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Generalized synchronization induced by noise and parameter mismatching in Hindmarsh–Rose neurons

Ying Wu [a,*], Jianxue Xu [a], Daihai He [b], David J.D. Earn [b]

[a] *Institute of Nonlinear Dynamics, School of Architectural Engineering and Mechanics, Xi'an Jiaotong University, Xi'an Shaanxi 710049, China*

[b] *Department of Mathematics and Statistics, McMaster University, Hamilton Canada L8S 4K1*

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Abstract

Synchronization of two simple neuron models has been investigated in many studies. Thresholds for complete synchronization (CS) and phase synchronization (PS) have been obtained for coupling by diffusion or noise. In addition, it has been shown that it is possible for directional diffusion to induce generalized synchronization (GS) in a pair of neuron models even if the neurons are not identical (and differ in a single parameter). We study a system of two uncoupled, nonidentical Hindmarsh–Rose (HR) neurons and show that GS can be achieved by a combination of noise and changing the value of a second parameter in one of the neurons (the second parameter mismatch cancels the first). The significance of this approach will be the greatest in situations where the parameter that is originally mismatched cannot be controlled, but a suitable controllable parameter can be identified.

1. Introduction

The potential for identical and sufficiently strongly coupled chaotic systems to synchronize is well known [1,2]. This phenomenon is important theoretically because it is unobvious that systems with sensitive dependence on initial conditions should be able to synchronize. It is also of possible practical importance in fields such as communications [3], neuroscience [4] and ecology [5,6].

Three types of synchronization are frequently discussed. The strongest form, *complete synchronization* (CS) means that the distinct oscillators asymptotically approach the same orbit. *Phase synchronization* (PS) refers to exact entrainment of the phases of distinct oscillators, but with no restriction on their amplitudes. *Generalized synchronization* (GS), which is normally defined only for systems involving "driving" and "receiving" components, refers to a situation in which the states of the "driver" and the "receiver" on their respective attractors are related by a continuous mapping from the driver state to the receiver state [7]. CS is, of course, the special case of GS in which the relevant mapping is the identity transformation. PS does not imply GS because the oscillator amplitudes can be uncorrelated in the case of PS. GS does not imply PS because it is not necessarily possible even to define the phase of a chaotic oscillator.

* Corresponding author.

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Each of these types of synchronization has been shown to occur in theoretical models [8] and in experiments [9], as a result of deterministic coupling between the oscillators. In addition, CS and PS have been shown to occur in uncoupled systems that are subject to common noise [10–13]. The model we focus on in this paper provides an example of noise-induced GS in a system where CS is impossible.

Synchronization in Hindmarsh–Rose (HR) neurons [14,15] has been studied widely [16,17,20] because of its physiological significance. Compared with many other mathematical models of excitable cells, the HR model is simpler. It consists of only three differential equations, which have relatively mild nonlinearities (quadratic and cubic terms). Like the Lorenz and Rössler equations, which are equally simple, the HR model can generate chaotic dynamics. However, the nature of the HR chaotic attractor is quite different from that of the Lorenz and Rössler attractors; it exhibits multi-time-scaled burst-rest behavior, a phenomenon that has great importance in neuroscience. Previous work has investigated variations in the periods of oscillation of two coupled identical HR neurons [16], noise-induced CS between two uncoupled identical HR neurons [17], noise-induced CS and PS in identical and non-identical chaotic Hodgkin–Huxley neurons [18,19] and PS in two coupled nonidentical HR neurons [20]. The first experimental observation of noise-induced PS in a pair of uncoupled sensory neurons was reported in Ref. [21].

The HR neuron model is characterized by three dependent variables: the membrane potential x, the recovery variable y, and a slow adaptation current z. Because the HR model has multiple time-scale dynamics [22], the burst of action potentials often consists of a group of spikes. It is believed that neuronal information is mainly transmitted using a code based on the time intervals between neuronal firings rather than the spiking amplitude [23]. Thus, an important concept in sensory information is the interspike interval (ISI), which is obtained from the successive spiking peaks of the membrane potential x.

