Math 4FT /Math 6 FT
Problem Set #2

Problem 1. Show that the heat operator satisfies the semigroup property; for all $0 < s < t$

$$\mathbb{H}(t) = \mathbb{H}(t-s)\mathbb{H}(s)$$

Problem 2. Justify on a rigorous level of analysis the exchange of integrations in the proof of Proposition 3.4(iii), therefore completing the rigorous proof of the proposition's three parts.

Problem 3. Solve the following initial value problems for the heat equation in explicit terms.

1. $f(x) = x$.
2. $f(x) = x^2$
3. $f(x) = 0$ for $x < 0$, and $f(x) = 1$ for $x \geq 0$.
4. $f(x) = e^{\alpha x}$
5. $f(x) = \sin(kx)$

What is the asymptotic behavior of $u(t, x)$ as $t \to +\infty$. Does it matter if $f(x) \notin L^1(\mathbb{R}^1)$

Problem 4. *(method of images)* Derive the heat kernel and the solution method for the initial boundary value problem on $\{(t, x) : x > 0, t > 0\}$ in the various cases of the boundary conditions given below.

1 Dirichlet boundary conditions:
$$u(t, 0) = 0 \ .$$

2 Neumann boundary conditions
$$\partial_x u(t, 0) = 0 \ .$$

3 Periodic boundary conditions over $0 \leq x < 2\pi$

$$u(t, x + 2\pi) = u(t, x) \ .$$