

# Potential–density basis sets in axisymmetric coordinates

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Accepted 1996 April 12. Received 1996 April 10; in original form 1995 September 13

## ABSTRACT

We present a general scheme for constructing potential–density basis sets in axisymmetric coordinates, for both infinite and finite systems. Several examples are given, starting from simple functions. The basis sets constructed are useful for galaxy modelling,  $N$ -body simulations and three-dimensional stability analyses of dynamical equilibria. Symbolic manipulation software implementing the technique is available.

**Key words:** instabilities – methods: numerical – celestial mechanics, stellar dynamics – galaxies: kinematics and dynamics – galaxies: structure.

## 1 INTRODUCTION

Potential–density (PD) pairs are the basic building blocks of galaxy models (e.g. Binney & Tremaine 1987, chapter 2). Due to the linearity of Poisson’s equation,  $\nabla^2\Phi=4\pi G\rho$ , complicated models can be constructed as linear combinations of simple PD pairs. More to the point, the potential and density of any reasonable mass configuration can be approximated arbitrarily well by sufficiently many terms of an expansion in a complete basis of PD pairs. Such expansions are exceedingly useful, for modelling of real objects,  $N$ -body simulations and stability analyses of dynamical equilibria.

Much effort has been devoted to finding PD basis sets where both the potential and density are simple expressions in elementary or special functions (e.g. Clutton-Brock 1972, 1973; Kalnajs 1976; Hernquist & Ostriker 1992; Qian 1992, 1993; Earn 1996; Zhao 1996). Discovering convenient, bi-orthogonal basis sets suited to particular problems is usually very difficult, and attempts can easily lead to intractable integrals or unmanageable expressions. However, every set of basis functions can be made orthogonal using the Gram–Schmidt algorithm. If the integrals that appear in the inner products are known analytically, a bi-orthonormal set can be constructed analytically, although the actual computations may be lengthy and tedious. Fortunately, modern computers running symbolic manipulation programs can perform the calculations, reducing the astronomer’s work to finding a suitable starting set of simple functions. After bi-orthonormal basis functions have been built from the original set, they can be tabulated and accurately interpolated for efficiency.

In this paper we describe several ways to generate PD basis sets starting from simple functions. The starting functions can be tailored to the problem at hand, which makes this approach very powerful. These methods are, in principle, applicable to all coordinate systems, but in this paper we concentrate on (orthogonal) axisymmetric coordinates. In the case of spherical coordinates, this type of basis construction has already been explored (Saha 1993) and applied to practical problems (Saha 1991). In many cases, spheroidal or cylindrical coordinates are more appropriate. In particular, some of the basis sets described below have been used in stability studies of oblate galaxy models (Robijn 1995; Robijn & de Zeeuw, in preparation). Syer (1995) has used Saha’s (1993) approach to construct a basis in prolate spheroidal coordinates, but his starting functions do not lend themselves to analytical orthogonalization.

Many PD sets in spherical coordinates are based on spherical harmonics. In Section 3 we generalize this and show how to build basis sets from spheroidal and cylindrical harmonics. Section 4 examines the use of harmonic PD sets to model finite systems.

## 2 COORDINATE SYSTEMS

The coordinates used in this paper are spherical (S), oblate spheroidal (OS), prolate spheroidal (PS) and cylindrical. All systems are denoted  $(u, v, \phi)$ , where  $u$  is the ‘radial’,  $v$  the ‘angular’ coordinate and  $\phi \in [0, 2\pi)$  the azimuth. In the case of cylindrical coordinates,  $u$  and  $v$  are Cartesian coordinates in the planes of fixed  $\phi$ . If we follow the deformation of the coordinates from spherical via spheroidal to cylindrical, we see that the lines of constant angular variable in prolate

coordinates become lines of constant  $z$  in the cylindrical limit; when viewed as the limiting case of an oblate spheroidal system, the ‘angular’ coordinate is  $R$ . The structure of the harmonic functions (Section 3) is different in each of these two limits, so we give them distinct labels in Table 1: CC is the cylindrical system where surfaces of constant ‘radial’ coordinate are cylinders, while in CP these surfaces are planes.

The metric coefficients  $h_\tau$  ( $\tau = u, v, \phi$ ) are defined by

$$h_\tau^2 = \left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial y}{\partial \tau}\right)^2 + \left(\frac{\partial z}{\partial \tau}\right)^2, \quad (2.1)$$

with  $(x, y, z)$  the standard Cartesian coordinates. The gradient and Laplacian operators are given by

$$\nabla = \left( \frac{1}{h_u} \frac{\partial}{\partial u}, \frac{1}{h_v} \frac{\partial}{\partial v}, \frac{1}{h_\phi} \frac{\partial}{\partial \phi} \right), \quad (2.2)$$

$$\nabla^2 = \frac{1}{h_u h_v h_\phi} \sum_{i=1}^3 \frac{\partial}{\partial \tau_i} \left( \frac{h_j h_k}{h_\tau} \frac{\partial}{\partial \tau_i} \right),$$

where  $(i, j, k)$  in the last equation is a cyclic permutation of  $(1, 2, 3)$ , and  $(\tau_1, \tau_2, \tau_3) = (u, v, \phi)$ .

### 3 PD PAIRS BASED ON HARMONICS

The potential  $\Phi$  that corresponds to a given density  $\rho$  is the unique solution of Poisson’s equation,  $\nabla^2 \Phi = 4\pi G \rho$  with appropriate boundary conditions. For galaxies,  $\rho$  must be non-negative, and it is customary to insist that  $\Phi$  be negative

and continuously differentiable. Satisfactory boundary conditions are that the spherical average of  $\Phi$  vanishes at infinity (e.g. Pfenniger 1984).

For the axisymmetric coordinate systems given in Table 1, the homogeneous Poisson equation (Laplace’s equation) can be separated into three ordinary differential equations (ODEs) when the potential is of the form  $\Phi_u(u)\Phi_v(v)\Phi_\phi(\phi)$ . In the S, OS and PS systems, the angular parts  $\Phi_v(v)\Phi_\phi(\phi)$  of the separated solutions (reviewed in the next section) are called spherical or spheroidal harmonics. Similarly, we group the  $v$  and  $\phi$  factors of the separated Laplace solutions together and call them ‘cylindrical harmonics’ in the CC and CP cases.

If a potential is separable with harmonic  $(v, \phi)$  factors, then the associated density is also of this form, and the radial functions for the PD pair are related through an ODE. Clutton-Brock (1973) and Hernquist & Ostriker (1992) solved the appropriate ODE in S coordinates to construct their basis sets; Earn (1996) solved the ODE in CP coordinates to build a class of basis sets for three-dimensional discs. In this paper, rather than try to find special solutions, we simply choose a set of radial basis functions for the potential and differentiate to find the corresponding density functions (the strategy of Saha 1991, 1993). Because the harmonics appear in both the potential and density, the inner product of two basis functions reduces to a one-dimensional integral, which can be evaluated rapidly numerically if not analytically.

The choice of radial basis set must be made with some care, since the boundary conditions must always be satisfied. Before formulating a useful condition for the radial functions alone, we take a closer look at Poisson’s equation.

**Table 1.** The axisymmetric coordinate systems used in this paper. The plots show the coordinate systems in the (Cartesian)  $x$ - $z$  plane, where  $z$  is the symmetry axis. The cylindrical system is present twice: surfaces of constant ‘radial’ coordinate  $u$  are cylinders (CC) or planes (CP). The scaling parameter  $e$  is determined by the position of the focal point  $z=e$  at  $\xi=\eta=1$  in PS and focal ‘circle’  $R=e$  at  $\xi=\eta=0$  in the OS system (cf. Abramowicz & Stegun 1972, section 21). For the plots  $e=1$  is used. Note that in CC and CP both  $u$  and  $v$  have dimensions (length); in S,  $v$  is dimensionless, while in PS and OS, both  $u$  and  $v$  are dimensionless ( $e$  has dimensions).

	CC	PS	S	OS	CP
	Cylindrical	Prolate Spheroidal	Spherical	Oblate Spheroidal	Cylindrical
$(u, v)$	$(R, z)$	$(\xi, \eta)$	$(r, \cos \theta)$	$(\xi, \eta)$	$(z, R)$
	$0 \leq u < \infty$ $-\infty < v < \infty$	$1 \leq u < \infty$ $-1 \leq v \leq 1$	$0 \leq u < \infty$ $-1 \leq v \leq 1$	$0 \leq u < \infty$ $-1 \leq v \leq 1$	$-\infty < u < \infty$ $0 \leq v < \infty$
$h_u$	1	$e\sqrt{\frac{u^2-v^2}{u^2-1}}$	1	$e\sqrt{\frac{u^2+v^2}{u^2+1}}$	1
$h_v$	1	$e\sqrt{\frac{u^2-v^2}{1-v^2}}$	$u$	$e\sqrt{\frac{u^2+v^2}{1-v^2}}$	1
$h_\phi$	$u$	$e\sqrt{(u^2-1)(1-v^2)}$	$u$	$e\sqrt{(u^2+1)(1-v^2)}$	$v$
$R$	$u$	$e\sqrt{(u^2-1)(1-v^2)}$	$u\sqrt{1-v^2}$	$e\sqrt{(u^2+1)(1-v^2)}$	$v$
$z$	$v$	$evv$	$uv$	$evv$	$u$

### 3.1 Separating Poisson’s equation

The separation of Poisson’s equation is treated in the literature (e.g. Morse & Feshbach 1953). In this subsection, we review some basic results to introduce our notation.

