for all $B_\rho(y)$ contained in the domain of definition of u. An alternative version of this statement is that

$$u(y) = \frac{n}{\omega_n \rho^n} \int_{B_\rho(y)} u(x) \, dx \ . \tag{4.21}$$

In words, this statement of this result is that a harmonic function $u(y)$ is equal to its average values over spheres $S_\rho(y)$ about y. This fact is consistent with discretizations of the Laplace operator in numerical simulations. For example, the simplest version of a finite difference approximation of the Laplacian at $y \in \mathbb{Z}^n$ is

$$\Delta_h u(y) := \frac{1}{h^2}\Big(\sum_{x \in \mathbb{Z}^n : |x-y|=1} u(x) - 2n u(y) \Big) \ .$$

A discrete harmonic function $\Delta_h u = 0$ explicitly satisfies the property that $u(y)$ is equal to the average of its values at the nearest neighbor points $|x - y| = 1$.

4.7. maximum principle again

The maximum principle is a recurring theme in the theory of elliptic and parabolic PDEs. The result above of Gauss' law of arithmetic mean (4.20) allows us to discuss this sort of behavior of functions which are not even $C^2(\Omega)$, but merely continuous.

Definition 4.11. A function $u \in C(\Omega)$ is *subharmonic* in Ω when for all $y \in \Omega$ and all $B_\rho(y) \subseteq \Omega$, one has

$$u(y) \le \frac{1}{\omega_n \rho^{n-1}} \int_{S_\rho(y)} u(x) \, dS_x \ .$$

A function $u \in C(\Omega)$ is *superharmonic* in Ω when for all $y \in \Omega$ and all $B_\rho(y) \subseteq \Omega$ the opposite inequality holds,

$$u(y) \geq \frac{1}{\omega_n \rho^{n-1}} \int_{S_\rho(y)} u(x)\, dS_x \ .$$

This nomenclature is motivated by the fact that the graph of a subharmonic function lies beneath the graph of the harmonic function with the same boundary values on $\partial\Omega$, as we will show below. Similarly, superharmonic functions lie above harmonic functions sharing their boundary values. Furthermore, if we also knew that $u \in C^2(\Omega)$ then the inequality

$$\Delta u(x) \geq 0 \ , \qquad \big(\text{ respectively } \Delta u(x) \leq 0\big) \ ,$$

implies that $u(x)$ is subharmonic (respectively, superharmonic).

It turns out that subharmonic functions $u \in C(\Omega)$ satisfy the maximum principle, a statement that substantially lessens the necessary hypotheses for the result, and which proves to be a very useful principle in nonlinear problems and in other generalizations of Laplace's equation.

Theorem 4.12. *Let Ω be a bounded domain and suppose that $u \in C(\overline{\Omega})$ is subharmonic. Then*

$$\max_{x\in\overline{\Omega}}(u(x)) = \max_{x\in\partial\Omega}(u(x)) \ . \tag{4.22}$$

Furthermore, if Ω is connected then either for all $x \in \Omega$ we have

$$u(x) < \max_{x\in\partial\Omega}(u(x)) \ , \tag{4.23}$$

or else, if at some point $x \in \Omega$ equality holds, then $u(x)$ is necessarily a constant function $u(x) \equiv M := \max_{x\in\partial\Omega}(u(x))$. This latter result is known as the strong maximum principle.

If $u(x)$ is superharmonic then $-u(x)$ is subharmonic, and therefore u satisfies a *minimum principle*. A corollary of Theorem 4.12 is that harmonic functions, which are both sub- and superharmonic, satisfy upper and lower estimates in the supremum norm. Namely, if $u(x)$ is harmonic in Ω and $u(x) = f(x)$ for $x \in \partial\Omega$, then

$$\min_{x\in\partial\Omega}(f(x)) \leq u(x) \leq \max_{x\in\partial\Omega}(f(x))$$

for all $x \in \overline{\Omega}$.

