Math 4FT /Math 6 FT
Final Exam

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Math 4FT students: Do at least four of the six problems below.
Math 6FT students: Do all six problems below.
Due date: Tuesday December 15, 2015.

Problem 1. *Equations with variable dispersion*

Consider the linear evolution equation

$$\begin{aligned} &\partial_t u + x\partial_x^3 u = 0 \ , \\ &u(0,x) = h(x) \in L^2(\mathbb{R}^1) \ , \end{aligned}$$

which is related to the KdV equation.

(a) Give an explicit expression for the solution, using the Fourier transform and the method of characteristics.

(b) For initial data $h(x)$ such that $\mathrm{supp}(\hat{h}(\xi)) \subseteq B_R(0)$, what is the (future) lifespan $[0, T^*)$ of the solution? What happens to the solution as $t \to T^*$?

(c) Is the solution unique?

(d) For $t < 0$ what is the lifespan of the solution?

Problem 2. *Convolutions and the central limit theorem*

Consider a function $h(x) \in L^1(\mathbb{R}^1)$ which satisfies

$$\int h(x)\, dx = 1 \ , \quad \int x h(x)\, dx = 0 \ , \quad \int x^2 h(x)\, dx = \sigma^2 < +\infty \ .$$

(a) Show that Show that $\hat{h}(0) = \frac{1}{\sqrt{2\pi}}$. Furthermore show that the multiple convolutions $h^{(n)}(x) := h * h \ldots (n\times) \ldots h(x)$ also satisfy

$$\int h^{(n)}\, dx = 1 \ ,$$

and therefore $\hat{h}^{(n)}(0) = \frac{1}{\sqrt{2\pi}}$.

(b) Explain why $\hat{h}$ is twice continuously differentiable at $\xi = 0$. It follows that in a neighborhood of $\xi = 0$ we have

$$\hat{h}(\xi) = \frac{1}{\sqrt{2\pi}}\big(1 + i\xi\hat{h}'(0) - \frac{\xi^2}{2}\hat{h}''(0) + O(\xi^3)\big) \ .$$

Furthermore $\hat{h}'(0) = 0$.

(c) Rescaling by $\sqrt{n}$ and taking the Fourier transform, show that

$$\mathcal{F}(h^{(n)})(\frac{\xi}{\sqrt{n}}) = (\hat{h}(\frac{\xi}{\sqrt{n}}))^n = \frac{1}{\sqrt{2\pi}}\Big(1 - \frac{\xi^2}{2n}\hat{h}''(0)\Big)^n + o(\frac{1}{n}) \ .$$

In the limit $n \to +\infty$ this quantity converges to

$$\frac{1}{\sqrt{2\pi}} e^{-(\xi^2/2)\sigma^2} \ .$$

This shows that repeated convolution, in the (appropriately rescaled) limit converges to the Gaussian

$$\lim_{n\to+\infty} h^{(n)}(\sqrt{n}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)} \ ,$$

which is one way to state the central limit theorem.

Problem 3. *Laplace's equation with a reentrant corner.* Consider domains Ω in $\mathbb{R}^2$ consisting of a disk $B_1(0)$ intersect the conic neighborhood $\{(r,\theta) : 0 < \theta < \pi/\alpha\}$, where $1/2 < \alpha$ is a constant. When $1/2 < \alpha < 1$ this is a domain with a corner removed.

(a) Show that $u(x,y) = \mathrm{im}(z^\alpha)$ is a harmonic function on Ω (where we are using complex notation). Show that u satisfies Dirichlet boundary conditions on the two boundary components $\{\theta = 0, 0 < r < 1\}$ and $\{\theta = \pi/\alpha, 0 < r < 1\}$, and that u is bounded on the third boundary component consisting of the arc $\{r = 1, 0 < \theta < \pi/\alpha\}$.

(b) Show that the gradient $\nabla u(x,y)$ is not in $L^p(\Omega)$ for some range of $2 < p \leq +\infty$.

Problem 4. *The Cauchy problem for the wave equation, and Duhamel's principle*
Consider the wave equation on $\mathbb{R}^1_t \times \mathbb{R}^n_x$,

$$\Box u = \partial_t^2 u - \Delta u = 0 \ ,$$

with the special initial data

$$u(0,x) = 0 \ , \quad \partial_t u(0,x) = g(x) \ .$$

Define the solution operator for this problem to be

$$\mathbf{W}(g)(t,x) = u(t,x) \ ,$$

so that

$$\mathbf{W}(g)(0,x) = 0 \ , \quad \partial_t \mathbf{W}(g)(0,x) = g(x) \ .$$

(a) Show that the solution with general initial data $u(0,x) = f(x)$ and $\partial_t u(0,x) = g(x)$ can be expressed in terms of the superposition

$$u(t,x) = \partial_t \mathbf{W}(f)(t,x) + \mathbf{W}(g)(t,x) \ .$$

(b) In the case $n = 1$ give an expression for the operator $\mathbf{W}(g)$ in terms of the d'Alembert formula.

In the case $n = 3$ give an expression for the operator $\mathbf{W}(g)$ in terms of the Kirchhoff formula and spherical means.

(c) Consider the inhomogeneous problem for the wave equation

$$\Box u = h(t,x) \ ,$$

where without loss of generality we may set $f = g = 0$. Show that the solution can be expressed in terms of $\mathbf{W}(g)(t,x)$ as follows:

$$u(t,x) = \int_0^t \mathbf{W}(h(s,\cdot))(t-s,x)\, ds \ . \tag{1}$$

This is the content of the *Duhamel principle* in the case of the wave equation. Give an explicit expression for (1) in the case that $n = 1$.

Problem 5. *Gaussian wave packets*

(a) Express the solutions of the free Schrödinger equation

$$\frac{1}{i}\partial_t \psi = -\tfrac{1}{2}\partial_x^2 \psi \ , \qquad x \in \mathbb{R}^1 \ ,$$

with the initial data

$$\psi_0(x) = e^{-Ax^2/2} e^{ikx} \ .$$

(b) Calculate the first several moments of the solution

$$m_0(\psi(t,\cdot) \ , \quad m_1(\psi(t,\cdot) \ , \quad \hat{m}_1(\psi(t,\cdot) \ , \quad m_2(\psi(t,\cdot)$$

Describe the trajectory of the solution.

Problem 6. *Soliton solutions of the KdV*
The KdV equation is

$$\partial_t q = -\frac{1}{6}\partial_x^3 q + 2q\partial_x q \ .$$

Soliton solutions (in the case of single solitons) are traveling waves, taking the form $q(t,x) = q(x-ct)$ for some velocity c.

(a) Show that such solutions satisfy

$$\frac{1}{6}\frac{d^2}{dx^2}q - q^2 = cq + Const.$$

(b) The *energy* of KdV solutions is defined as

$$E(q) = \int_{-\infty}^{+\infty} \frac{1}{12}(\partial_x q)^2 + \frac{1}{3}q^3 \, dx$$

and the *momentum* is

$$I(q) = \int_{-\infty}^{+\infty} \tfrac{1}{2}q^2 \, dx \ .$$

Show that single soliton solutions of the KdV are critical points of the energy $E(q)$ for fixed momentun $I(q)$. What is the associated Lagrange multiplier?

(c) Solve the equations to find the one parameter family of single soliton solutions.