Chapter 5

Properties of the Fourier transform

The purpose of this section is to raise our level of sophistication of the analysis of the Fourier transform, and to make up our backlog of analytic justification of our work in the previous several sections. The Fourier transform plays a very important role in analysis, and for this reason it has been thoroughly analyzed from many points of view, by many people in many different settings. We will work through the bodies of two of the principal settings in this subsection, the ones most important to analysis and PDE. The cases are

(1) Hilbert spaces: covered in Section 5.1.

(2) Schwartz class: covered in Section 5.2.

5.1. Hilbert spaces

Definition 5.1. The usual Hilbert space for us is $L^2(\mathbb{R}^n)$, which consists of the square-integrable measurabe functions on $\mathbb{R}^n$;

$$L^2(\mathbb{R}^n) = \{f(x) \text{ Lebesgue measurable in } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^2 dx < +\infty\}.$$

Proposition 5.2 (Elementary properties of $L^2(\mathbb{R}^n)$)**.** (1) The space of functions $L^2(\mathbb{R}^n)$ is a linear space, namely if $f, g \in L^2$, then this implies that $\alpha f + \beta g \in L^2$.

(2) The topology on $L^2(\mathbb{R}^n)$ is given by a norm, which is homogenous of degree 1.

$$\|f\|_{L^2} = \Big(\int_{\mathbb{R}^n} |f(x)|^2\, dx\Big)^{1/2}$$
$$\|\alpha f\|_{L^2} = |\alpha| \|f\|_{L^2} \ , \quad \|f\|_{L^2} = 0 \Leftrightarrow f = 0 \ .$$

(3) The norm satisfies the triangle (or Minkowski) inequality:

$$\|f + g\|_{L^2} \leq \|f\|_{L^2} + \|g\|_{L^2} \ .$$

(4) The linear space $L^2(\mathbb{R}^n)$ is *complete* with respect to the norm

$$\|f\|_{L^2} = (\int_{\mathbb{R}^n} |f(x)|^2 dx|)^{\frac{1}{2}} \ .$$

That is, if $\{f_n\}_{n=1}^{\infty} \subseteq L^2(\mathbb{R}^n)$ is a Cauchy sequence of functions, then there exists a limit function $f(x) \in L^2(\mathbb{R}^n)$ such that $f_n(x) \to_{L^2} f(x)$ (meaning $\|f_n(x) - f(x)\|_{L^2} \to 0$).

(5) Furthermore, the norm on $L^2(\mathbb{R}^n)$ is given by an inner product:

$$\langle f, g\rangle = \int f(x)\overline{g(x)}\, dx \ , \qquad \langle f, f\rangle = \|f\|_{L^2}^2 \ . \tag{5.1}$$

The inner product satisfies the Schwartz inequality:

$$|\langle f, g\rangle| \leq \|f\|_{L^2} \|g\|_{L^2} \ .$$

(6) The "dual space", or space of bounded linear functionals on $L^2(\mathbb{R}^n)$, is $L^2(\mathbb{R}^n)$ itself.

Definition 5.3 (Banach and Hilbert spaces)**.** A linear space with a norm, which is complete, is a *Banach space*. A Banach space whose norm is given by an inner product is a *Hilbert space*.

Our favourite Hilbert space $L^2(\mathbb{R}^n)$ has its inner product defined in (5.1). One reason that $L^2(\mathbb{R}^n)$ is a natural setting for the Fourier transform is that it is preserved under the transform. In $\mathbb{R}^n$, we define the transform as follows:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x) dx = (\mathcal{F}f)(\xi) \ .$$

Theorem 5.4. *The Fourier transform is an isometry of* $L^2(\mathbb{R}^n)$*. That is,* $\|f\|_{L^2(\mathbb{R}^n_x)} = \|\hat{f}\|_{L^2(\mathbb{R}^n_\xi)}$*, or stated in other words,*

$$\int_{\mathbb{R}^n_x} |f(x)|^2\, dx = \int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2\, d\xi \ .$$

This equality between the L^2 norms of a function and its Fourier transform is known as the Plancherel identity; it is a general fact about the Fourier transform that holds in many settings. The proof of Theorem 5.4 is deferred until the end of our discussion of Schwartz class.