Proof. We will give two proofs of this theorem, for the methods are interesting in their own right. Firstly, we will give a proof of the weak maximum principle, which is that of statement (4.22), under the stronger hypothesis that $u(x) \in C^2(\Omega) \cap C(\Omega)$. As a preliminary step, start with the case in which strict inequality holds; $\Delta u(x) > 0$ in Ω. Suppose that at some point

$x_0 \in \Omega$ the function $u(x)$ achieves its maximum, $u(x_0) = \max_{x\in\overline{\Omega}}(u(x))$. Then $\nabla u(x_0) = 0$ and the Hessian matrix of u satisfies

$$H(u) = \big(\partial_{x_j}\partial_{x_\ell} u(x_0)\big)_{j\ell=1}^n \le 0 \ .$$

However this contradicts the fact that u is subharmonic, indeed

$$\Delta u(x) = \mathrm{tr}\big(H(u)\big) > 0 \ .$$

Hence no such maximum point can exist in Ω, and even rules out local maxima. For the general case where we assume that $\Delta u \ge 0$ in Ω, consider the subharmonic function $v(x) = u(x) + \varepsilon|x|^2$, which satisfies $\Delta v(x) = \Delta u(x) + 2\varepsilon n > 0$. Therefore v satisfies the hypotheses of the first case, from which we conclude

$$\begin{aligned}\max_{x\in\overline{\Omega}}(u(x)) &\le \max_{x\in\overline{\Omega}}(u(x) + \varepsilon|x|^2) \\ &= \max_{x\in\partial\Omega}(u(x) + \varepsilon|x|^2) \le \max_{x\in\partial\Omega}(u(x)) + \varepsilon \max_{x\in\partial\Omega}(|x|^2) \ .\end{aligned}$$

Since Ω is bounded, $\max_{x\in\partial\Omega}(|x|^2)$ is finite, and since ε is arbitrary, we have shown that

$$\max_{x\in\overline{\Omega}}(u(x)) \le \max_{x\in\partial\Omega}(u(x)) \ .$$

The advantage of this proof is that it generalizes to many other elliptic equations, including

$$\sum_{j\ell=1}^n a^{j\ell}(x)\partial_{x_j}\partial_{x_\ell} u + \sum_{j=1}^n b^j(x)\partial_{x_j} u = 0 \ ,$$

where the matrices $(a^{j\ell}(x))_{j\ell=1}^n$ are positive definite.

The second proof is more specific to Laplace's equation, using Gauss' law of arithmetic mean. Consider Ω a domain which is connected, $u(x)$ a subharmonic function, and set $M := \sup_{x\in\Omega}(u(x))$. Decompose the domain into two disjoint subsets,

$$\Omega = \{x \in \Omega \,:\, u(x) = M\} \cup \{x \in \Omega \,:\, u(x) < M\} := \Omega_1 \cup \Omega_2 \ .$$

Since $u(x) \in C(\Omega)$, then Ω_2 is open as a subset of Ω, since inequality is an open condition. The claim is that the set Ω_1 is also open in Ω, which we will prove using the property of subharmonicity. Therefore, because of the fact of being connected, either $\Omega = \Omega_2$ and $\Omega_1 = \emptyset$, whereupon strict inequality holds throughout Ω. Or else $\Omega = \Omega_1$ and $\Omega_2 = \emptyset$, in which case $u(x) = M$ a constant.

To prove the claim above that Ω_1 is open, consider $x_0 \in \Omega_1$ and refer to Gauss' law of mean on all sufficiently small spheres $S_\rho(x_0)$ about x_0 which lie in Ω. We have that

$$0 \le \frac{1}{\omega_n \rho^{n-1}} \int_{|x-x_0|=\rho} u(x) - u(x_0)\, dS_x \ .$$

The integrand is nonpositive because $u(x) \le M = u(x_0)$ in Ω. Buth the integrand cannot be negative anywhere near x_0 either, for that would violate the above inequality. Therefore we must have $S_\rho(x_0) \subseteq \Omega_1$ for all sufficiently small ρ, which is the statement that Ω_1 is open. □

4.8. Green's functions and the Dirichlet – Neumann operator

As we observed in Section 4.6 we may add any harmonic function to the fundamental solution $\Gamma(|x-y|)$ and retain the property that $\Delta(\Gamma + w) = \delta_y$. Thus we may add a function $w(x, y) \in C^2(\Omega \times \Omega)$ that satisfies

$$\Delta_x w(x, y) = 0 \text{ for } x \in \Omega \ ,$$
$$w(x, y) = -\Gamma(|x-y|) \text{ for } x \in \partial\Omega \ .$$