A separable solution of Poisson’s equation,  $\nabla^2\Phi = 4\pi G\rho$ , is of the form

$$\Phi(u, v, \phi) = \Phi_u(u) \Phi_v(v) \Phi_\phi(\phi). \quad (3.1)$$

A separable potential is *harmonic* if  $\Phi_\phi(\phi) = e^{im\phi}$  for an integer  $m$  and  $\Phi_v(v)$  is a solution of the ODE

$$\nabla_{v,m}^2 \Phi_v(v) + \alpha^2 \Phi_v(v) = 0, \quad (3.2)$$

where the one-dimensional ‘angular’ operator  $\nabla_{v,m}^2$  is given in Table 2 for each coordinate system. We take the separation constant  $\alpha$  to be real, in keeping with the standard convention for spherical coordinates (taking  $\alpha$  imaginary yields no additional independent solutions). For the S, OS and PS coordinates  $\alpha^2 = l(l+1)$ , where  $l \geq 0$  is an integer; for CC and CP,  $\alpha = l \geq 0$  is a real number.

We define the one-dimensional ‘radial’ operator  $\nabla_{u,\alpha,m}^2$  such that for a harmonic potential

$$\nabla^2\Phi = \frac{1}{c(u, v)} \Phi_v(v) e^{im\phi} \nabla_{u,\alpha,m}^2 \Phi_u(u), \quad (3.3)$$

where  $c(u, v)$  is a metric factor given in Table 2.  $\nabla_{u,\alpha,m}^2$  depends on both separation constants  $m$  and  $\alpha$ . There are two solutions  $H_i^{lm}(u)$  and  $H_o^{lm}(u)$  of the homogeneous equation

$$\nabla_{u,\alpha,m}^2 \Phi_u(u) = 0, \quad (3.4)$$

of which  $H_i^{lm}(u)$  is regular for  $u = u_{\min}$  (the lowest value of the  $u$ -coordinate: 1 for PS and 0 for the others) and  $H_o^{lm}(u)$  is regular as  $u \rightarrow \infty$ .

Table 2 summarizes the solutions  $\Phi_v^{lm}$ ,  $H_i^{lm}$  and  $H_o^{lm}$  for the five coordinate systems. The  $\Phi_v^{lm}$  functions have been normalized so that an arbitrary function  $f(v, \phi)$  can be written

$$f(v, \phi) = \sum_{l,m} c_{lm} \Phi_v^{lm}(v) e^{im\phi}, \quad (3.5)$$

with

$$c_{lm} = \iint w_v(v) \Phi_v^{lm}(v)^* e^{-im\phi} f(v, \phi) dv d\phi, \quad (3.6)$$

where the weight function  $w_v(v) = v$  for CP, and  $w_v(v) = 1$  for all the other coordinate systems.

### 3.2 Sample PD basis sets

The harmonics  $\Phi_v^{lm}(v) e^{im\phi}$  form a complete, orthonormal set on the ‘sphere’  $u = \text{constant}$  (Morse & Feshbach 1953; Hobson 1965). They can be extended to a basis for all space by appending a complete set of radial factors  $\{F_u^{lmn}(u): n = 0, 1, 2, \dots\}$  to each  $(l, m)$ -harmonic. It is most convenient to start with a basis of potential functions and derive the corresponding ‘radial’ density factors  $\{D_u^{lmn}(u)\}$  by differentiation:

$$D_u^{lmn} = \nabla_{u,l,m}^2 F_u^{lmn}. \quad (3.7)$$

Any potential can be written as a linear combination of the basis functions:

$$\Phi = 4\pi \sum_{l,m,n} d^{lmn} F_u^{lmn}(u) \Phi_v^{lm}(v) e^{im\phi} \quad (3.8a)$$

(we set  $G=1$  for convenience), and the corresponding density is

$$\rho = \frac{1}{c(u, v)} \sum_{l,m,n} d^{lmn} D_u^{lmn}(u) \Phi_v^{lm}(v) e^{im\phi}, \quad (3.8b)$$

where  $c(u, v)$  is given in Table 2 for the different coordinate systems. Note that  $D_u^{lmn}(u)$  is itself a density function only in the CC and CP systems: in all systems

**Table 2.** The separated Laplace operator for harmonic functions and the solutions of the homogeneous Laplace equation for the coordinate systems of Table 1. The norm

$$n^{lm} = \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} (-1)^{\max(0,m)}.$$

The prefactors for  $H_i^{lm}$  and  $H_o^{lm}$  in the OS system have been chosen to make the functions real-valued. We adhere to the notation of Abramowitz & Stegun (1972) for Legendre and Bessel functions.

	CC $u = R; v = z$	PS $u = \xi; v = \eta$	S $u = r; v = \cos \theta$	OS $u = \xi; v = \eta$	CP $u = z; v = R$
$\nabla_{v,m}^2$	$\frac{d^2}{dv^2}$	$\frac{d}{dv}(1-v^2)\frac{d}{dv} - \frac{m^2}{1-v^2}$	$\frac{d}{dv}(1-v^2)\frac{d}{dv} - \frac{m^2}{1-v^2}$	$\frac{d}{dv}(1-v^2)\frac{d}{dv} - \frac{m^2}{1-v^2}$	$\frac{1}{v}\frac{d}{dv}v\frac{d}{dv} - \frac{m^2}{v^2}$
$\nabla_{u,l,m}^2$	$\frac{1}{u}\frac{d}{du}u\frac{d}{du} - l^2 - \frac{m^2}{u^2}$	$\frac{d}{du}(u^2-1)\frac{d}{du} - l(l+1) - \frac{m^2}{u^2-1}$	$\frac{d}{du}u^2\frac{d}{du} - l(l+1)$	$\frac{d}{du}(u^2+1)\frac{d}{du} - l(l+1) - \frac{m^2}{u^2+1}$	$\frac{d^2}{du^2} - l^2$
$c(u, v)$	1	$e^2(u^2 - v^2)$	$u^2$	$e^2(u^2 + v^2)$	1
$\Phi_v^{lm}(v)$	$\frac{1}{2\pi} e^{ilv}$ $l \in \mathbb{R}$ $m \in \mathbb{Z}$	$n^{lm} P_l^m(v)$ $l \in \{0\} \cup \mathbb{N}$ $m = -l, \dots, l$	$n^{lm} P_l^m(v)$ $l \in \{0\} \cup \mathbb{N}$ $m = -l, \dots, l$	$n^{lm} P_l^m(v)$ $l \in \{0\} \cup \mathbb{N}$ $m = -l, \dots, l$	$\sqrt{\frac{l}{2\pi}} J_m(lv)$ $l \in \mathbb{R}$ $m \in \mathbb{Z}$
$H_i^{lm}(u)$	$I_m(lu)$	$P_l^m(u)$	$u^l$	$(-i)^{l+m} P_l^m(iu)$	$e^{lu}$
$H_o^{lm}(u)$	$K_m(lu)$	$Q_l^m(u)$	$u^{-l-1}$	$i^{l+m} Q_l^m(iu)$	$e^{-lu}$

$$\frac{1}{c(u, v)} D_u^{lmn}(u)$$

is a density function.

There is considerable freedom in choosing the radial basis  $\{F_u^{lmn}(u)\}$ . To be formally complete, the basis must be able to represent any ‘reasonable’ function of  $u$  on the (infinite)  $u$ -domain. The classical approach is to solve for the eigenfunction of the  $\nabla_{u,l,m}^2$  operator. This yields a (unique) complete, bi-orthogonal basis set, but one that is not necessarily suitable for galactic problems. Furthermore, in the OS and PS systems the ‘radial’ eigenfunctions are complicated to compute, and they are not available in standard numerical libraries.

Instead, a useful ansatz is the form

$$F_u^{lmn}(u) = W^{lm}(u) f^{lmn}(u), \quad (3.9)$$

where  $W^{lm}(u)$  is a fixed function that is suggested by the nature of the physical problem, and  $f^{lmn}$  are polynomials of degree  $n$  in a finite variable derived from  $u$ . The forms of  $W^{lm}$  and  $f^{lmn}$  should be chosen, if possible, so that the inner product integrals discussed below can be performed easily analytically. For OS we shall use the functions

$$F_u^{lmn}(u) = \frac{1}{(u+h)^p} \times \frac{u^n}{(u+h)^n}, \quad n \geq 0, \quad (3.10)$$

where  $h > 0$  and  $p \geq 1$  are free parameters. It is then possible to represent any realistic galactic potential factor. The function  $W^{lm}(u) \equiv W^{00}(u) = (u+h)^{-p}$  with  $p \geq 1$  ensures that the total mass associated with a basis function is finite, since the spherical average  $\langle \Phi \rangle \sim r^{-p}$  as  $r \rightarrow \infty$ . Our  $f^{lmn}$  functions (3.10) are polynomials in  $t_u = u/(u+h)$ , which increases monotonically with  $u$ , ranging from 0 to 1. Useful choices for  $W^{lm}$  and  $f^{lmn}$  in each of the five coordinate systems are given in Table 3; the choice for S resembles the basis functions found by Hernquist & Ostriker (1992) and Zhao (1996). For  $p=1$ , the starting function for S is a Hernquist (1990) model.

Because the functions (3.10) differ from each other most significantly for small  $u$ , they are most efficient at representing potential functions that differ from the asymptotic form

$[(u+h)^{-p}]$  for small  $u$  (examples are given in later sections). There has been some emphasis in the literature on basis sets with a lowest order member that corresponds to the exact initial PD pair used in, say, a normal mode analysis. We wish to emphasize that there is no need for this stringent restriction. The goal is to be able to represent perturbations of the initial model and their form will not be known a priori. It is best to choose a *simple* basis set that can represent both the initial model and its perturbations with a small number of terms. The density factors  $D_u^{lmn}$  corresponding to our chosen potential factors (Table 3) are not unduly complicated and are given in the same table.