Other examples of Hilbert spaces and Banach spaces as tools of analysis include the following:

(1) Sobolev spaces $H^s(\mathbb{R}^n)$ (Hilbert spaces based on L^2 norms):

$$H^s(\mathbb{R}^n) = \{f(x) \,:\, \sum_{0\le|\alpha|\le s} \Big(\int |\partial_x^\alpha f(x)|^2\,dx\Big)^{\frac{1}{2}} < +\infty\} \ .$$

This is a *scale of spaces*, which are nested in terms of decreasing index s: $H^s \subseteq H^{s-1} \subseteq \cdots \subseteq L^2(\mathbb{R}^n)$.

(2) L^p spaces (Banach spaces):

$$L^p(\mathbb{R}^n) = \{f(x) \,:\, \Big(\int_{\mathbb{R}^n} |f(x)|^p\,dx\Big)^{\frac{1}{p}} < +\infty\} \ .$$

For $p = \infty$ the space of bounded measurable functions on $\mathbb{R}^n$ is denoted by $L^\infty(\mathbb{R}^n)$, and has a norm given by

$$\|f\|_{L^\infty} := \sup_{x\in\mathbb{R}^n} |f(x)| \ .$$

(3) Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ (Banach spaces modeled on L^p norms):

$$W^{s,p}(\mathbb{R}^n) = \{f(x) \,:\, \sum_{0\le|\alpha|\le s} \Big(\int |\partial_x^\alpha f(x)|^p\,dx\Big)^{\frac{1}{p}} < +\infty\} \ .$$

(4) Schauder spaces $C^{s,\gamma}(\Omega)$ (Banach spaces modeled on C^0 norms):

$$C^{s,\gamma}(\Omega) = \{f(x) \,:\, \sup_{x\in\Omega} |\partial_x^\alpha f(x)| < +\infty \ \forall|\alpha| \le s$$
$$\text{and } \forall|\alpha| = s, \ \sup_{x,y\in\Omega, x\ne y} \frac{|\partial_x^\alpha f(x) - \partial_x^\alpha f(y)|}{|x-y|^\gamma}\} < +\infty\} \ .$$

The many relationships of inclusion among these spaces, and their associated inequalities of norms, are important information for the analysis of PDE. A principal one that is very often used in PDEs is the Sobolev Embedding Theorem. A version of this result is as follows:

Theorem 5.5 (Sobolev embedding)**.** *If $f(x) \in H^s(\mathbb{R}^n)$ for a Sobolev index $s > \frac{n}{2}$ then*

$$|f(x)|_{L^\infty} \le C_n \|f\|_{H^s} \ .$$

In other words $H^s \subseteq L^\infty$ for $s > \frac{n}{2}$, and there is a bound on the inclusion operator $\iota \,:\, H^s \to L^\infty$ given by the constant C_n.

Proof. Consider the value of $f(x)$ at an arbitrary point of $\mathbb{R}^n$,

$$\begin{aligned}|f(x)| &= \Big|\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\hat{f}(\xi)\,d\xi\Big| \\ &= \Big|\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\frac{1}{(1+|\xi|^2)^{\frac{s}{2}}}(1+|\xi|^2)^{\frac{s}{2}}\hat{f}(\xi)\,d\xi\Big| \ .\end{aligned}$$

Using the Cauchy – Schwarz inequality on this last expression

$$|f(x)| \le \frac{1}{\sqrt{2\pi}^n}\Big(\int \frac{1}{(1+|\xi|^2)^s}\,d\xi\Big)^{\frac{1}{2}}\Big(\int (1+|\xi|^2)^s|\hat{f}(\xi)|^2\,d\xi\Big)^{\frac{1}{2}} \ . \tag{5.2}$$