The function $w(x, y)$ is harmonic in $x \in \Omega$ and depends parametrically on $y \in \Omega$. Then

$$G(x, y) := \Gamma(|x-y|) + w(x, y)$$

is still a fundamental solution in the distributional sense (4.19), meaning that $\Delta_x G(x, y) = \delta_y(x)$. Furthermore $G(x, y) = 0$ for $x \in \partial\Omega$, and therefore

$$u(x) := \int_\Omega G(x, y) h(y)\, dy$$

satisfies

$$\begin{aligned} \Delta_x u(x) &= \Delta_x \int_\Omega G(x, y) h(y)\, dy \\ &= \Delta_x \int_\Omega \Gamma(|x-y|) h(y)\, dy + \int_\Omega \Delta_x w(x, y) h(y)\, dy = h(x) \ . \end{aligned}$$

It also satisfies Dirichlet boundary conditions; indeed for $x \in \partial\Omega$

$$u(x) = \int_\Omega G(x, y) h(y)\, dy = 0 \ .$$

The function $G(x, y)$ is called the *Green's function* for the domain Ω; it is the integral kernel of the solution operator $\mathbf{P}$ for the Poisson problem (4.2),

$$u(x) = \int_\Omega G(x, y) h(y)\, dy := \mathbf{P}h(x) \ . \tag{4.24}$$

While it is usually not possible to have explicit formulae for the Green's function $G(x, y)$ for a general domain, it is straightforward for the domain consisting of the half space $\mathbb{R}^n_+$. Let $y = (y', y_n) \in \mathbb{R}^n_+$ so that $y_n > 0$, denote the point that is its reflection through the boundary $\{x_n = 0\}$ by

$y^* := (y', -y_n) \in \mathbb{R}^n_-$. The function $w(x,y) = -\Gamma(|x-y^*|)$ satisfies the property that it is harmonic in $\mathbb{R}^n_+$ (since its singularity at $x = y^*$ is in $\mathbb{R}^n_- 0$), and $\Gamma(|x-y^*|) = \Gamma(|x-y|)$ when $x = (x', 0) \in \partial\mathbb{R}^n_+$. This gives rise to the following expression for the Green's function for the domain $\mathbb{R}^n_+$:

$$
\begin{aligned}
G(x,y) &= \Gamma(|x-y|) - \Gamma(|x-y^*|) \\
&= \frac{1}{(2-n)\omega_n}\Big(\frac{1}{(|x'-y'|^2 + (x_n-y_n)^2)^{(n-2)/2}} \\
&\qquad - \frac{1}{(|x'-y'|^2 + (x_n+y_n)^2)^{(n-2)/2}}\Big) ,
\end{aligned} \tag{4.25}
$$

when $n \geq 3$. When $n = 2$ then

$$
G(x,y) = \frac{1}{4\pi} \log\Big(\frac{(x_1-y_1)^2 + (x_2-y_2)^2}{(x_1-y_1)^2 + (x_2+y_2)^2}\Big) . \tag{4.26}
$$

This procedure is similar to the method of images, as one can see. Actually, in two dimensions there is a connection with the theory of complex variables; because of the Riemann mapping theorem there is an expression for the Green's function for arbitrary simply connected domains in terms of the formula (4.26) and the Riemann mapping of the domain to the upper half plane $\mathbb{R}^2_+$.

Although it is not so evident from the way we constructed the Green's function for a domain Ω, the Green's function has the symmetry that $G(x,y) = G(y,x)$. This reflects the property of the Laplace operator with Dirichlet boundary conditions being a self-adjoint operator[1]. This property can be verified explicitly for the Green's functions for the upper half-space upon inspecting the formulae in (4.25) and (4.26).

The Poisson kernel: The Green's function $G(x,y)$ is the integral kernel for the solution of the Poisson problem (4.2), but it also is relevant for the Dirichlet problem (4.1). Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be harmonic. the by Green's identities

$$
\begin{aligned}
0 &= \int_\Omega G(x,y)\Delta u(x)\, dx \\
&= u(y) + \int_{\partial\Omega} \big(G(x,y)\partial_N u(x) - \partial_{N_x} G(x,y) u(x)\big)\, dS_x .
\end{aligned}
$$

Using the fact that $G(x,y) = 0$ for $x \in \partial\Omega$ we deduce that

$$
\begin{aligned}
u(y) &= \int_{\partial\Omega} u(x)\partial_{N_x} G(x,y)\, dS_x \\
&= \int_{\partial\Omega} f(x)\partial_{N_x} G(x,y)\, dS_x := \mathbf{D}f(y) ,
\end{aligned} \tag{4.27}
$$

