Math 4FT /Math 6 FT
Problem Set #4

Problem 1. The $L^p(\Omega)$ norm for functions on a domain $\Omega \subseteq \mathbb{R}^n$ is defined as

$$\|f\|_{L^p} = \Big(\int_\Omega |f(x)|^p \, dx\Big)^{1/p} ,$$

for $1 \le p < +\infty$. The Hölder inequality states that

$$\Big|\int_\Omega f(x)g(x)\, dx\Big| \le \|f\|_{L^p}\|g\|_{L^{p'}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ are dual indices. A special case for $p = p' = 2$ is the Cauchy − Schwarz inequality.

Give a proof of the Hölder inequality for the range of p, p' given above.

Problem 2. This problem concerns the case of domains $\Omega = \mathbb{R}^n$ and of bounded domains $\Omega \subseteq \mathbb{R}^n$.

1. In the case of $\Omega = \mathbb{R}^n$, for which indices p and q does the following hold:

$$L^p(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n) \ ?$$

2. In the case of bounded Ω, use the Hölder inequality to show that

$$L^p(\Omega) \subseteq L^q(\Omega) \ .$$

   for $q \le p$.

3. In the case of bounded Ω, is it true that

$$L^\infty(\Omega) = \bigcap_{1\le p<+\infty} L^p(\Omega) \ .$$

Problem 3. Prove the part (i) of the technical lemma on Schwartz class. Namely show that for $g \in \mathcal{S}(\mathbb{R}^n)$ such that $g(0) = 1$, then for all $f \in \mathcal{S}(\mathbb{R}^n)$

$$\mathcal{S} - \lim_{\varepsilon\to 0} \big(g(\varepsilon x)f(x)\big) = f(x) \ .$$

Problem 4. We have described the Fourier transform $\mathcal{F}$ as a unitary operator on $L^2(\mathbb{R}^n)$. What are the eigenvalues and eigenfunctions $(\lambda_k, \psi_k(x))$ of this operator. For simplicity you may consider only the case $n = 1$.