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Arnold Diffusion in Symplectic Lattice Maps

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Abstract. In numerical experiments with Hamiltonian systems, roundoff errors can be avoided by using symplectic lattice maps. This increases the accuracy of long-term studies of individual orbits. However, it is shown here that estimates of the mean local Arnold diffusion rate (in coupled standard maps) do not change when a lattice map is used. This is important because it reinforces the belief that statistical properties of ensembles of orbits are not significantly affected by roundoff error.

1 Introduction

Most numerical studies of Hamiltonian systems involve iterating a symplectic map on a computer using floating-point arithmetic. Although maps take discrete jumps, in most cases roundoff error prevents them from being iterated exactly. The resulting trajectories are different from those of the original system and, moreover, they are not exactly symplectic. The basic reason is that floating-point operations can map many distinct points to the same place even when the exact mapping is one-to-one. This can significantly alter long-term behaviour. In particular, a typical trajectory does not return to its initial point; instead it eventually joins a periodic orbit not including its initial point. Earn and Tremaine ([1991] [1992]; hereafter ET) showed that roundoff errors cause artificial drifting across invariant curves and can even lead to confusion between regularity and chaos.

A *lattice map* is a function that maps a discrete subset of phase space (a lattice) into itself. Any continuous function f on a phase space defines a natural lattice map F on a given lattice via "$F(p,q)$ is the nearest lattice point to $f(p,q)$". Rescaling makes it possible to ensure that all lattice points have integer phase coordinates.

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This means that error-free addition and multiplication of phase coordinates can be done on any computer. Because of this the lattice maps considered below can be iterated exactly without errors of any sort. Rannou [1974] used this method to study area-preserving maps of the plane.

We say that a lattice map is symplectic (or Hamiltonian) if there is a symplectic diffeomorphism that, when applied to points on the lattice, agrees with the lattice map. The natural lattice maps associated with a large class of symplectic maps (and symplectic integration algorithms) are in fact symplectic lattice maps (ET; Scovel [1991]). This is the principal motivation for using lattice maps to study Hamiltonian systems: spurious dissipative behaviour is impossible because the computed system is exactly symplectic.

Using lattice maps rather than ordinary floating-point maps reduces qualitative errors in orbits of Hamiltonian systems (ET). Quantitative errors (in known integrals of motions) are also significantly reduced, even in short trajectory computations (Earn [1994]). Nevertheless, while individual trajectories are better represented with lattice maps, it is not clear that statistical properties of ensembles of orbits are significantly affected by roundoff error.

An important measure of the average behaviour in a particular region of phase space in a nearly integrable Hamiltonian system (with three or more degrees of freedom) is the local Arnold diffusion rate. This is the (typically slow) rate at which a collection of orbits diffuses *across* stochastic layers (as opposed to motion *along* stochastic layers, which does not require three or more degrees of freedom and is typically fast). (See, *e.g.*, Lichtenberg and Lieberman [1992], Chapter 6.)

In this paper we compare the local Arnold diffusion rates computed numerically with floating-point and lattice versions of a map for which the local Arnold diffusion rate has been calculated analytically.

2 Coupled standard maps

The standard map (*e.g.*, Lichtenberg and Lieberman [1992], Chapter 4) is the plane area-preserving map given by

$$\begin{aligned} x_{n+1} &= x_n + y_{n+1}\,, \\ y_{n+1} &= y_n + \tfrac{K}{2\pi}\sin(2\pi x_n)\,, \end{aligned} \tag{2.1}$$

where K is the stochasticity parameter. This map is spatially periodic with period 1 in both x and y. The floating-point version of this map results from iterating (2.1) with floating-point arithmetic. The natural lattice map is obtained by multiplying (2.1) by a (large) integer, say m, and then replacing the sine term with the nearest integer to its value. The integer m gives the number of lattice points per unit in phase space and thus defines the resolution of the lattice map. Floating-point and lattice versions of the standard map were studied in detail by ET.

A simple four-dimensional Hamiltonian map can be constructed by coupling two standard maps,

$$\begin{aligned} u_{n+1} &= u_n + v_{n+1}\,, \\ v_{n+1} &= v_n + \tfrac{K_1}{2\pi}\sin(2\pi u_n) + \tfrac{\mu}{2\pi}\sin 2\pi(u_n + x_n)\,, \\ x_{n+1} &= x_n + y_{n+1}\,, \\ y_{n+1} &= y_n + \tfrac{K_2}{2\pi}\sin(2\pi x_n) + \tfrac{\mu}{2\pi}\sin 2\pi(u_n + x_n)\,. \end{aligned} \tag{2.2}$$

Here K_1 and K_2 are the stochasticity parameters for each standard map and μ is the coupling constant. The two maps are uncoupled for $\mu = 0$. Equation (2.2) can be thought of as a Poincaré return map for a continuous Hamiltonian system with three degrees of freedom. The natural lattice map for (2.2) is derived in the same way as for the ordinary standard map above.

3 Arnold diffusion rate

Wood, Lichtenberg and Lieberman [1990] derived an analytic approximation for the local Arnold diffusion rate in the system of two coupled standard maps [(2.2)] and compared their results with numerical orbit calculations (using floating-point arithmetic). The agreement was good, but it is important to know if it can be improved further by using a lattice map to iterate the orbits.