[1]Technically, the Laplace operator is a symmetric operator which is self-adjoint when restricted to an appropriate subspace $H^1_0(\Omega) \subseteq L^2(\Omega)$

where $f(x)$ is the Dirichlet data for $u(x)$ on the boundary $\partial\Omega$. One may check that in the case of $\mathbb{R}^n_+$ this is indeed the formula for the Poisson kernel (4.12), namely that $-\partial_{x_n} G(x,y)\big|_{y_n=0} = D(x'-y',x_n)$.

There is a related expression for the analog of the Green's function for the Neumann problem, choosing another function $w(x,y)$ which is harmonic in x, and forming the fundamental solution

$$N(x,y) = \Gamma(|x-y|) + w(x,y) \ .$$

If $w(x,y)$ is chosen so that $\partial_N \Gamma(|x-y|) = -\partial_N w(x,y)$ for all $x \in \partial\Omega$, then the function $N(x,y)$ is the integral kernel for the solution operator of the Neumann problem (4.3). That is, the analog of the formula (4.27) holds. On general domains, given Neumann data $g(x)$ such that $\int_{\partial\Omega} g(x)\, dS_x = 0$, then

$$\begin{aligned} u(y) &= -\int_{\partial\Omega} N(x,y)\partial_N u(x)\, dS_x \\ &= -\int_{\partial\Omega} N(x,y) g(x)\, dS_x := \mathbf{S}g(y) \ , \end{aligned} \tag{4.28}$$

analogous to (4.27), as one shows with a calculation using Green's second identity. In the case that $\Omega = \mathbb{R}^n_+$ the choice is that $w(x,y) = \Gamma(|x-y^*|)$, and

$$\begin{aligned} N(x,y) &= \Gamma(|x-y|) + \Gamma(|x-y^*|) \\ &= \frac{1}{(2-n)\omega_n}\Big(\frac{1}{(|x'-y'|^2 + (x_n-y_n)^2)^{(n-2)/2}} \\ &\qquad + \frac{1}{(|x'-y'|^2 + (x_n+y_n)^2)^{(n-2)/2}} \Big) \ , \end{aligned} \tag{4.29}$$

when $n \geq 3$. When $n=2$ then

$$N(x,y) = \frac{1}{4\pi} \log\Big(\big((x_1-y_1)^2 + (x_2-y_2)^2\big)\big((x_1-y_1)^2 + (x_2+y_2)^2\big) \Big) \ . \tag{4.30}$$

Setting $x_n = 0$ this gives the single layer potential

$$S(x'-y',y_n) = \frac{2}{(2-n)\omega_n}\Big(\frac{1}{(|x'-y'|^2 + y_N^2)^{(n-2)/2}} \Big) \ , \tag{4.31}$$

with which one solves the Neumann boundary value problem with data $g(x)$, namely

$$u(y) = \int_{x' \in \mathbb{R}^{n-1}} S(x'-y',x_n) g(x')\, dx' := \mathbf{S}g(x) \ .$$

The Dirichlet – Neumann operator: Given Dirichlet data on the boundary of a domain Ω, it is often the most important part of the solution process for $u(x)$ of the Dirichlet problem to recover the normal derivatives of the solution $\partial_N u(x)$ on the boundary. For Ω a conducting body this would be the map from applied voltage $u(x) = f(x)$ on $\partial\Omega$ to the resulting current $\partial_N u(x) = g(x)$ across the boundary. This map can be expressed in terms of the Green's function, in particular using the Poisson kernel we can express the solution to the Dirichlet problem

$$u(y) = \int_{\partial\Omega} (N_x \cdot \nabla_x) G(x,y) f(x)\, dS_x \ .$$

Therefore we have an expression for its normal derivative on $\partial\Omega$, namely

$$(N_y \cdot \nabla_y) u(y)\Big|_{y\in\partial\Omega} = \int_{\partial\Omega} (N_x \cdot \nabla_x)(N_y \cdot \nabla_y) G(x,y) f(x)\, dS_x := \mathbf{G} f(y) \ , \tag{4.32}$$

where $\mathbf{G}$ is the Dirichlet – Neumann operator for the domain Ω. There is symmetry in the exchange of x with y in the integrand of (4.32), from which we deduce that the Dirichlet – Neumann operator is self-adjoint[2]; $\mathbf{G}^T = \mathbf{G}$.