### 3.3 Orthonormalization

Ideally, the basis set should be orthonormal with respect to the inner product

$$\langle \Phi_1, \Phi_2 \rangle = -\frac{1}{4\pi} \int \int \int \Phi_1^* \nabla^2 \Phi_2 h_u h_v h_\phi \, du \, dv \, d\phi, \quad (3.11)$$

for two potential functions. The inner product of two harmonic basis functions reduces to

$$\langle F_u^{lmk}, F_u^{lmn} \rangle_u = - \int w_u(u) F_u^{lmk}(u)^* D_u^{lmn}(u) \, du, \quad (3.12)$$

with  $w_u(u) = u$  for the CC system,  $e$  for OS and PS and 1 for S and CP. For the suggested forms given in Table 3, these integrals (3.12) can be performed analytically in all cases (see Appendix B).

It is not immediately clear that (3.12) is indeed an inner product. Three conditions must be satisfied for all ‘radial’ potential factors  $F_1, F_2, F_3$ :

- (i)  $\langle \alpha F_1 + \beta F_2, F_3 \rangle_u = \alpha \langle F_1, F_3 \rangle_u + \beta \langle F_2, F_3 \rangle_u$ ;
- (ii)  $\langle F_1, F_2 \rangle_u = \langle F_2, F_1 \rangle_u^*$ , and
- (iii)  $\langle F_1, F_1 \rangle_u \geq 0$ .

Condition (i) is manifest, but (ii) and (iii) are non-trivial. For the OS coordinates, we can integrate (3.12) by parts to obtain

**Table 3.** Suggested forms of the factors  $W^{lm}$  and  $f^{lmn}$  for starting functions (3.9), and the corresponding density factors  $D_u^{lmn}$  (equation 3.7). The full potential and density basis functions are  $\Phi^{lmn} = 4\pi \Phi_v^{lm}(v) e^{im\phi} F_u^{lmn}(u)$  and  $\rho^{lmn} = [1/c(u, v)] \Phi_v^{lm}(v) e^{im\phi} D_u^{lmn}(u)$ , where  $\Phi_v^{lm}(v)$  and  $c(u, v)$  are given in Table 2 for each coordinate system. The free parameter  $h > -1$  for PS, and  $h > 0$  for the other systems. For CC,  $p > 1$ ; otherwise  $p \geq 1$ . In all cases, the  $n=0$  function is included only for  $m=0$  (since for  $m > 0$  the potential must be zero at the origin to be continuous). For PS, the following constants appear in  $D_u^{lmn}$ :  $C_{0,n} = 2(h+1)^2 n^2$ ,  $C_{1,n} = [1 + (h+1)n(n+1) - (2n+1)(2p+1)](h+1)$ ,  $C_{2,n} = 2p[p - (h+1)(n+1)]$ ,  $C_3 = p(p-1)$ .

	$W^{lm}$	$f^{lmn}$	$D_u^{lmn}$
CC	$\frac{1}{(u^2 + h^2)^{p/2}}$	$\frac{u^{2n}}{(u^2 + h^2)^n}$	$\frac{u^{n-2}}{(u^2 + h^2)^{(n+p+4)/2}} [(n^2 - m^2)h^4 - (2(m^2 + n + p + np) + l^2 h^2)h^2 u^2 - (m^2 - p^2 + 2l^2 h^2)u^4 - l^2 u^6]$
PS	$\frac{1}{(u+h)^p}$	$\frac{(u-1)^n}{(u+h)^n}$	$\frac{(u-1)^{n-1}}{(u+h)^{n+p+2}} (C_{0,n} + C_{1,n}(u-1) + C_{2,n}(u-1)^2 + C_3(u-1)^3) - \frac{(u-1)^{n-1}}{(u+h)^{p+n}} (l(l+1)(u-1) + \frac{m^2}{u+1})$
S	$\frac{1}{(u+h)^p}$	$\frac{u^n}{(u+h)^n}$	$-\frac{u^n}{(u+h)^{n+p+2}} [(n(n+1) - l(l+1))h^2 - 2(l(l+1) + p(n+1))hu + (p(p-1) - l(l+1))u^2]$
OS	$\frac{1}{(u+h)^p}$	$\frac{u^n}{(u+h)^n}$	$\frac{u^{n-2}}{(u+h)^{p+n+2}} [n(n-1)h^2 - 2n(p+1)hu + p(p+1)u^2](u^2 + 1) - \frac{u^n}{(u+h)^{n+p}} (l(l+1) + \frac{m^2}{u^2 + 1})$
CP	$\frac{1}{(u^2 + h^2)^{p/2}}$	$\frac{u^n}{(u^2 + h^2)^{n/2}}$	$\frac{u^{n-2}}{(u^2 + h^2)^{(n+p+4)/2}} [n(n-1)h^4 - h^2(h^2 l^2 + 3n + p(2n+1))u^2 + (p(p+1) - 2h^2 l^2)u^4 - l^2 u^6]$

$$\begin{aligned}
\langle F_1, F_2 \rangle_u &= \frac{1}{4\pi} \left\{ -F_1^*(0) \frac{dF_2(0)}{du} \right. \\
&+ \left. \int_0^\infty (u^2 + 1) \frac{dF_1^*}{du} \frac{dF_2}{du} + \left[ l(l+1) + \frac{m}{1+u^2} \right] F_1^* F_2 du \right\} \\
&= \frac{1}{4\pi} \left[ \frac{dF_1^*(0)}{du} F_2(0) - F_1^*(0) \frac{dF_2(0)}{du} \right] + \langle F_2, F_1 \rangle_u^*.
\end{aligned} \tag{3.13}$$

Condition (ii) holds only for radial functions  $F$  that satisfy vanishing conditions,  $F'(0)=0$  and/or  $F(0)=0$ . For CP coordinates the  $u$ -integral extends from  $-\infty$  to  $\infty$ , so the equivalent of (3.13) implies that condition (ii) is always satisfied if the radial functions drop as  $1/u$  or faster. For the other three coordinate systems it is sufficient that  $F'(u_{\min})$  is bounded for all radial functions: the equivalent of the first term in (3.13) is proportional to

$$\lim_{u \downarrow u_{\min}} (u - u_{\min}) F_1^*(u) \frac{dF_2(u)}{du} = 0. \tag{3.14}$$

From (3.13) and its forms in the other coordinate systems, it is evident that if the radial functions are such that condition (ii) holds, then condition (iii) is satisfied as well.

The  $n=0, 1$ -functions (3.10) do not satisfy the vanishing conditions required for OS: for  $n=0$  both  $F_u^{lm0}(0)$  and  $(d/du)F_u^{lm0}(0)$  are non-zero, while for  $n=1$  the derivative does not vanish. In physical terms, a non-zero  $F'(0)$  in OS means that the  $u = \xi$ -component of the force field corresponding to the potential  $F$  is not continuous for all points in the equatorial plane within the ‘focal circle’ ( $\xi=0, \eta=0$ ). For physically relevant potentials, the coefficients of  $F_u^{lm0}$  and  $F_u^{lm1}$  are related to produce a zero derivative at  $\xi=0$ . Hence we modify (3.10) slightly:

$$\tilde{F}_u^{lm0} = F_u^{lm0} + \frac{P}{h} F_u^{lm1}, \tag{3.15}$$

$$\tilde{F}_u^{lmn} = F_u^{lm, n+1} \quad \text{for } n > 0.$$

When a basis is not orthogonal (as for all the suggested forms in Table 3) the Gram–Schmidt algorithm can be applied to make it orthonormal. An arbitrary potential component  $\Phi_u^{lm}(u)$  can then be written as

$$\Phi_u^{lm}(u) = \sum_n \langle \Phi_u^{lm}, F_u^{lmn} \rangle_u F_u^{lmn}(u). \tag{3.16}$$

Since  $\langle \Phi_1, \Phi_2 \rangle = \langle \Phi_2, \Phi_1 \rangle^*$ , the expression (3.16) allows us to compute the expansion coefficients from a given mass density to obtain the potential, as well as from a given potential using the density basis functions.

Saha (1991) used this technique to construct a basis set for perturbations of spherical galaxies. Saha (1993) pointed out that it is not necessary for the basis to be orthogonalized; the coefficients of an expansion (3.16) can be found by solving a set of linear equations, which is equivalent to orthonormalization. In practice, the difference is whether to solve the linear equations before (Gram–Schmidt) or after (Saha 1993) the expansion coefficients have been found.

### 3.4 Example: an oblate galaxy model

One of the applications of PD basis sets used in many  $N$ -body calculations is to compute the gravitational force  $-\nabla\Phi$  when a density is known. The potential itself is also used to check the conservation of energy in the system. As an example, consider the perfect oblate spheroid (de Zeeuw 1985) with density

$$\rho_p(R, z) = \frac{M}{\pi^2 \sqrt{1-e^2}} \left( 1 + R^2 + \frac{z^2}{1-e^2} \right)^{-2} \tag{3.17a}$$

and potential

$$\Phi_p = -\frac{2GM}{\pi} \frac{\Phi_\tau(\xi) - \Phi_\tau(\eta)}{\xi^2 - \eta^2}, \tag{3.17b}$$

where

$$\Phi_\tau(\tau) = \frac{\tau}{e} \arctan \frac{e\tau}{\sqrt{1-e^2}}, \tag{3.17c}$$

and  $(\xi, \eta)$  are PS coordinates. The flattening varies from  $e=0$  (sphere) to  $e=1$  (flat disc). Fig. 1 shows the density and potential of this model for  $e=0.5$ . Throughout this paper we use  $G=1$  and  $M=1$  when plotting functions. It is not obvious a priori that the perfect spheroid can be well represented with a small number of functions for any of our suggested basis sets, since its potential is not one of our simple starting functions.