By the binomial theorem

$$\begin{aligned}\Big(\int (1+|\xi|^2)^s|\hat{f}(\xi)|^2\,d\xi\Big)^{\frac{1}{2}} &= \Big(\sum_{j=0}^{s}\int \binom{s}{j}|\xi|^{2j}|\hat{f}(\xi)|^2\,d\xi\Big)^{\frac{1}{2}} \\ = \Big(\sum_{j=0}^{s}\int \binom{s}{j}|\nabla^j f(x)|^2\,dx\Big)^{\frac{1}{2}} &\le C\|f\|_{H^s} \ ,\end{aligned}$$

where we have used the Plancherel identity for $f(x)$ and its derivatives in the last line. This holds of course for any integer s. It remins to bound the RHS of (5.2);

$$\Big(\int \frac{1}{(1+|\xi|^2)^s}\,d\xi\Big)^{\frac{1}{2}} = \int_{\mathbb{S}^{n-1}}\int_0^{+\infty}\frac{1}{(1+r^2)^s}\,r^{n-1}dr\,dS_\varphi \ ,$$

and if $s > \frac{n}{2}$ then this quantity is finite, which finishes the proof. □

5.2. Schwartz class

This linear space of functions was introduced by Laurent Schwartz (Ecole Polytechnique - Paris) in order to embody a very convenient class of functions with which to work; for which one can interchange integrations with differentiations and integrate by parts with impunity. Colloquially, $f(x)$ is a Schwartz class function if it is infinitely differentiable, and if it decays along with all of its derivatives as $|x| \to \infty$ faster than any polynomial.

Definition 5.6 (Schwartz class)**.** Consider the functions $f(x)$ defined over $\mathbb{R}^n$ such that for all α, β,

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)| < +\infty.$$

The linear space of such functions is called Schwartz class.

Recall our previously introduced notation that is a convenient notational device for multivariable processes:; the quantities α, β are multi-indices, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\beta = (\beta_1, \beta_2, ..., \beta_n)$, with each $\alpha_i, \beta_j \in \mathbb{N}$. Thus, we have

$|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ so that $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ a monomial of degree $|\alpha|$ and $\partial_x^\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \ldots \partial_{x_n}^{\beta_n} = \Pi \partial_{x_j}^{\beta_j}$ a differential operator of order $|\beta|$.

With this notation, many combinatorially complex objects can be written very conveniently. For example, Taylor series in n dimensions can be stated

$$f(x) \sim \sum_\alpha \frac{1}{\alpha!} \partial_x^\alpha f(y)(x-y)^\alpha$$

with $\alpha! = \alpha_1! \alpha_2! \ldots \alpha_n!$.

Here are several examples of particular functions, where we can compare their properties of smoothness and decay in $x \in \mathbb{R}^n$ to those of Schwartz class:

$$f(x) = e^{-\frac{x^2}{2}} \in \mathcal{S} \ ,$$
$$f(x) = (1 + |x|^2)^{-\frac{b}{2}} \notin \mathcal{S} \text{ but } f(x) \in L^2(\mathbb{R}^n) \text{ for } b > n/2 \ ,$$
$$f(x) = e^{-|x|} \notin \mathcal{S} \text{ for a different reason .}$$

The space C_0^∞ of infinitely differentiable functions with compact support is a subset of $\mathcal{S}$ as a subspace, $C_0^\infty \subseteq \mathcal{S}$, and the space C^∞ of bounded infinitely differentiable functions contains Schwartz class; $\mathcal{S} \subseteq C^\infty$. The space $C^\omega(\mathbb{R}^n)$ of bounded *real analytic function* on $\mathbb{R}^n$ (the restriction to $x \in \mathbb{R}^n$ of functions that are analytic in a neighborhood of $\mathbb{R}^n \subseteq \mathbb{C}^n$) is a subset of C^∞, but not every C^∞ function is analytic.

The topology of $\mathcal{S}$ is defined using a countable family of seminorms

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)| \ .$$

That is to say, a sequence $\{f_n(x)\}_{n=1}^\infty$ converges to $f(x)$ if and only if for all multi-indices α, β the seminorms

$$\|f_n - f\|_{\alpha,\beta} = \sup_x |x^\alpha \partial_x^\beta (f_n(x) - f(x))| \to 0$$

with $n \to \infty$. No uniformity in α, β is implied, however. Some functional analytic remarks are in order at this point. The space $\mathcal{S}$ is a linear space, but not a Banach space, and there is no way to describe this sense of convergence by using a norm. It is, however, a metric space, with the distance between two points given by