Recall the energy identity for a harmonic function $u(x)$,

$$0 \le \int_{\Omega} |\nabla u(x)|^2\, dx = \int_{\partial\Omega} u(x) \partial_N u(x)\, dS_x = \int_{\partial\Omega} f(x) (\mathbf{G} f)(x)\, dS_x \ . \tag{4.33}$$

This is to say that the operator $\mathbf{G}$ is nonnegative definite and self-adjoint. The formula (4.33) also exhibits the reelation between the Dirichlet integral of a harmonic function over a domain Ω and the boundary integral over $\partial\Omega$ involving the Dirichlet – Neumann operator.

It is useful to work this out on the domain $\mathbb{R}^n_+$. The solution to the Dirichlet problem for $\mathbb{R}^n_+$ is given in (4.13) in terms of the Fourier transform of the Poisson kernel;

$$u(x) = \frac{1}{\sqrt{2\pi}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'} e^{-|\xi'| x_n} \hat{f}(\xi')\, d\xi' \ .$$

Recall that $\widehat{\frac{1}{i}\partial_{x_j} u}(\xi) = \xi_j \hat{u}(\xi)$, which motivates the notation for differential operators that $D_j = \frac{1}{i}\partial_{x_j}$ and the definition of a general Fourier multiplier operator

$$\begin{aligned}\big(m(D')f\big)(x') &= \frac{1}{\sqrt{2\pi}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'} m(\xi') \hat{f}(\xi')\, d\xi' \\ &= \frac{1}{(2\pi)^{n-1}} \iint_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} e^{i\xi'\cdot(x'-y')} m(\xi') f(y')\, d\xi' dy' \ .\end{aligned}$$

In these terms the harmonic extension $u(x)$ of the boundary conditions $f(x')$ can be written as

$$u(x) = e^{-x_n|D'|} f(x') := (\mathbf{D} f)(x', x_n) \ ,$$

[2]Analogous to the case above, technically the operator $\mathbf{G}$ is symmetric on $C^1(\partial\Omega)$, and is self-adjoint on an appropriate subspace $H^{1/2}(\partial\Omega) \subseteq L^2(\partial\Omega)$

which is an interpretation of the formula (4.13) which used a Fourier integral expression for the Poisson kernel. The operator $\mathbf{D}$ extended the Dirichlet data $f(x')$ to a harmonic function $u(x) = \mathbf{D}f(x', x_n)$ in the upper half space $\mathbb{R}^n_+$.

Differentiating (4.13) with respect to x_n and evaluating the result on the boundary $\{x_n = 0\}$, the Dirichlet – Neumann operator has a related Fourier integral expression, namely

$$\begin{aligned}\mathbf{G}f(x') &= -\partial_{x_n}\big|_{x_n=0}\Big(\frac{1}{\sqrt{2\pi}^{n-1}}\int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'}e^{-|\xi'|x_n}\hat{f}(\xi')\,d\xi'\Big) \\ &= \big(|D'|f\big)(x')\ .\end{aligned} \tag{4.34}$$

Finally, the Dirichlet integral can be expressed in terms of Fourier multipliers, using that

$$\begin{aligned}\partial_{x'}u(x', x_n) &= \frac{1}{\sqrt{2\pi}^{n-1}}\int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'} i\xi' e^{-|\xi'|x_n}\hat{f}(\xi')\,d\xi'\ , \\ \partial_{x_n}u(x', x_n) &= \frac{1}{\sqrt{2\pi}^{n-1}}\int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'}|\xi'|e^{-|\xi'|x_n}\hat{f}(\xi')\,d\xi'\ .\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^n_+}|\nabla u(x)|^2\,dx = \int_0^{+\infty}\Big(\int_{\mathbb{R}^{n-1}}|i\xi'\hat{f}(\xi')|^2e^{-2|\xi'|x_n} + |\xi'|^2|\hat{f}(\xi')|^2e^{-2|\xi'|x_n}\,d\xi'\Big)dx_n\ ,$$

where we have used the Plancherel identity on the hyperplanes $\{(x', x_n) : x_n = \text{Constant}\}$. Thus

$$\begin{aligned}\int_{\mathbb{R}^n_+}|\nabla u(x)|^2\,dx &= \int_{\mathbb{R}^{n-1}} 2|\xi'|^2|\hat{f}(\xi')|^2\Big(\int_0^{+\infty} e^{-2|\xi'|x_n}\,dx_n\Big)d\xi \\ &= \int_{\mathbb{R}^{n-1}} f(x')\big(|D'|f(x')\big)\,dx'\ .\end{aligned}$$