We use the global basis functions (3.15) in OS coordinates to reproduce the density and potential of the perfect oblate model. The OS system is used because it matches the contours of the model density and potential more closely than the PS system, as can be seen in Fig. 1. In fact, the density contour of  $\rho = \rho_{\text{centre}}/4$  coincides with the  $\xi = 1/e^2 - 1$  contour. We construct two basis subsets, GOS1 and GOS2, of 10 and 30 elements, respectively (the G in GOS stands for global and distinguishes these sets from the local basis functions discussed later). Each basis element consists of a single function (3.15):

$$\text{GOS1: } (l, m) = (0, 0) \quad n = 0 \dots 5 \quad \text{and}$$

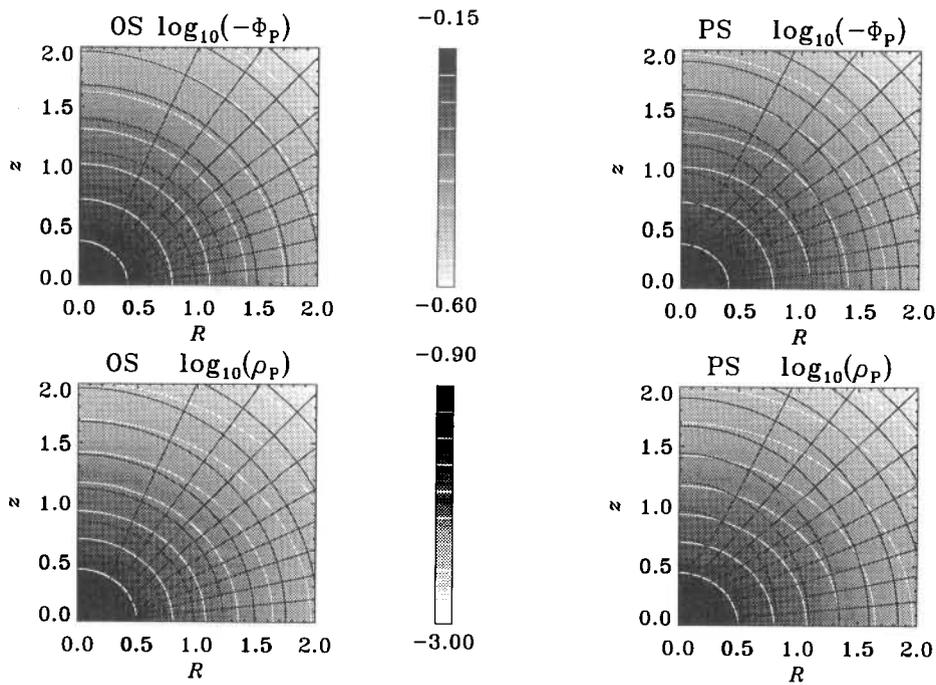
$$(l, m) = (2, 0) \quad n = 0 \dots 5.$$

$$\text{GOS2: } (l, m) = (0, 0) \quad n = 0 \dots 9,$$

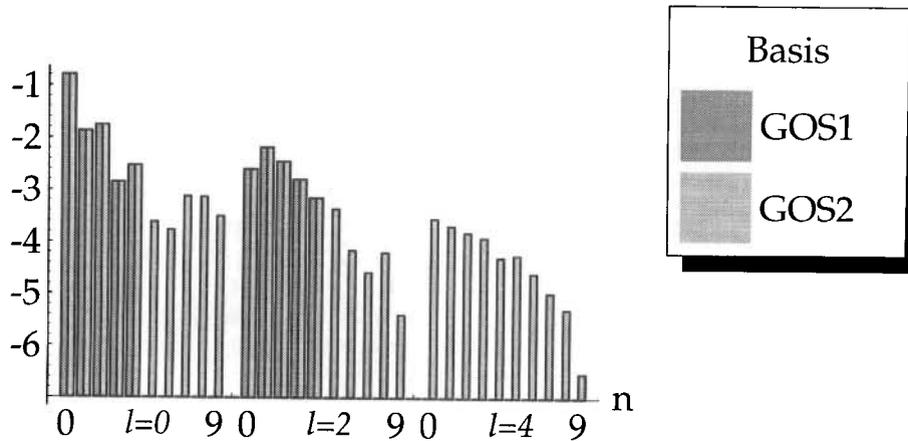
$$(l, m) = (2, 0) \quad n = 0 \dots 9 \quad \text{and}$$

$$(l, m) = (4, 0) \quad n = 0 \dots 9.$$

The parameter  $h$  is set to 1. As a result of the orthonormalization, the orthonormal basis element derived from the original  $n=N > 0$  function contains a linear combination of the  $n=0, \dots, N-1$  functions as well; we continue to label this function as the  $n=N$  basis function. Fig. 2 shows the logarithm of the expansion coefficients of the orthogonalized basis functions [which are linear combinations of elementary functions (3.15) with the same  $l$ -value]. The errors in the potential and density of the fits are displayed in Fig. 3. The potential is fitted within 0.3 per cent by GOS1 and within 0.07 per cent by GOS2 in the region shown in Fig. 3. Deviation of the fitted potential from the model potential



**Figure 1.** The potential (top) and density (bottom) of the perfect oblate spheroid for  $e=0.5$ , an E3 galaxy, as logarithmic grey-scale and (white) contour plot. The black contours show the OS (left) and PS (right) coordinate system. The OS system matches the equipotential and equidensity contours of the model more closely.

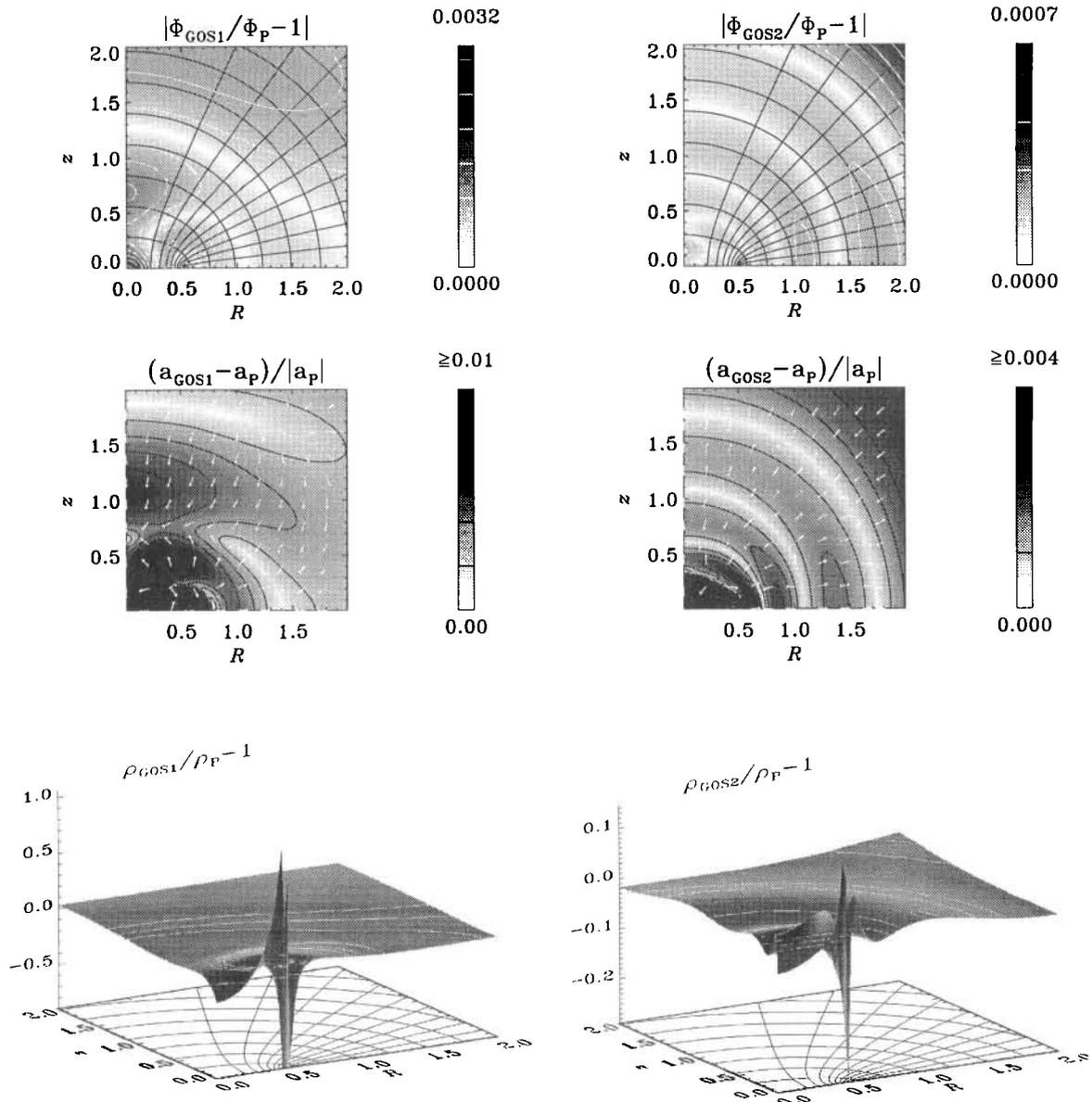


**Figure 2.** The coefficients  $c_l$  for the expansion of the E3 perfect oblate model in the orthonormalized basis sets GOS1 and GOS2. Plotted is  $\log|c_l|$  as a function of the values of  $(l, n)$  for the starting basis function (before Gram–Schmidt).

introduces an additional acceleration, which is also shown in Fig. 3. Away from the origin  $R=z=0$ , where the acceleration vanishes, the mean relative error in the acceleration is 1 per cent for GOS1 and 0.2 per cent for GOS2.