$$\rho(f, g) = \sum_{\alpha,\beta} \frac{1}{2^{n(|\alpha|+|\beta|)}} \frac{\|f-g\|_{\alpha,\beta}}{1 + \|f-g\|_{\alpha,\beta}} \ ,$$

and it is complete with respect to this metric. Clearly $\{f_n(x)\}_{n=1}^\infty$ converges in $\mathcal{S}$ to $f(x)$ if and only if $\rho(f_n, f) \to 0$ if and only if all $\|f_n - f\|_{\alpha,\beta} \to 0$ as $n \to \infty$. The notation for limits of sequences of functions in this Schwartz class sense is to write $\mathcal{S} - \lim f_n = f$. Metric spaces such as this one whose topology is given by a countable number of semi-norms are called *Fréchet spaces.*

Lemma 5.7 (technical lemma)**.** This result quantifies the propertes of scaling limits of two quantities, giving rise to well defined functionals on $\mathcal{S}$:
(i) Let $g \in \mathcal{S}$ with $g(0) = 1$. Then for all $f \in \mathcal{S}$,

$$\mathcal{S} - \lim_{\varepsilon \to 0} (g(\varepsilon x) f(x)) = f(x) \ .$$

(ii) Let $h(x) \in L^1(\mathbb{R}^n)$, with $\int h(x) dx = 1$, and suppose that $f(x)$ is bounded, and continuous at $x = 0$. Then

$$\lim_{\varepsilon \to 0} \int f(x) \frac{1}{\varepsilon} h(\frac{x}{\varepsilon}) dx = f(0) \ .$$

For $h(x) \in L^1 \mathbb{R}^n$, with $\int h(x)\, dx = 1$, the limit operation in (5.7) describes the approximation of the Dirac δ-function by rescaling a L^1 function $h(x)$.

Proof of Lemma 5.7(ii). By change of variables,

$$\int \frac{1}{\varepsilon^n} g(\frac{x}{\varepsilon})\, dx = \int g(x')\, dx' = 1 \ ,$$

with $x' = \frac{x}{\varepsilon}$. Given $\delta > 0$, choose $r > 0$ such that $|f(x) - f(0)| < \delta$ for all $|x| < r$. Then write

$$\int f(x) \frac{1}{\varepsilon} h(\frac{x}{\varepsilon})\, dx = \int f(0) \frac{1}{\varepsilon^n} h(\frac{x}{\varepsilon})\, dx + \int (f(x) - f(0)) \frac{1}{\varepsilon^n} h(\frac{x}{\varepsilon})\, dx \ .$$

The first term is $f(0)$. Split the second term in two:

$$\Big|\int_{|x|<r} (f(x) - f(0)) \frac{1}{\varepsilon^n} h(\frac{x}{\varepsilon})\, dx\Big| \le \delta \Big|\int_{|x|<r} \frac{1}{\varepsilon^n} h(\frac{x}{\varepsilon})\, dx\Big| \le \delta \|h\|_{L^1}$$

$$\Big|\int_{|x|\ge r} (f(x) - f(0)) \frac{1}{\varepsilon^n} h(\frac{x}{\varepsilon})\, dx\Big| \le 2 \sup_x |f(x)| \Big|\int \frac{1}{\varepsilon^n} |h(\frac{x}{\varepsilon})|\, dx\Big|$$

$$= 2 \sup_x |f(x)| \Big|\int_{|x'| \ge \frac{r}{\varepsilon}} \frac{1}{\varepsilon^n} |h(x')| dx'\Big|$$

and the RHS of the last term with r fixed tends to zero with $\varepsilon \to 0$. □

The proof of Lemma 5.7(ii) appears in the exercises of this Chapter.

Corollary 5.8. The space $C_0^\infty \subseteq \mathcal{S}$ is a dense subspace of $\mathcal{S}$.