4.9. Hadamard variational formula

It is rare to have such explicit formulae as (4.10) for the Dirichlet problem or (4.34) for the Dirichlet Neumann operator as is the case for the domain $\mathbb{R}^n_+$. More often one considers general domains, with less explicit solution procedures, so that it is reasonable to think to develop more general methods in order to understand the solution operators. In particular, it is relevant to ask whether the Green's function or the Poisson kernel vary continuously under perturbation of the domain. For the Dirichlet problem there is an elegant idea due originally to Hadamard, to describe the Taylor expansion of the Green's function with respect to variations of the domain itself. We will describe this idea in a specific case, namely in terms of the Poisson kernel $\partial_N G(x', y)$ for domains which are perturbations of the upper half space $\mathbb{R}^n_+$.

Consider $\eta(x') \in C^1(\mathbb{R}^{n-1})$, whose graph defines the boundary of a domain $\Omega(\eta) = \{(x', x_n) \in \mathbb{R}^n : x_n > \eta(x')\}$. In this notation $\Omega(0) = \mathbb{R}^n_+$. The domain $\Omega(\eta)$ has its Green's function $G_{\Omega(\eta)}(x', y)$, from which we obtain the Poisson kernel $\partial_N G_{\Omega(\eta)}(x', y) := D_{\Omega(\eta)}(x', y)$. The solution to the Dirichlet problem with Dirichlet data is thus given by the integral operator

$$u(x) = \mathbf{D}_{\Omega(\eta)} f(x) = \int_{\mathbb{R}^{n-1}} D_{\Omega(\eta)}(x', y) f(y) \sqrt{1 + |\nabla_{y'}\eta|^2}\, dy'$$

where $\sqrt{1 + |\nabla\eta|^2} dy' = dS_y$. From (4.13) we see that $\mathbf{D}_{\Omega(0)} = e^{-x_n|D'|}$, a Fourier multiplier operator. In general $u(x) = \mathbf{D}_{\Omega(\eta)} f(x)$ is the bounded harmonic extension to the domain $\Omega(\eta)$ of the boundary data $f(x)$ defined on $\partial\Omega(\eta) = \{(x', x_n) : x_n = \eta(x')\}$. The solution operator $\mathbf{D}_{\Omega(\eta)}$ clearly depends upon the domain given by $\eta(x')$ in a nonlinear, global and possibly complicated way. The Hadamard variational formula expresses the derivative of the operator $\mathbf{D}_{\Omega(\eta)}$ with respect to variations of η, giving a linear approximation to changes of the Green's function under perturbation of the domain.

Definition 4.13. Let $\mathbf{D}(\eta)$ be a bounded linear operator from the space L^2 to L^2 that depends upon functions $\eta(x) \in C^1$. The bounded linear operator $\mathbf{A}(\eta)$ is the *Fréchet derivative* of $\mathbf{D}(\eta)$ with respect to $\eta(x)$ at the point $\eta = 0$ if it satisfies

$$\mathbf{A}(\lambda\eta) = \lambda\mathbf{A}(\eta) \quad \forall\lambda \in \mathbb{R}$$
$$\|\big(\mathbf{D}(\eta)f - \mathbf{D}(0)f\big) - \mathbf{A}(\eta)f\|_{L^2} \le C|\eta|^2_{C^1}\|f\|_{L^2} \ .$$

This definition can of course be adapted to the more general situation of the operators $\mathbf{D}(\eta)$ mapping f in a Banach X to a Banach space Y, for η varying over a neighborhood of a third Banach space Z.