Unfortunately, the density is not fitted accurately near the focal ‘circle’ of the coordinate system. The problem is that a spheroidal harmonic times a single elementary radial function (3.10) for low  $n$  is not a smooth potential: it is not twice differentiable at the focal circle, causing the corresponding density to diverge. The singular density error disappears in the limit  $l_{\max}, n_{\max} \rightarrow \infty$ , but is present for any finite  $l_{\max}$  and  $n_{\max}$  (as in GOS1 and GOS2). The singularity is due to the

singular factor  $c(u, v)^{-1}$  in the Laplace operator, which is a sum rather than a product of  $u$  and  $v$  factors, and therefore never cancels out in equation (3.3). Nevertheless, at the cost of a very small extra error away from the focal circle, the singularity can always be removed by including a few additional (higher  $l$  and low- $n$ ) elementary functions (3.15) with weights determined by the coefficients of the singular terms in the rest of the expansion. This problem is unique to the OS and PS systems. In the S system,  $c(u, v)^{-1} = u^{-2}$  and the radial potential form can be chosen so that  $c^{-1}$  always cancels out; in the CC and CP systems,  $c(u, v) = 1$  and the problem does not arise.



**Figure 3.** The errors in the potential (top, linear grey-scale plots) and density (bottom) of the expansion of the model density from Fig. 1 in the GOS1 (left) and GOS2 (right) bases. The white contours in the density plots are at 0.01, 0.1, 2. Note that the vertical scales are different in the left- and right-hand panels. The error in the accelerations is depicted in the middle left (GOS1) and right (GOS2) panels: the linear grey-scale plot and black contours give its magnitude, the arrows its direction. Near the origin the ratio diverges because  $a_P$  vanishes.

### 3.5 Behaviour at large radii

The potential and density can be represented very accurately out to a finite radius using functions of the form (3.9), but at sufficiently large radii the deviations may be significant. The asymptotic form of  $W^{lm}$  for large  $u$  contributes little to the inner product of a model density and a basis potential function: the largest contribution comes from the region where the density is large, i.e., the centre. Hence the coefficient of a basis function in an expansion of the density does not change much when the form of  $W^{lm}$  is changed at large radii. The  $W^{lm}$  function should match the large-scale behaviour of the potential multipole moments  $\Phi_u^{lm}$  whenever

it is known. This is not crucial, since the potential outside a sizeable finite volume is never of interest.

It is tempting to let  $W^{lm}$  have the same behaviour at large radii as the homogeneous solution  $H^{lm}$  of the Laplace equation, but this will fail in general. As can be seen from the multipole expansion (which is derived in Appendix A; the results are listed in Table 4), it will be meaningful only when the integrals in the expression for  $\Phi_u^{lm}$  are bounded on the  $u$ -domain. This can fail to occur, even in common physical situations. As an example, consider a spheroidal mass distribution

$$\rho(r, \theta) = \rho_s(\theta)(1+r)^{-4}, \quad (3.18)$$

**Table 4.** The components  $\Phi_u^{lm}$  in equation (A1) as obtained by the multipole expansion (Appendix A) are listed in this table. These expressions can be used to find the behaviour of the potential on the boundary of the  $u$ -domain or as  $u \rightarrow \infty$ . The expression for  $n^{lm}$  is given in the caption of Table 2.

CC	$u = R$	$\sigma_{lm}(R) = \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi \frac{1}{2\pi} e^{-ilz} e^{-im\phi} \rho(R, z, \phi)$
	$v = z$	$\Phi_u^{lm}(R) = 4\pi \left[ K_m(lR) \int_0^R u I_m(lu) \sigma_{lm}(u) du + I_m(lR) \int_R^{\infty} u K_m(lu) \sigma_{lm}(u) du \right]$
PS	$u = \xi$	$\sigma_{lm}(\xi) = \int_{-1}^1 d\eta \int_0^{2\pi} d\phi n^{lm} P_l^m(\eta) e^{-im\phi} (\xi^2 - \eta^2) \rho(\xi, \eta, \phi)$
	$v = \eta$	$\Phi_u^{lm}(\xi) = \frac{4\pi}{g_{lm}} \left[ Q_l^m(\xi) \int_1^{\xi} P_l^m(u) \sigma_{lm}(u) du + P_l^m(\xi) \int_{\xi}^{\infty} Q_l^m(u) \sigma_{lm}(u) du \right]$
S	$u = r$	$\sigma_{lm}(r) = \int_{-1}^1 dv \int_0^{2\pi} d\phi n^{lm} P_l^m(v) e^{-im\phi} \rho(r, v, \phi)$
	$v = \cos \theta$	$\Phi_u^{lm}(r) = \frac{4\pi}{2l+1} \left[ r^{-l-1} \int_0^r u^{l+2} \sigma_{lm}(u) du + r^l \int_r^{\infty} u^{1-l} \sigma_{lm}(u) du \right]$
OS	$u = \xi$	$\sigma_{lm}(\xi) = \int_{-1}^1 d\eta \int_0^{2\pi} d\phi n^{lm} P_l^m(\eta) e^{-im\phi} (\xi^2 + \eta^2) \rho(\xi, \eta, \phi)$
	$v = \eta$	$\Phi_u^{lm}(\xi) = \frac{4\pi}{ig_{lm}} \left[ Q_l^m(i\xi) \int_0^{\xi} P_l^m(iu) \sigma_{lm}(u) du + P_l^m(i\xi) \int_{\xi}^{\infty} Q_l^m(iu) \sigma_{lm}(u) du \right]$
CP	$u = z$	$\sigma_{lm}(z) = \int_0^{\infty} dR \int_0^{2\pi} d\phi R \sqrt{\frac{1}{2\pi}} J_m(lR) e^{-im\phi} \rho(R, z, \phi)$
	$v = R$	$\Phi_u^{lm}(z) = \frac{2\pi}{l} \left[ e^{-lz} \int_{-\infty}^z e^{lu} \sigma_{lm}(u) du + e^{lz} \int_z^{\infty} e^{-lu} \sigma_{lm}(u) du \right]$

in spherical (S) coordinates. Because of the symmetry, only the functions  $\sigma_{lm}(r)$  with  $m=0$  are non-zero (cf. Table 4). Furthermore, it is clear that

$$\sigma_{l0}(r) = \sigma_{l0}^0 (1+r)^{-4}, \quad (3.19)$$

where  $\sigma_{l0}^0$  depends on  $\rho_v$ . The potential multipole moments can readily be computed. To lowest order in  $1/r$  they read:

$$\begin{aligned} \Phi_u^{00}(r) &\simeq 4\pi \sigma_{00}^0 r^{-1}, \\ \Phi_u^{10}(r) &\simeq \frac{4}{3} \pi \sigma_{10}^0 r^{-2} \log r, \\ \Phi_u^{l0}(r) &\propto 4\pi \sigma_{l0}^0 r^{-2} \quad (l \geq 2). \end{aligned} \quad (3.20)$$

In this case, the basis set should consist of functions (3.10) with  $W^{l0} \simeq (u+h)^{-2}$  for every  $l \geq 2$ , and  $W^{l0} \simeq (u+h)^{-2} \log(u+h)$  for  $l=1$ . Logarithmic large-radius behaviour is quite common in expansion procedures (see, e.g., Qian 1992, Appendix B).

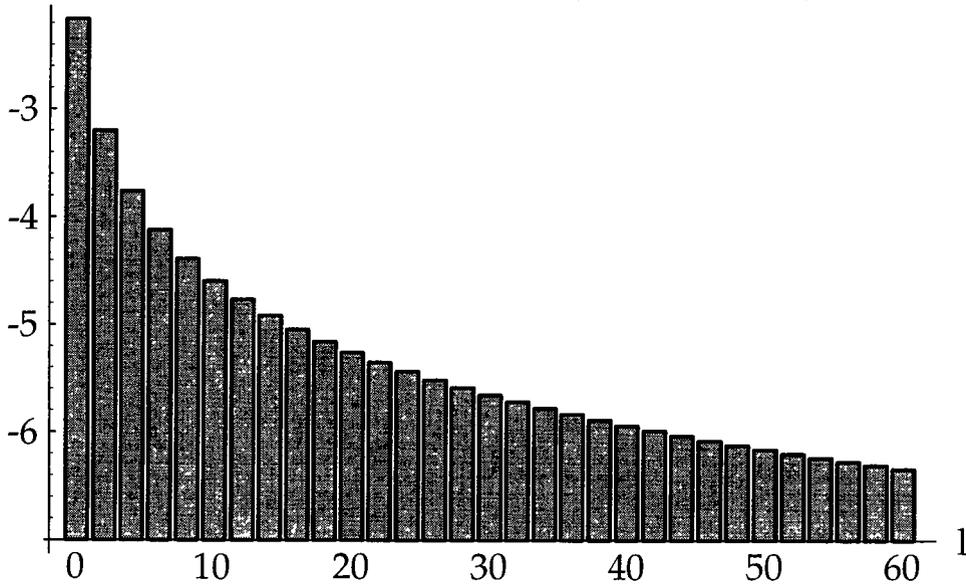
### 3.6 Flattened models in prolate coordinates

Although OS coordinates are the most obvious choice for describing an oblate galaxy model, there are situations

where one is forced to use a different system. For example, the Kuzmin–Kutuzov model (Kuzmin 1956; Kuzmin & Kutuzov 1962; for density and potential see equations (4.2) and (4.3) of de Zeeuw & Hunter 1990) is a Stäckel model: the equations of motion separate in PS coordinates. In semi-analytical stability studies (e.g. Robijn 1995; Robijn & de Zeeuw, in preparation) which involve the use of Hamilton–Jacobi theory, it is therefore preferable to use PS coordinates.