Proof. This uses 5.7: Take any $g \in C_0^\infty$ with $g(0) = 1$. Then for $f \in \mathcal{S}$ arbitrary,

$$g(\varepsilon x) f(x) \in C_0^\infty$$

and

$$\mathcal{S} - \lim_{\varepsilon \to 0} (g(\varepsilon x) f(x)) = f(x) \ .$$

□

The space of *distributions* is the space of continuous linear functionals on C_0^∞; $\mathcal{D}' = (C_0^\infty)'$, otherwise known as its dual space. The space of *tempered distributions* is the dual space of $\mathcal{S}$. Also, we write $\mathcal{E}'$ for the dual space to C^∞, the space of bounded infinitely differentiable functions. Since $C_0^\infty \subseteq \mathcal{S} \subseteq C^\infty$, then $\mathcal{E} \subseteq \mathcal{S}' \subseteq \mathcal{D}'$.

The Fourier transform acts nicely on $\mathcal{S}$; for $f(x) \in \mathcal{S}$ define the Fourier transform as usual

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int e^{-i\xi\cdot y} f(y)\, dy = \mathcal{F}(f)(\xi) \ .$$

One checks that this integral converges absolutely because $f(x) \in \mathcal{S}$; indeed

$$|e^{-i\xi\cdot x} f(x)| \le \|f\|_{N,0} \frac{1}{(1+|x|^2)^{\frac{N}{2}}} \ ,$$

and $N > n+1$ will do for this. For Schwartz class $f(x)$ we may exchange integrations and differentiations as much as we wish, therefore the following list of properties holds for $\hat{f}(\xi)$.

Proposition 5.9 (Elementary properties of the Fourier transform on $\mathcal{S}$)**.**
(1) $\widehat{\partial_x f}(\xi) = (i\xi)\hat{f}(\xi)$.

(2) $\widehat{xf}(\xi) = i\partial_\xi \hat{f}(\xi)$.
Notice the vectorial notation; x and $i\partial_\xi$ are vector operations.

Define two operations on functions: $\tau_h f(x) = f(x-h)$ (translations) and $(\sigma_\lambda f)(x) = f(\lambda x)$ (dilations).

(3) $\widehat{\tau_h f}(\xi) = e^{-ih\xi}\hat{f}(\xi)$ a *position boost.*

(4) $\widehat{e^{ihx} f}(\xi) = \hat{f}(\xi - h)$ a *momentum boost.*

(5) $\widehat{\sigma_\lambda f}(\xi) = \frac{1}{|\lambda|^n}\sigma_{(1/\lambda)}(\hat{f})(\xi) = \frac{1}{|\lambda|^n}\hat{f}(\xi/\lambda)$.

The principal reason why Schwartz class is well-suited for Fourier analysis is that it is invariant under the Fourier transform.

Theorem 5.10. *The Fourier transform maps $\mathcal{S}$ onto itself; that is, whenever $f \in \mathcal{S}$, then $\hat{f} = \mathcal{F}(f) \in \mathcal{S}$.*

Proof. For $f(x) \in \mathcal{S}$, at the very least $\hat{f}(\xi)$ makes sense as a convergent integral. In order to check that $\hat{f}(\xi) \in \mathcal{S}$, we have to check that all of the seminorms are finite, which we do using the properties described in

Proposition 5.9.

$$\begin{aligned}\|\hat{f}\|_{\alpha,\beta} &= \sup_{\xi} |\xi^{\alpha}\partial_{\xi}^{\beta}\hat{f}(\xi)| \\ &= \sup_{\xi} \Big| \frac{1}{\sqrt{2\pi}^{n}} \int e^{-i\xi\cdot x}((\frac{1}{i}\partial_x)^{\alpha}(\frac{1}{i}x)^{\beta} f(x))\, dx \Big| .\end{aligned}$$

This integrand is still bounded by $C_{\beta+N,\alpha}(1+|x|^2)^{-\frac{N}{2}}$, and hence the supremum over $\xi \in \mathbb{R}^n$ is finite. Furthermore, the Fourier transform is continuous on $\mathcal{S}$. Suppose that $\{f_n(x)\}_{n=1}^{\infty}$, and $f_n \underset{\mathcal{S}}{\rightarrow} f$. This implies that for each α, β,

$$\|\hat{f}_n - \hat{f}\|_{\alpha,\beta} = \sup_{\xi} \Big| \frac{1}{\sqrt{2\pi}^{n}} \int e^{-i\xi\cdot x}((\frac{1}{i}\partial_x)^{\alpha}(\frac{1}{i}x)^{\beta}(f_n(x) - f(x)))\, dx \Big| \rightarrow 0$$

with $n \rightarrow \infty$ as well. Hence $\hat{f}_n(\xi) \underset{\mathcal{S}}{\rightarrow} \hat{f}(\xi)$, and therefore the mapping $\hat{f} : \mathcal{S} \rightarrow \mathcal{S}$ is continuous. □

Finally we are prepared to prove the central theorem of Fourier inversion on $\mathcal{S}$.