We choose the three parameters in (2.2) so that the map will display observable Arnold diffusion. Thus we take K_1 large enough to ensure that there are large stochastic regions in u-v space, K_2 small enough to ensure that there are large regular regions in x-y space, and $0 < \mu \ll 1$ so the coupling is weak. Initial conditions are selected in a primary stochastic region of u-v space and a region of x-y space dominated by invariant curves (far from sizable island chains that would distort the diffusion calculation). To achieve the desired property in x-y space we place the initial points as close as possible to the KAM curve that is the last to survive as K is increased in (2.1). Greene [1979] found that the "last KAM curve" has rotation number $\frac{1}{2}(1+\sqrt{5})$, the golden mean, or $1/2\pi$ times this value in our units in which the spatial period of the standard map (2.1) is 1 rather than 2π. [The rotation number or frequency of an invariant curve of the standard map is equal to the average action (y-value) of any orbit on the curve. (See, *e.g.*, Meiss [1992]).]

To estimate the local Arnold diffusion rate, we iterate a sample of trajectories and look at the RMS change in the action y until this change is of order 0.01. Our initial configuration consists of 256 points on a 16^2 grid in a primary stochastic region of u-v space, each associated with the same initial x-y coordinates. The initial (x, y) is chosen by taking $x_0 = 0$ and searching (by bisection) for y_0 such that the average y value after 10^5 iterations of (2.1) agrees with the golden mean to 10 significant figures.

Figure 1 shows the RMS change in y as a function of time n, for four and eight byte floating-point maps and a lattice map with $m = 2^{52}$ points per unit. Wood *et. al.* [1990] predict that $\Delta y_{\text{rms}} \sim n^{1/2}$ for the exact map (2.2) so the curves drawn in Figure 1 should be lines with slope $\frac{1}{2}$. This is in reasonable agreement with all our numerical results; the agreement is no better in the cases in which the lattice map has been used.

4 Discussion

Tennyson, Lichtenberg and Lieberman [1979] investigated Arnold diffusion in a different four-dimensional symplectic map. ET repeated part of their study with a lattice map and found a significantly different global Arnold diffusion rate, which suggested that some of the diffusion seen by Tennyson *et. al.* [1979] might have been due to roundoff errors. In this case the diffusion rate was estimated crudely from the evolution of a single long trajectory; it emphasizes that the long-term qualitative features of individual trajectories can be strongly influenced by numerical errors.

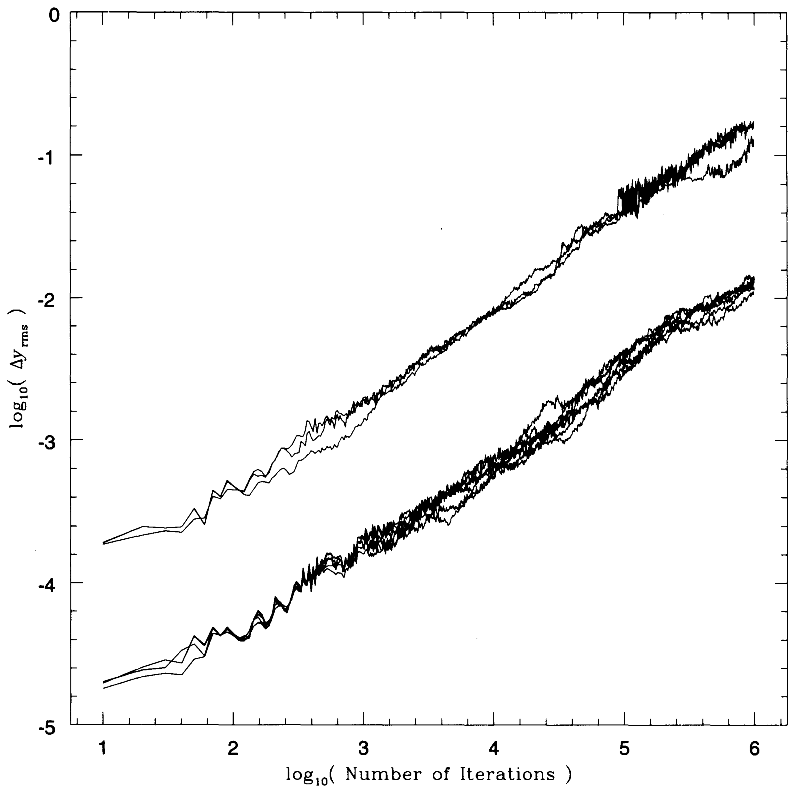

Figure 1 A log-log plot of the RMS action change (due to Arnold diffusion) as a function of time in the system of two coupled standard maps, (2.1). There are 3 curves in the upper group. These all have coupling constant $\mu = 0.001$ and were all obtained using floating-point arithmetic. Two values of K_2 (0.2 and 0.4) were used and the $K_2 = 0.4$ run was done with both 4 byte and 8 byte floating-point arithmetic. There are 6 curves in the lower group. These all have $\mu = 0.0001$. The same values of K_2 and floating-point number size were used as for the upper curves, but this time each run was repeated with a lattice map with $m = 2^{52}$ points per unit. There is nothing to choose from between the various different ways of computing these curves (all are consistent with a slope of 1/2) so this calculation is clearly not seriously affected by roundoff error.

However, precisely because individual trajectories vary so much, it is essential to use a sizable sample to estimate a diffusion rate reliably. The present study has shown that the errors introduced by floating-point arithmetic do not corrupt estimates of the *local* Arnold diffusion rate based on an *ensemble* of trajectories. Strictly, this conclusion is valid only for the specific examples considered here; it remains possible that with sufficiently weak coupling (small enough μ) or with a much more complicated map (requiring many more arithmetic operations per iteration) significant differences may appear between ensemble estimates of Arnold diffusion rates based on ordinary floating-point versus lattice map computations. Nevertheless, the present results are likely to be typical of many symplectic maps and Hamiltonian systems evolved with symplectic integrators, and they strengthen the general belief that statistical properties of ensembles of orbits are not significantly affected by roundoff error.

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