Without actually proving that $\mathbf{D}(\eta)$ is analytic with respect to $\eta \in C^1$ (which it is), we will derive a formula for the Fréchet derivative of $\mathbf{D}(\eta)$ for the domain that is the upper half space, for small $|\eta|_{C^1}$. A standard harmonic function on $\mathbb{R}^n_+$, and indeed on any of the domains $\omega(\eta)$, is of course $\varphi_k(x) = e^{ik\cdot x'}e^{-|k|x_n}$ for each parameter $k \in \mathbb{R}^{n-1}$ fixed. Therefore

$$\mathbf{D}(0)\big(e^{ik\cdot x'}\big) = e^{ik\cdot x'}e^{-|k|x_n} = e^{-x_n|D'|}e^{ik\cdot x'} \ .$$

Given one of the domains $\Omega(\eta)$, the boundary values of $\varphi_k(x)$ on $\partial\Omega(\eta)$ are $e^{ik\cdot x'}e^{-|k|\eta(x')}$, therefore for $x_n > \eta(x')$ we know that

$$\mathbf{D}(\eta)\big(e^{ik\cdot x'}e^{-|k|\eta(x')}\big) = e^{ik\cdot x'}e^{-|k|x_n} \ .$$

Thus taking any point $(x', x_n) \in \Omega(0) \cap \Omega(\eta)$,

$$\begin{aligned} 0 &= \mathbf{D}(\eta)\big(e^{ik\cdot x'}e^{-|k|\eta(x')}\big) - \mathbf{D}(0)\big(e^{ik\cdot x'}\big) \\ &= \mathbf{D}(\eta)\big((1 - \eta(x')|k| + \tfrac{1}{2}\eta^2(x')|k|^2 + \dots)e^{ik\cdot x'}\big) - \mathbf{D}(0)\big(e^{ik\cdot x'}\big) \ . \end{aligned}$$

This is to say that

$$\mathbf{D}(\eta)\big((e^{ik\cdot x'}\big) - \mathbf{D}(0)\big(e^{ik\cdot x'}\big) - \mathbf{D}(\eta)\big(\eta(x')|k|e^{ik\cdot x'}\big) = \mathbf{D}(\eta)\big((\tfrac{1}{2}\eta^2(x')|k|^2 + \dots)e^{ik\cdot x'}\big) \ . \tag{4.35}$$

The facts are that the operator $\mathbf{D}(\eta)$ does have a Fréchet derivative $\mathbf{A}(\eta)$ at $\eta = 0$, and that the RHS is bounded by $C|\eta|^2_{C^1}|k|^2$ for small $|\eta|_{C^1}$. This allows us to compute $\mathbf{A}(\eta)$ from (4.35). Namely,

$$\mathbf{D}(\eta)(e^{ik\cdot x'}) - \mathbf{D}(0)(e^{ik\cdot x'}) = \mathbf{D}(0)\big(\eta(x')|D'|e^{ik\cdot x'}\big) + \mathcal{O}(|\eta|^2_{C^1}) \ ,$$

from which we read that

$$\mathbf{A}(\eta)e^{ik\cdot x'} = \mathbf{D}(0)\big(\eta(x')|D'|e^{ik\cdot x'}\big) = e^{-x_n|D'|}\big(\eta(x')|D'|\big)e^{ik\cdot x'} \ .$$

In other words, $\partial_\eta \mathbf{D}(\eta)\big|_{\eta=0} f = \mathbf{A}(\eta)f = e^{-x_n|D'|}\big(\eta(x')|D'|f\big)(x)$.

Now consider the Dirichlet – Neumann operator $\mathbf{G}(\eta)$ on domains $\Omega(\eta)$ which are perturbations of $\Omega(0) = \mathbb{R}^n_+$, and its Fréchet derivative at $\eta = 0$. Recall that

$$\mathbf{G}(\eta)f(x') = N_x \cdot \nabla u(x)\big|_{x_n=\eta(x')} \ ,$$

where $u(x)$ is the bounded harmonic extension of the Dirichlet data $f(x')$ to the domain $\Omega(\eta)$. Using again the family of harmonic functions $\varphi_k(x) = e^{ik\cdot x'}e^{-|k|x_n}$ we compute its boundary values $f(x')$ and its normal derivative $N_x \cdot \nabla\varphi_k(x)$ on $\{x_n = \eta(x')\}$:

$$f(x') = e^{ik\cdot x'}e^{-|k|\eta(x')} = \big(1 - \eta(x')|k| + \mathcal{O}(|\eta|^2_{C^1})\big)e^{ik\cdot x'}$$

$$\begin{aligned} \mathbf{G}(\eta)f(x') &= N_x \cdot \nabla\varphi_k(x) \\ &= \frac{1}{\sqrt{1+|\nabla\eta|^2}}(\partial_{x'}\eta, -1)\cdot(ik, -|k|)\nabla\varphi_k(x)\big|_{x_n=\eta(x')} \ . \end{aligned} \tag{4.36}$$