Theorem 5.11 (Fourier inversion theorem on $\mathcal{S}$)**.** *Suppose that $f \in \mathcal{S}$, then*

$$f(x) = \frac{1}{\sqrt{2\pi}^{n}} \int e^{i\xi\cdot x}\hat{f}(\xi)\, d\xi = \mathcal{F}^{-1}(\hat{f})(x) \ .$$

Proof. Lets first note that this works if $f(x)$ is a Gaussian. For $G(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{|x|^2}{2}}$ then $\hat{G}(\xi) = \frac{1}{\sqrt{2\pi}}e^{-\frac{|\xi|^2}{2}}$, by explicit calculation. One interpretation of this fact is that $G(x)$ is an eigenfunction of the Fourier transform, $\mathcal{F}(G) = G$, with eigenvalue 1. Now for the case of general $f(x) \in \mathcal{S}$, use properties given in (5.7),

$$\begin{aligned} f(0) &= \lim_{\varepsilon\rightarrow 0} \int f(x)\frac{1}{\varepsilon^n}G(\frac{x}{\varepsilon})\, dx = \lim_{\varepsilon\rightarrow 0} \int f(x)\frac{1}{\varepsilon^n}\sigma_{\frac{x}{\varepsilon}}(G)\, dx \\ &= \lim_{\varepsilon\rightarrow 0} \int f(x)\frac{1}{\varepsilon^n}\sigma_{\frac{x}{\varepsilon}}(\hat{G})\, dx = \lim_{\varepsilon\rightarrow 0} \int f(x)\widehat{\sigma_\varepsilon(G)}\, dx \\ &= \lim_{\varepsilon\rightarrow 0} \int \hat{f}(\xi)\sigma_\varepsilon(G)(\xi)\, d\xi \ . \end{aligned}$$

We have to verify the last step of this sequence of calculations.

Lemma 5.12. For $f, g \in \mathcal{S}$, then

$$\int f(x)\overline{\hat{g}(x)}dx = \int \hat{f}(\xi)\overline{g(\xi)}d\xi \ .$$

Proof of Lemma 5.12.

$$\int f(x)\frac{1}{\sqrt{2\pi}^n}\int e^{+i\xi\cdot x}\,dx\,\overline{g(\xi)}\,d\xi = \frac{1}{\sqrt{2\pi}^n}\int f(x)\int e^{+i\xi\cdot x}\,dx\,\overline{g(\xi)}\,d\xi$$
$$= \int (\hat{f}(\xi)\overline{g(\xi)})\,dx\ .$$

□

Now apply (5.7) to the result. Since $G(\xi)\big|_{\xi=0} = \frac{1}{\sqrt{2\pi}^n}$;

$$f(0) = \lim_{\varepsilon\to 0}\int \hat{f}(\varepsilon)G(\varepsilon/\xi)\,d\xi = \frac{1}{\sqrt{2\pi}^n}\int \hat{f}(\xi)\,d\xi\ .$$

This recovers $f(0) = \frac{1}{\sqrt{2\pi}^n}\int \hat{f}(\xi)\,d\xi$. To generalize this procedure to cover all $x \in \mathbb{R}^n$ is in fact simple, again using the list of properties in Proposition 5.9:

$$f(x) = (\tau_{-x}f)(0) = \frac{1}{\sqrt{2\pi}^n}\int \mathcal{F}(\tau_{-x}f)(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}^n}\int e^{i\xi\cdot x}\hat{f}(\xi)\,d\xi\ .$$