In this paper, we study Gaussian white noise-induced GS between two uncoupled nonidentical HR neuron models. The mapping that defines the GS is very simple: two of the variables completely synchronize while the third approaches a constant difference associated with parameter mismatching.

2. Noise induced GS in nonidentical neurons

We consider a pair of Hindmarsh–Rose (HR) [14] neuron models ($i = 1, 2$) as follows:

$$\begin{aligned} \dot{x}_i &= y_i - a_i x_i^3 + b_i x_i^2 - z_i + I_i + D^x \xi(t) \\ \dot{y}_i &= c_i - d_i x_i^2 - y_i \\ \dot{z}_i &= r[S(x_i - \chi_i) - z_i] \end{aligned} \tag{1}$$

With the standard parameter values $a_i = 1.0$, $b_i = 3.0$, $d_i = 5.0$, $S = 4.0$, $r = 0.006$, $\chi_i = -1.56$, $c_i = 1.0$ and $I_i = 3.0$, the two neuron models are identical and chaotic. The noise term $\xi(t)$ is Gaussian with $\langle \xi(t)\xi(t-\tau)\rangle = \delta(\tau)$. The parameter I represents the applied current while χ is the threshold above which x displays spike behavior. The other parameters do not have obvious biological meanings; they were introduced in Ref. [14] in order to mimic the dynamical behavior of the Hodgkin–Huxley neuron model [18] using a simpler mathematical model.

In Ref. [17] it was shown that common additive Gaussian white noise can induce CS between two uncoupled identical HR neuron models with critical noise intensity $D_c^x = 2.4$ on $\dot{x}$, or $D_c^y = 6$ on $\dot{y}$. We now consider the implications of parameter mismatching for synchronization of this two-neuron model.

As a first example, we consider a small mismatching of the parameter c between the two neurons ($c_1 = 1.05$, $c_2 = 0.95$). With common additive Gaussian white noise acting on $\dot{x}$, CS cannot be achieved when the noise intensity $D^x < 5.0$. This is clear in Fig. 1a, which shows (solid line) the "synchronization error" for x, i.e., time-averaged absolute difference $\langle |x_1 - x_2| \rangle$, as a function of noise intensity D^x. The figure also shows the synchronization errors for y (dot-dash line) and z (dotted line). A transition occurs near $D^x = 2.4$, which is the critical point for CS between identical neurons [17]. Above this intensity, synchronization errors are greatly reduced (note that the errors shown are time-averaged and, in fact, the two systems intermittently synchronize). As a function of D^x, the synchronization error in x decreases much more rapidly than the synchronization error in y and z, and beyond the critical point $D^x = 2.4$, x displays the smallest synchronization error of the three variables. Since x represents the variable that is usually observed in experiments, above the critical noise intensity we would expect to observe nearly complete synchronization in practice. A sample time series, $x(t)$ (for $D^x = 2.5$), is shown in Fig. 2.

As a second example, we consider the effect of slightly different applied currents on the two neurons ($I_1 = 3.05$, $I_2 = 2.95$). Again, common additive Gaussian white noise acts on $\dot{x}$; the results, shown in Fig. 1b, resemble those in Fig. 1a (in panel b, $\langle |y_1 - y_2| \rangle$ has a slightly steeper slope near, and a lower value after, the critical noise intensity).