In principle, expansion of the model in PS coordinates is identical to the example presented in the previous section. Because the shape of the coordinate system does not match the shape of the model as close as OS, more spheroidal harmonics are needed to reach a similar accuracy. As the model becomes more flattened, the density is more concentrated towards the equatorial plane. The angular functions become sharply peaked near  $v=0$ , and the number of spheroidal harmonics needed increases dramatically.

In the limiting case of a flat-disc galaxy it is possible to do part of the expansion analytically, and indicate the severity of the problem in the nearly flat limit. As an example, consider the disc limit  $e=1$  of the Kuzmin–Kutuzov model, which is the Kuzmin (1956) disc,



**Figure 4.** The coefficient  $d^{l00}$  for the lowest order basis function in an expansion of the Kuzmin disc model for  $l=0, 2, \dots, 60$ . The basis is a harmonic PD set in PS coordinates. Plotted is  $\log|d^{l00}|$ .

$$\Phi_{\kappa}(\xi, \eta) = -\frac{GM}{\xi + |\eta|}, \quad (3.21)$$

$$\Sigma_{\kappa}(\xi) = \frac{M}{2\pi\xi^3},$$

where  $\Sigma_{\kappa}$  is the surface density of the disc. The form of the potential suggests a lowest order radial function

$$F_u^{l00} = \frac{\sqrt{3}}{\sqrt{2+3l(l+1)}} u^{-1}, \quad (3.22)$$

which has already been normalized. It is straightforward to compute the expansion coefficients of the set  $\{F_u^{l00}\Phi_v^{l0}\}_l$  (for even  $l \geq 0$ ) by computing the inner product of the potential basis functions and the model density:

$$\begin{aligned} d^{l00} &= \langle 4\pi F_u^{l00}\Phi_v^{l0}, \Phi_{\kappa} \rangle \\ &= (-1)^{l/2} \frac{\sqrt{3}\sqrt{2l+1}(l-1)!!}{6\sqrt{\pi}l!!\sqrt{2+3l(l+1)}}, \end{aligned} \quad (3.23)$$

which has a limiting behaviour of  $l^{-1}$  for large values of  $l$ . The  $d^{l00}$  coefficient is plotted in Fig. 4. Because the convergence is so slow, it is clear that any practical use of the *radial function*  $\times$  *spheroidal harmonic* series is excluded in this limiting case.

#### 4 HARMONIC PD SETS FOR FINITE SYSTEMS

There are many examples where the region of interest is limited in the  $u$ -coordinate: finite models for galaxies, localized perturbations, etc. A useful feature of the *radial basis*  $\times$  *angular harmonic* approach is that we are free to choose a basis that has a local support, i.e., is zero outside a finite volume. To illustrate this point, assume that we are studying a shell or ring-like perturbation that is concen-

trated within  $u_1 + \varepsilon < u < u_2 - \varepsilon$  for some small  $\varepsilon > 0$ . We introduce a radial basis set

$$F_u^{lmn}(u) = \begin{cases} (1-y^2)^3 y^n, & y \in (-1, 1), \\ 0, & |y| > 1, \end{cases} \quad (4.1)$$

for  $n \geq 0$ , with

$$y = \frac{u - \frac{1}{2}(u_1 + u_2)}{u_2 - u_1}. \quad (4.2)$$

The potential functions are twice differentiable and form a complete basis for functions with support in  $(u_1, u_2)$ . Any combination of these functions yields a potential that is zero at infinity. One would be inclined to accept the set (4.1) as a suitable basis.

However, the set (4.1) cannot represent the potential of a general ring-like perturbation, so it is not complete. From the multipole expansion (derived in Appendix A and summarized in Table 4) it is clear that a ring-like density has a potential proportional to  $H_l^m(u)$  for  $u < u_1$  and  $H_l^m(u)$  for  $u > u_2$ . The set (4.1) cannot represent the non-zero potential of the perturbation outside the domain  $(u_1, u_2)$ . This problem was also encountered (in PS coordinates) by de Zeeuw & Schwarzschild (1991).

There seems to be a paradox here. It is a well-known fact that the potential of a given density is uniquely determined (e.g. Jackson 1975). Since the multipole expansion disagrees with a potential expansion in the basis (4.1), the latter must be wrong. On the other hand, the  $H_l^m$  functions, as solutions of the homogeneous Laplace equation, correspond to zero density. Hence adding  $H_l^m$  to the basis does not improve the ability of the basis to represent the density. What is the status of these extra functions?

There are two answers to this problem. First, when solving for the potential of a given density, we are in fact solving a second-order partial differential equation. The general solution includes the two  $H_l^m$  functions with free coefficients that are determined by the boundary conditions. The

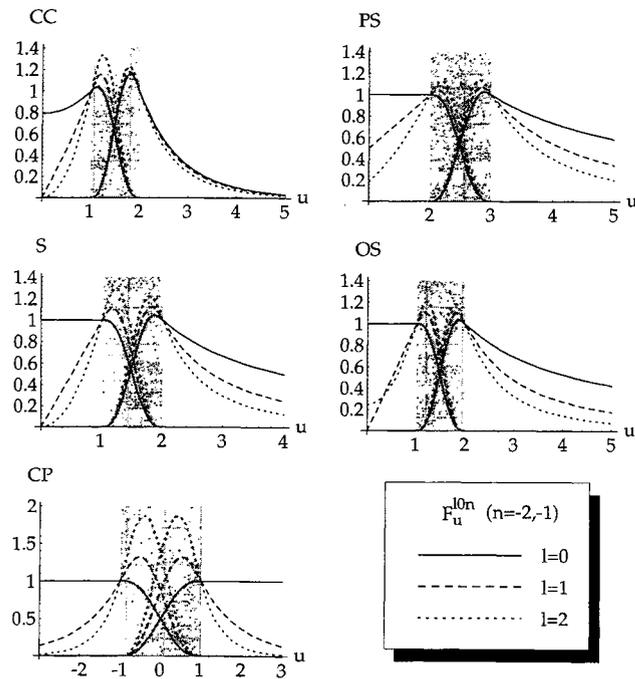
coefficients have a physical meaning: e.g., the coefficient of  $H_0^{00}$  is connected to the total mass. An expansion (3.16) of a given density using the basis (4.1) still solves the differential equation, but does not satisfy the appropriate boundary conditions: it is a solution to a different problem.

Although the  $H_{\bullet}^{lm}$  functions are solutions of the homogeneous Laplace equation, they do not correspond to zero density on the infinite  $u$ -domain:  $H_i^{lm}(u)$  is not bounded for  $u \rightarrow \infty$ , and  $H_o^{lm}(u)$  is unbounded for  $u \downarrow u_{\min}$ . Hence they are not proper potential functions. Any valid potential function that has a limiting behaviour of  $H_i^{lm}$  for  $u < u_1$  and  $H_o^{lm}$  for  $u > u_2$  must deviate from either function in the domain  $\mathcal{D} = (u_1, u_2)$ , and hence contribute to the density.

The boundary conditions can be incorporated in the set (4.1) by adding two functions

$$F_u^{lm, -2}(u) = \begin{cases} \left[ \sum_{j=0}^2 a_{-2,j}^{lm} (1+y)^j \right] (1-y)^3 & u \in \mathcal{D}, \\ H_i^{lm}(u) & u \leq u_1, \\ 0 & u \geq u_2; \end{cases} \quad (4.3a)$$

$$F_u^{lm, -1}(u) = \begin{cases} \left[ \sum_{j=0}^2 a_{-1,j}^{lm} (1-y)^j \right] (1+y)^3 & u \in \mathcal{D}, \\ 0 & u \leq u_1, \\ H_o^{lm}(u) & u \geq u_2, \end{cases} \quad (4.3b)$$



**Figure 5.** The functions  $F_u^{lm, -2}$  and  $F_u^{lm, -1}$  for the five coordinate systems, with  $m=0$  and  $l=0$  (solid),  $l=1$  (dashed) and  $l=2$  (dotted). The functions are normalized by  $F_u^{lm, -2}(u_1) = F_u^{lm, -1}(u_2) = 1$ . The grey area indicates the domain where the basis functions  $F_u^{lmn}$  with  $n \geq 0$ , as given in (4.1), are non-zero.