□

We can finish this circle of ideas with a proof of the $L^2(\mathbb{R}^n)$ Fourier inversion theorem. We have shown that Schwartz class $\mathcal{S} \subseteq L^2(\mathbb{R}^n)$ is a dense subspace. The Lemma 5.12 states that for $f, g \in \mathcal{S}$, then

$$\langle f, \mathcal{F}^{-1}g\rangle = \langle \mathcal{F}f, g\rangle\ ,$$

where we also note that

$$(\mathcal{F}^{-1}g)(x) = \hat{g}(-\xi)\ .$$

Therefore for every $f \in \mathcal{S}$,

$$\begin{aligned}\|f\|_{L^2} &= \langle f, \mathcal{F}^{-1}(\mathcal{F}f)\rangle \\ &= \langle \mathcal{F}(f), \mathcal{F}(f)\rangle \\ &= \|\hat{f}\|_{L^2}^2.\end{aligned}$$

so the Fourier transform acts on the the subspace $\mathcal{S} \subseteq L^2(\mathbb{R}^n)$ isometrically (in the sense of the L^2-norm). Since $\mathcal{S}$ is dense, given an arbitrary $f \in L^2(\mathbb{R}^n)$, there is a sequence $f_n \to f$ in the $L^2(\mathbb{R}^n)$ sense of convergence, with $f_n \in \mathcal{S}$. Then we define

$$\mathcal{F}(f) = \lim_{n\to\infty}\mathcal{F}(f_n).$$

Theorem 5.13 (of functional analysis)**.** *: Suppose that an operator* $\mathbf{T}$ *is defined on a dense subspace* $\mathcal{S} \subseteq \mathcal{B}$ *a Banach space, and it is bounded;* $\|\mathbf{T}f\|_{\mathcal{B}} \le C\|f\|_{\mathcal{B}}$ *for all* $f \in \mathcal{S}$. *Then there exists a unique extension* $\mathbf{T}^{(1)}$ *of* $\mathbf{T}$ *to all of* $\mathcal{B}$*, which is bounded with the same bound.*

Proof. For arbitrary $f \in \mathcal{B}$ consider a sequence $\{f_n\}_{n=1}^{\infty} \to f$, with $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$. Then $\{f_n\}_{n=1}^{\infty}$ is Cauchy, and so is the sequence $\{\mathbf{T}f_n\}_{n=1}^{\infty}$ because

$$\|\mathbf{T}f_n - \mathbf{T}f_m\|_{\mathcal{B}} \leq C\|f_n - f_m\|_{\mathcal{B}} \to 0 \ .$$

Thus $\{\mathbf{T}f_n\}_{n=1}^{\infty}$ has a limit $g \in \mathcal{B}$, which we define to be $\mathbf{T}^{(1)}f := g$. The extension is clearly linear. It is also well-defined, for if $\{h_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$ is another sequence such that $\{h_n\}_{n=1}^{\infty} \to f$, but $Th_n \to g_1$, then

$$\|g - g_1\|_{\mathcal{B}} = \lim_{n\to\infty} \|\mathbf{T}f_n - \mathbf{T}h_n\|_{\mathcal{B}} \leq C \lim_{n\to\infty} \|f_n - h_n\|_{\mathcal{B}} = 0 \ .$$

□

The conclusion that is relevant to the Fourier transform is that $\mathcal{F}$ restricted to $\mathcal{S}$ (such that it has norm $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$ on $\mathcal{S}$) extends uniquely to $\mathcal{F}^{(1)}$ defined on all of $L^2(\mathbb{R}^n)$. Furthermore, this extension is an isometry of $L^2(\mathbb{R}^n)$, as for $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{S} \subseteq L^2(\mathbb{R}^n)$, with $f_n \to f$;

$$\|f\|_{L^2}^2 = \lim_{n\to\infty} \|f_n\|_{L^2}^2 = \lim_{n\to\infty} \|\hat{f}_n\|_{L^2}^2 = \|\mathcal{F}^1(f)\|_{L^2}^2 \ .$$

This completes the proof of 5.4. We will redefine the extension notation $\mathcal{F}^{(1)} := \mathcal{F}$ at this time, dropping the extra superscript. The statement is then that $\|f\|_{L^2} = \|\mathcal{F}f\|_{L^2}$ for all $f \in L^2$, which is the desired result.