The Dirichlet – Neumann operator is not bounded on L^2, but it is bounded from $H^1(\mathbb{R}^{n-1})$ to L^2, as can be seen by its expression (4.34) as a Fourier multiplier when $\Omega = \mathbb{R}^n_+$. We seek the linear approximation to it among domain perturbations $\Omega(\eta)$ at the point $\eta = 0$. To calculate $\mathbf{B}(\eta) := \partial_\eta \mathbf{G}(\eta)\big|_{\eta=0}$, compare the first two terms of of the LHS with the RHS of (4.36) in powers of η:

$$\begin{aligned} \mathbf{G}(\eta)f(x') &= \big(\mathbf{G}(0) + \mathbf{B}(\eta)\big)e^{ik\cdot x'} + \mathbf{G}(0)\big(-\eta(x')|k|e^{ik\cdot x'}\big) \\ &= |D'|e^{ik\cdot x'} + \mathbf{B}(\eta)e^{ik\cdot x'} - |D'|\big(\eta(x')|D'|e^{ik\cdot x'}\big) \end{aligned}$$

which is the LHS, and where the RHS is

$$\begin{aligned} N_x \cdot \nabla\varphi_k(x) &= |k|e^{ik\cdot x'} + \partial_{x'}\eta(x')\cdot ik\, e^{ik\cdot x'} - |k|^2e^{ik\cdot x'} + \mathcal{O}(|\eta|^2_{C^1}) \\ &= |D'|e^{ik\cdot x'} + \partial_{x'}\big(\eta(x')\partial_{x'}\, e^{ik\cdot x'}\big) + \mathcal{O}(|\eta|^2_{C^1}) \ . \end{aligned}$$

Equating these expressions, solving for $\mathbf{B}(\eta)$, and applying the operators to a general $f(x')$ rather than the particular family of functions $e^{ik\cdot x'}$, we obtain

$$\mathbf{B}(\eta)f(x') = \partial_{x'}\big(\eta(x')\partial_{x'}f(x')\big) + |D'|\big(\eta(x')|D'|f(x')\big)\ . \tag{4.37}$$

Notice the symmetry under adjoints that is evident in this expression for $\mathbf{B}(\eta)$, reflecting the self-adjoint property of the Dirichlet – Neumann operator itself.

Exercises: Chapter 4

Exercise 4.1. Derive the expression (4.12) for the case $n = 2$ from the Fourier integral, and show that $\omega_2 = 2\pi$. *Hint:* Complex variables techniques would be useful.

Derive the expression (4.12) in the general case for $n \geq 3$ and show that ω_n is the surface area of the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. *Hint:* A good starting place would be to adapt the method of images to the situation.

Exercise 4.2. In the case $n = 2$ the fundamental solution has the property that $\Delta\Gamma(|x-y|) = \delta_y(x)$. Show this by proving that the limit of the expression in (4.17) vanishes, and that the limit in (4.18) holds.

Exercise 4.3. Prove the second version of the Gauss' law of arithmetic mean (4.21).

Exercise 4.4. Prove that a function $u(x) \in C^2(\Omega)$ that satisfies

$$\Delta u \geq 0 \tag{4.38}$$

is subharmonic in Ω in the sense of Definition 4.11. Prove that if $u(x) \in C^2(\Omega)$ is subharmonic then it satisfies the inequality (4.38).

Exercise 4.5. Show that the Green's function $G(x, y)$ on a domain $\Omega \subseteq \mathbb{R}^n$ satisfies the property of symmetry $G(x, y) = G(y, x)$. Conclude that the resulting operator Δ^{-1} with Dirichlet boundary conditions is self-adjoint on $L^2(\Omega)$.

Exercise 4.6. Derive the Green's function and the Poisson kernel for the disk $B_R(0) \subseteq \mathbb{R}^n$.

Exercise 4.7. For simply connected domains Ω in $\mathbb{R}^2$ describe the Green's function in terms of the Riemann map of Ω to $\mathbb{R}^2_+$.