with

$$\begin{aligned} a_{-2,0}^{lm} &= \frac{1}{8} H_i^{lm}(u_1); \\ a_{-2,1}^{lm} &= \frac{3}{16} H_i^{lm}(u_1) + \frac{u_2 - u_1}{16} \frac{d}{du} H_i^{lm}(u_1); \\ a_{-2,2}^{lm} &= \frac{3}{16} H_i^{lm}(u_1) + \frac{3(u_2 - u_1)}{32} \frac{d}{du} H_i^{lm}(u_1) \\ &\quad + \frac{(u_2 - u_1)^2}{64} \frac{d^2}{du^2} H_i^{lm}(u_1); \\ a_{-1,0}^{lm} &= \frac{1}{8} H_o^{lm}(u_2); \\ a_{-1,1}^{lm} &= \frac{3}{16} H_o^{lm}(u_2) - \frac{u_2 - u_1}{16} \frac{d}{du} H_o^{lm}(u_2); \\ a_{-1,2}^{lm} &= \frac{3}{16} H_o^{lm}(u_2) - \frac{3(u_2 - u_1)}{32} \frac{d}{du} H_o^{lm}(u_2) \\ &\quad + \frac{(u_2 - u_1)^2}{64} \frac{d^2}{du^2} H_o^{lm}(u_2). \end{aligned} \quad (4.4)$$

These (fixed) coefficients  $a_{n,j}^{lm}$  for the  $n < 0$  basis functions are determined by the multipole expansion (Table 4), not by equation (3.16). If there is no inner boundary for the  $u$ -domain (e.g., in the case of a finite galaxy model), the  $n = -2$  function can be omitted. Examples of  $F_u^{lmn}$  functions with  $n < 0$  are given in Fig. 5. The set (4.1), completed with (4.3), has been used in the stability analysis of oblate galaxy models (Robijn 1995; Robijn & de Zeeuw, in preparation).

There is a minor issue concerning the application of Gram–Schmidt when the sets (4.1) and (4.3) are used. In all numerical applications the basis set is truncated to include  $N$  elements. It is favourable to have a basis set for which the expansion coefficients do not depend strongly on  $N$ , i.e., when  $N + 1$  elements are present, the coefficients of the first  $N$  elements should not change much. This will not occur if we use the  $n < 0$  fractions above as our lowest order elements. The Gram–Schmidt procedure then yields a basis set where all elements are non-zero outside the finite  $u$ -domain. It follows that, by adding an extra element to the basis, the total change in the first  $N$  coefficients must be equal to the  $(N + 1)$ th.

A better approach is to consider the  $n < 0$  functions as the highest order elements of the basis. Then adding another element to the basis does not change the first  $N$  coefficients at all: the  $n \geq 0$  elements are unchanged in shape. The coefficients for  $n < 0$  are determined by the density or potential being expanded and do not depend on the other basis functions, but the *shape* of the  $n < 0$  functions within the  $u$ -domain does change in this case.

## 5 NON-HARMONIC PD PAIRS

The method described thus far is not useful if many ‘angular’ functions have to be included in typical expansions, such as when using the PS system in the disc limit (Section 3.6).

A cylindrical coordinate system is probably a better choice for discs, but an inconvenient feature of harmonic basis sets in both CC and CP is that the index  $l$  associated with the ‘angular’  $v$  coordinate is continuous rather than discrete.

If we do not use harmonic ( $v, \phi$ )-functions, the Laplace equation,  $\nabla^2\Phi=0$ , does not separate into three ODEs. However, we can use a different set of orthonormal  $v$ -functions and still profit from the separability of the Laplace operator. Each function of the new set  $\{F_v^{km}(v)\}_{km}$  can be expanded in harmonics:

$$F_v^{km}(v) = \sum_l S_{km}(l) \Phi_v^{lm}(v), \quad (5.1)$$

where the summation is an integration in the CC and CP cases. The inner product of two  $v$ -functions reduces to

$$\langle F_v^{jm}, F_v^{km} \rangle_v = \sum_l S_{jm}(l) S_{km}(l) = \delta_{jk}. \quad (5.2)$$

The new basis functions can be specified directly by prescribing  $F_v^{km}$ , or by assuming a basis for the  $S_{jm}$  transforms. If a  $v$ -basis is not orthonormal by itself, it can be modified by Gram–Schmidt. We can proceed as in Section 3 by appending a complete set of radial factors  $\{F_u^{kmn}(u, l): n=0, 1, 2, \dots\}$  to each  $(k, m)$  angular function. The inner product of two radial functions, the equivalent of (3.12), becomes

$$\begin{aligned} \langle F_u^{kmj}, F_u^{k'mn} \rangle_u &= - \sum_l S_{km}(l) S_{k'm}(l) \\ &\times \int w_u(u) F_u^{kmj}(u, l) * \nabla_{u,\alpha,m}^2 F_u^{k'mn}(u, l) du, \end{aligned} \quad (5.3)$$

where  $\alpha=\alpha(l)$  is the separation constant given in Section 2.

As an example, consider a flat disc in CP coordinates. The ‘radial’ coordinate is  $u=z$  and the density is proportional to  $\delta(u)$ . A single ‘radial’ function is sufficient to form a basis:

$$F_u^{km0}(z, l) = -e^{-l|z|/\sqrt{2}},$$

$$D_u^{km0}(z, l) = \delta(z)\sqrt{2},$$

which has already been normalized. [The vanishing conditions  $F(0)=0$  and/or  $F'(0)=0$  are not required when there is only one function.] In the equatorial plane  $z=0$  the potential and surface density  $\Sigma$  are given by

$$\Phi(R, \phi) = -2\sqrt{\pi}G \sum_{k,m} d^{km0} e^{im\phi} \int_0^\infty S_k(l) J_m(lR) dl, \quad (5.4)$$

$$\Sigma(R, \phi) = \frac{1}{\sqrt{\pi}} \sum_{k,m} d^{km0} e^{im\phi} \int_0^\infty l S_k(l) J_m(lR) dl,$$

by analogy with (3.8). The  $l$ -integrals are recognized as Hankel transforms. This result was first used to construct galaxy models by Toomre (1963). Clutton-Brock (1972) found a discrete set of  $S_k(l)$  functions for which the Hankel transform of each member can be evaluated analytically; he thereby designed a discrete basis for flat discs. The same

approach can be used in principle for three-dimensional systems, but would probably require some ingenious integrations.

## 6 SUMMARY AND DISCUSSION

We have presented a general method for constructing a bi-orthonormal basis of PD pairs in five types of axisymmetric coordinate systems. In each case, the basis is built from simple functions. Considerable freedom in the choice of the starting functions is one of the key features of the method and makes it possible to adapt the basis to the application. We have demonstrated that gravitational systems with an infinite extent can be fitted accurately. The same method has been used to set up a basis to describe the density in a constrained region; some simple conditions, formulated in Section 4, have to be satisfied to ensure that the potential and density are physical.

We have generally avoided the delicate question of completeness. In numerical applications, completeness is not crucial, because we shall never use more than a few functions and the precise space that is spanned by the full set will not be important. Our aim is merely to produce a reasonably smooth approximation to the function being expanded.

Our software, which implements the general construction method and all the particular PD basis sets we have discussed, is freely available (see Appendix C).

## ACKNOWLEDGMENTS

It is a pleasure to thank Tim de Zeeuw for careful reading of the manuscript. DJDE gratefully acknowledges the hospitality of the Sterrewacht Leiden, where the initial part of the work was done; he was supported in Jerusalem by a Lady Davis Fellowship.

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## APPENDIX A: MULTIPOLE EXPANSION

For reference we derive the multipole expansion for axisymmetric coordinate systems, following the derivation given by Binney & Tremaine (1987, section 2.4) for spherical systems.

The principle of the multipole expansion is that the potential  $\Phi$  corresponding to a general mass distribution  $\rho(u, v, \phi)$  is expanded in harmonics  $\Phi_v^{lm}(v)e^{im\phi}$ :

$$\Phi(u, v, \phi) = \sum_{l,m} \Phi_u^{lm}(u) \Phi_v^{lm}(v) e^{im\phi}. \quad (\text{A1})$$

Note that for CC and CP the summation over  $l$  is actually an integration. We want to find an expression relating the coefficients  $\Phi_u^{lm}$  to  $\rho$ .

Formally, we can describe the mass distribution as a collection of ‘shells’  $u = u_0$  (cylinders in CC and planes in CP):

$$\rho(u, v, \phi) = \int \rho(u_0, v, \phi) \delta(u_0 - u) du_0. \quad (\text{A2})$$

For each of the shells with mass density  $\rho_s(u, v, \phi) = \delta(u - u_0) \sigma(v, \phi)$ , the corresponding potential  $\Phi_s$  satisfies the homogeneous Laplace equation outside the shell; hence it must be a linear combination of the  $H_i^{lm}$  and  $H_o^{lm}$  functions:

$$\Phi_s(u, v, \phi) = \begin{cases} \sum_{l,m} c_{lm} H_i^{lm}(u) \Phi_v^{lm}(v) e^{im\phi} & u < u_0, \\ \sum_{l,m} d_{lm} H_o^{lm}(u) \Phi_v^{lm}(v) e^{im\phi} & u > u_0. \end{cases} \quad (\text{A3})$$