To finish our remarks we should note that complex isometries $\mathbf{U}$ have the property that $\mathbf{U}^* = \mathbf{U}^{-1}$; namely they are unitary operators. Lets verify this with the Fourier transform.

Definition 5.14. The adjoint $\mathbf{T}^*$ of an operator $\mathbf{T}$ on a Hilbert space H satisfies $\langle \mathbf{T}^*f, g\rangle = \langle f, \mathbf{T}g\rangle$ for all $f, g \in H$.

Applying this definition of the adjoint to $\mathcal{F}$, take any two $f, g \in L^2$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} g(\xi)\overline{\mathcal{F}(f)(\xi)}\, d\xi &= \int_{\mathbb{R}^n} g(\xi)\overline{\frac{1}{\sqrt{2\pi}^n}\int e^{-i\xi\cdot x} f(x)\, dx d\xi} \\
&= \int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi}^n} e^{i\xi\cdot x} g(\xi)\, d\xi)\overline{f(x)}\, dx \\
&= \int_{\mathbb{R}^n} \mathcal{F}^{-1}(g)(x)\overline{(f(x))}\, dx \ ,
\end{aligned}$$

which states that $\mathcal{F}^* = \mathcal{F}^{-1}$, as desired.

Exercises: Chapter 5

Exercise 5.1. Show that the Fourier transform preserves angles between vectors. For $f, g \in L^2$ then,

$$Re\langle f, g\rangle = \|f\|_{L^2}\|g\|_{L^2}\cos\theta$$

and

$$Re\langle \hat{f}, \hat{g}\rangle = \|\hat{f}\|_{L^2}\|\hat{g}\|_{L^2}\cos\varphi \ .$$

Show that $\theta = \varphi$.

Exercise 5.2. The $L^p(\Omega)$ norm for functions on a domain $\Omega \subseteq \mathbb{R}^n$ is defined as

$$\|f\|_{L^p} = \Big(\int_\Omega |f(x)|^p\,dx\Big)^{1/p} \ ,$$

for $1 \le p < +\infty$. The Hölder inequality states that

$$\Big|\int_\Omega f(x)g(x)\,dx\Big| \le \|f\|_{L^p}\|g\|_{L^{p'}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ are dual indices. A special case for $p = p' = 2$ is the Cauchy – Schwarz inequality.

Give a proof of the Hölder inequality for the range of p, p' given above.

Exercise 5.3. This problem concerns the case of domains $\Omega = \mathbb{R}^n$ and of bounded domains $\Omega \subseteq \mathbb{R}^n$.

(1) In the case of $\Omega = \mathbb{R}^n$, for which indices p and q does the following hold:

$$L^p(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n) \ ?$$

(2) In the case of bounded Ω, use the Hölder inequality to show that

$$L^p(\Omega) \subseteq L^q(\Omega) \ .$$

for $q \le p$.

(3) In the case of bounded Ω, is it true that

$$L^\infty(\Omega) = \bigcap_{1\le p<+\infty} L^p(\Omega) \ .$$

Exercise 5.4. Prove the part (i) of the technical lemma 5.7 on Schwartz class functins. Namely show that for $g \in \mathcal{S}(\mathbb{R}^n)$ such that $g(0) = 1$, then for all $f \in \mathcal{S}(\mathbb{R}^n)$

$$\mathcal{S} - \lim_{\varepsilon\to 0}\big(g(\varepsilon x)f(x)\big) = f(x) \ .$$

Exercise 5.5. Give examples of non-analytic, C^∞ functions on $\mathbb{R}^n$.

Exercise 5.6. We have described the Fourier transform $\mathcal{F}$ as a unitary operator on $L^2(\mathbb{R}^n)$. What are the eigenvalues and eigenfunctions $(\lambda_k, \psi_k(x))$ of this operator. For simplicity you may consider only the case $n = 1$.

Exercise 5.7. The Schrödinger operator that describes quantum harmonic oscillator is given by

$$\tfrac{1}{2}\Delta\psi - \tfrac{1}{2}|x|^2\psi = \lambda\psi \ .$$

Show that it is invariant under the Fourier transform, and describe its eigenfunctions and eigenvalues.