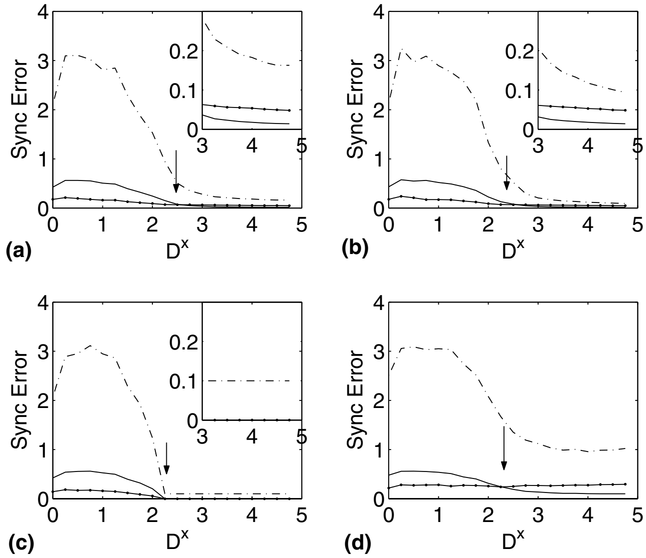

Fig. 1. Synchronization errors of two uncoupled HR neurons with mismatched parameters versus additive noise intensity D^x. (a) Single mismatching $c_1 = 1.05$, $c_2 - 0.95$; (b) single mismatching $I_1 = 3.05$, $I_2 = 2.95$; (c) $I_1 = 3.05$, $I_2 = 2.95$, $c_1 = 0.95$, $c_2 = 1.05$; (d) $a_1 = 1.05$, $a_2 = 0.95$, $I_1 = I_2 = 3$ and $c_1 = c_2 = 1$. In all panels, average synchronization error $\langle |x_1 - x_2| \rangle$ is denoted by the solid line, $\langle |y_1 - y_2| \rangle$ by the dot-dash line, and $\langle |z_1 - z_2| \rangle$ by the dotted line.

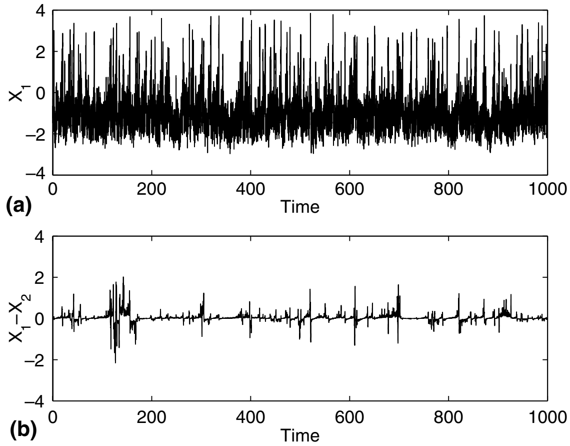

Fig. 2. Time evolution of the model with two HR neurons, subject to common additive noise with intensity $D^x = 2.5$ and mismatched parameters $c_1 = 0.95$ and $c_2 = 1.05$. (a) A sample time series of the membrane potential x of one of the two neurons. (b) The difference in membrane potentials of the two neurons.

Surprisingly, if we impose simultaneously the above parameter mismatchings in c and I ($c_1 = 1.05$, $c_2 = 0.95$, $I_1 = 2.95$, $I_2 = 3.05$), GS is achieved above the critical point $D^x = 2.4$; see Fig. 1c. In this case, the synchronization errors in x and z are both exactly zero, while $\langle |y_1 - y_2| \rangle = 0.1$. It is worth noting here that the synchronization error in y is equal to the parameter mismatchings ($|c_1 - c_2| = 0.1$ and $|I_1 - I_2| = 0.1$).

As a further example, we considered mismatch of the coefficient of the cubic term ($a_1 = 1.05$, $a_2 = 0.95$). The synchronization error curves are shown in Fig. 1d. The reduction in the average synchronization error in y (dot-dash curve) near the critical noise intensity is much less dramatic than for the parameter mismatchings considered in panels a–c, while the average synchronization error in z displays a slight increase rather than a decrease. The apparent inability to achieve GS in this case may be related to the fact that a appears in a nonlinear term in the equations, whereas I