At  $u = u_0$  the potential is continuous, but not differentiable. The jump in the first derivative can be computed by applying the Gauss theorem:

$$\begin{aligned} \int_{\partial V} \nabla \Phi_s(u, v, \phi) \cdot d\mathbf{S} &= \int \nabla^2 \Phi_s dV \\ &= 4\pi \int_V \rho_s(u, v, \phi) dV, \end{aligned} \quad (\text{A4})$$

where  $d\mathbf{S}$  is the normal to the surface  $\partial V$  of the volume  $V$ , and  $dV$  is the volume element. If we apply this to a small volume  $V$  that contains the point  $(u, v, \phi)$  and is infinitely thin in the  $u$ -direction, we find

$$\int \frac{1}{h_u} \left[ \frac{\partial \Phi_s(u^+, v, \phi)}{\partial u} - \frac{\partial \Phi_s(u^-, v, \phi)}{\partial u} \right] h_v h_\phi dv d\phi \quad (\text{A5})$$

$$= 4\pi \int \sigma(v, \phi) h_u h_v h_\phi dv d\phi,$$

where  $f(u^+) = \lim_{u' \rightarrow u^+} f(u')$  and  $f(u^-) = \lim_{u' \rightarrow u^-} f(u')$ . For an infinitesimally small volume  $V$  this simplifies to

$$\frac{\partial \Phi_s(u^+, v, \phi)}{\partial u} - \frac{\partial \Phi_s(u^-, v, \phi)}{\partial u} = 4\pi \sigma(v, \phi) h_u^2(u, v). \quad (\text{A6})$$

If we now substitute equation (A3) into (A6), multiply by  $w_v(v) \Phi_v^{lm}(v)^* e^{-im\phi}$  and integrate over  $v$  and  $\phi$ , we arrive at

$$c_{lm} \frac{dH_o^{lm}(u)}{du} - d_{lm} \frac{dH_i^{lm}(u)}{du} = 4\pi h_{uu}(u) \sigma_{lm}(u), \quad (\text{A7})$$

where the scale factor  $h_u(u, v) = h_{uu}(u) h_{uv}(u, v)$  has been split, and

$$\sigma_{lm}(u) = \iint h_{uv}(u, v) \Phi_v^{lm}(v)^* e^{-im\phi} \sigma(v, \phi) dv d\phi. \quad (\text{A8})$$

The factors  $h_{uu}$  and  $h_{uv}$  depend on the coordinate system:

$$\begin{aligned} \text{CC, S, CP: } h_{uu}^2 &= 1, & h_{uv}^2 &= 1; \\ \text{PS: } h_{uu}^2 &= (u^2 - 1)^{-1}, & h_{uv}^2 &= u^2 - v^2; \\ \text{OS: } h_{uu}^2 &= (u^2 + 1)^{-1}, & h_{uv}^2 &= u^2 + v^2. \end{aligned} \quad (\text{A9})$$

The constants  $c_{lm}$  and  $d_{lm}$  have been determined from (A7) using the properties of the special functions  $H_\bullet^{lm}$ :

$$H_i^{lm}(u) \frac{d}{du} H_o^{lm}(u) - H_o^{lm}(u) \frac{d}{du} H_i^{lm}(u) = \begin{cases} -u^{-1} & \text{CC (GR 8.486.4, 8.486.13, 8.477.2)} \\ -\frac{g_{lm}}{u^2 - 1} & \text{PS (GR 8.741.2)} \\ -\frac{2l+1}{u^2} & \text{S} \\ -i \frac{g_{lm}}{u^2 + 1} & \text{OS (GR 8.741.2)} \\ -2l & \text{CP} \end{cases} \quad (\text{A10})$$

where the GR numbers refer to Gradshteyn & Ryzhik (1994), and

$$g_{lm} = 2^{2m} \frac{\Gamma\left(\frac{l+m+1}{2}\right) \Gamma\left(\frac{l+m+2}{2}\right)}{\Gamma\left(\frac{l-m+1}{2}\right) \Gamma\left(\frac{l-m+2}{2}\right)}. \quad (\text{A11})$$

As a last step, the contributions  $\Phi_s$  of the shells to the potential  $\Phi$  can be integrated; the resulting expressions for  $\Phi_u^{lm}$  are given in Table 4.

## APPENDIX B: INNER PRODUCT INTEGRALS

For each of the basis forms we have suggested (Table 3 and equation 4.1) the inner product integrals required for orthonormalization can be performed analytically. At worst, these integrals can always be reduced to simple recurrence relations, which we describe below. Our symbolic software (Appendix C) runs efficiently due to fast evaluation of these recurrence relations.

For the local basis functions in Section 4, the potentials and densities are all polynomials or simple rational functions, so the inner product integrals are trivial. The inner products of the global basis functions that are given in Table 3 involve integrals of six forms.

### CC coordinates

In the CC system only a single type of integral is involved:

$$I^{CC}(p_1, p_2) \equiv \int_0^\infty \frac{u^{p_1}}{(u^2 + h^2)^{p_2}} du, \quad (B1)$$

where  $p_1 \geq 1$  is an odd integer, and  $p_2 \geq p > 1$ . We find

$$I^{CC}(p_1, p_2) = \frac{\Gamma\left(\frac{p_1+1}{2}\right)\Gamma\left(p_2 - \frac{p_1+1}{2}\right)}{2\Gamma(p_2)} h^{p_1-2p_2+1}. \quad (B2)$$

### PS coordinates

Two integral forms show up in the PS system. The first one is

$$I_1^{PS}(p_1, p_2) \equiv \int_1^\infty \frac{(u-1)^{p_1}}{(u+h)^{p_2}} du, \quad (B3)$$

where  $p_1 \geq 0$  is an integer, and  $p_2 \geq p \geq 1$ . The integral can be evaluated immediately:

$$I_1^{PS}(p_1, p_2) = B(p_1+1, p_2-p_1-1)(h+1)^{p_1-p_2+1}, \quad (B4)$$

where  $B$  is the beta function. The second integral occurs only for  $m \neq 0$ :

$$I_2^{PS}(p_1, p_2) \equiv \int_1^\infty \frac{(u-1)^{p_1}}{(u+1)(u+h)^{p_2}} du,$$

which is equal to  $I_1^{PS}(p_1, p_2+1)$  for  $h=1$ . When  $h \neq 1$ , recurrence relations for  $I_2^{PS}(0, p_2)$  can still be found. We restrict ourselves to the case where  $p_2$  is an integer; the recurrence relations then read for  $p_1=0$ :

$$I_2^{PS}(0, 1) = \frac{\ln(h+1) - \ln 2}{h-1};$$

$$I_2^{PS}(0, p_2) = \frac{1}{h-1} [I_2^{PS}(0, p_2-1) - I_1^{PS}(0, p_2)], \quad (B5a)$$

where the relation  $h-1 = (u+h) - (u+1)$  has been used. For all values of  $p_2$  the next relation can be derived using  $u-1 = (u+1) - 2$ :

$$I_2^{PS}(p_1, p_2) = I_1^{PS}(p_1-1, p_2) - 2I_2^{PS}(p_1-1, p_2). \quad (B5b)$$

### S coordinates

All integrals in the S system are of the form

$$I^S(p_1, p_2) \equiv \int_0^\infty \frac{u^{p_1}}{(u+h)^{p_2}} du, \quad (B6)$$

which is equivalent to  $I_1^{PS}$  with  $h+1$  replaced by  $h$ :

$$I^S(p_1, p_2) = B(p_1+1, p_2-p_1-1)h^{p_1-p_2+1}. \quad (B7)$$

### OS coordinates

In the OS system we encounter the  $I^S$  integral, as well as an integral of the form

$$I^{OS}(p_1, p_2) \equiv \int_0^\infty \frac{u^{p_1}}{(u+h)^{p_2}(u^2+1)} du, \quad (B8)$$

where  $p_1 \geq 0$  is an integer, and  $p_2 \geq p \geq 1$ . This integral can be performed analytically for arbitrary  $p_2$ , but we restrict ourselves to integer  $p$  values only. In this case  $p_2$  is an integer, and for  $p_1=0$  we obtain the recurrence relations

$$I^{OS}(0, 0) = \frac{1}{2}\pi;$$

$$I^{OS}(0, 1) = \frac{1}{2} \frac{h\pi - 2 \ln h}{h^2 + 1}; \quad (B9a)$$

$$I^{OS}(0, p_2) = \frac{I^S(0, p_2) - I^{OS}(0, p_2-2) + 2I^{OS}(0, p_2-1)}{h^2 + 1},$$

where the last equation has been derived using the identity  $h^2 + 1 = (u^2 + 1) - (u+h)^2 + 2h(u+h)$ . For all  $p_2$  values the following two recurrence relations hold. Since  $u = (u+h) - h$ , we have

$$I^{OS}(1, p_2) = I^{OS}(0, p_2-1) - hI^{OS}(0, p_2) \quad (B9b)$$

and, using  $u^2 = (u^2+1) - 1$ , we arrive at

$$I^{OS}(p_1, p_2) = I^S(p_1-2, p_2) - I^{OS}(p_1-2, p_2). \quad (B9c)$$

### CP coordinates

The integral that occurs in the inner products in the CP system is

$$I^{\text{CP}}(p_1, p_2) \equiv \int_{-\infty}^{\infty} \frac{u^{p_1}}{(u^2 + h^2)^{p_2}} du, \quad (\text{B10})$$

with  $p_1 \geq 0$  an even integer and  $p_2 \geq p_1 \geq 1$ . This is related simply to  $I^{\text{CC}}$ ,

$$I^{\text{CP}}(p_1, p_2) = 2I^{\text{CC}}(p_1, p_2), \quad p_1 \text{ even.} \quad (\text{B11})$$

## APPENDIX C: THE SOFTWARE

We have implemented the construction of PD pairs as described in this paper in a MATHEMATICA package called PDPAIRS. It is available via the Internet: use anonymous ftp to <ftp.Strw.LeidenUniv.NL> and get the file `pub/dynamics/Software/PDPairs.tar.z`.

WWW users should visit the URL

<http://WWW.Strw.LeidenUniv.NL/dynamics/>