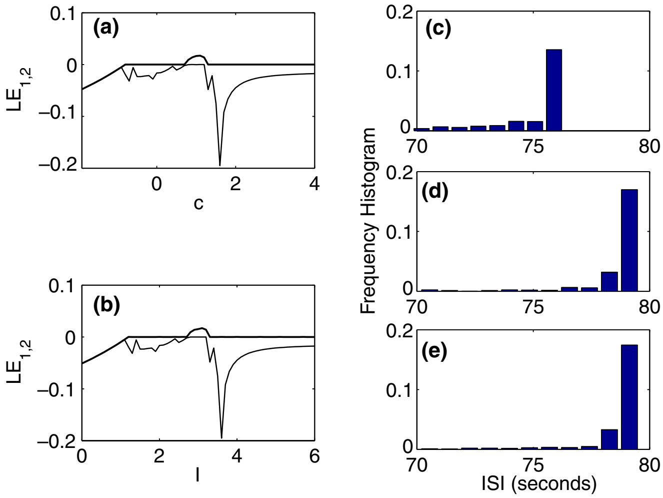

Fig. 3. The first two Lyapunov exponents of the system versus different parameters (left panels) and distributions of interspike intervals (right panels). (a) The first two Lyapunov exponents versus c. (b) The first two Lyapunov exponents versus I. It can be seen that the LEs in (a) and (b) one related by a horizontal shift. The ISI distribution is shown in three cases: (c) with all parameters taking standard values; (d) $I = 2.9$ with others parameters taking standard values; (e) $c = 0.9$ with others parameters taking standard values.

and c occur in linear terms. Note, however, that the approximate synchronization of x is still achieved beyond the critical noise intensity of $D^x = 2.4$; again, this implies that we would expect to observe approximate CS in practice.

All the above results, including the existence of a threshold noise intensity at which $\langle|x_1 - x_2|\rangle = \langle|z_1 - z_2|\rangle$, are reproduced if common additive noise is imposed on $\dot{y}$ rather than $\dot{x}$.

To understand why the dual parameter mismatching in I and c leads to GS while individual mismatchings do not, we explored the dynamics of the single neuron HR model as a function of each of these parameters. Figs. 3a and b show the first two Lyapunov exponents (LEs) of the HR model versus c and I, respectively. The only difference between panels (a) and (b) in this figure is a horizontal shift by 2 units. We also compared the ISI distributions of the model when c or I is changed by the same amount; Figs. 3c–e show ISI histograms for all parameters at their standard values ($I = 3$, $c = 1$; panel c), for $I = 2.9$ (standard value minus 0.1; panel d), and for $c = 0.9$ (standard value minus 0.1; panel e). The ISI distributions in Fig. 3d and e are nearly identical, and both are different from the standard ISI distribution in panel c.

Thus, it appears that the same quantitative change on c or I yields the same dynamical change on the neuron's behavior. With hindsight, we therefore looked for a simple relationship between the parameters c and I. If we rewrite the HR equations, replacing the variable y by $w = y - c$, we obtain

$$\begin{aligned} \dot{x} &= w - ax^3 + bx^2 - z + I + c \\ \dot{w} &= -dx^2 - w \\ \dot{z} &= r[S(x - \chi) - z] \end{aligned} \tag{2}$$

Because the two parameters I and c appear in these equations only in the combination $I + c$, it is clear that they can be considered a single parameter and that changes in I can always be eliminated by equal and opposite changes in c. This demystifies the results in Figs. 1 and 3.

While the simple transformation $w = y - c$ shows that constant changes in c or I are equivalent, the same is not true of time-dependent changes of these parameters. To see this, consider the full equations including a differentiable driving term ($D^y\eta(t)$ on $\dot{y}$), and let $w = y - c + D^y\eta(t)$. Then the transformed equations are

$$\begin{aligned} \dot{x} &= w - ax^3 + bx^2 - z + I + c - D^y\eta(t) \\ \dot{w} &= -dx^2 - w + D^y\dot{\eta}(t) \\ \dot{z} &= r[S(x - \chi) - z]. \end{aligned} \tag{3}$$

While the driving term $D^y\eta(t)$ is transferred to the $\dot{x}$ equation by this transformation, a new term $D^y\dot{\eta}(t)$ appears in the $\dot{w}$ equation. If the perturbation $\eta(t)$ were smooth and nearly constant (so that $|\dot{\eta}| \ll 1$) then this term could be neglected, but this is never the case when $\eta(t)$ (which is not even differentiable) represents noise. Consequently, it is not surprising that the critical noise intensity on c and I differ ($D^x_c = 2.4$ and $D^y_c = 6$).

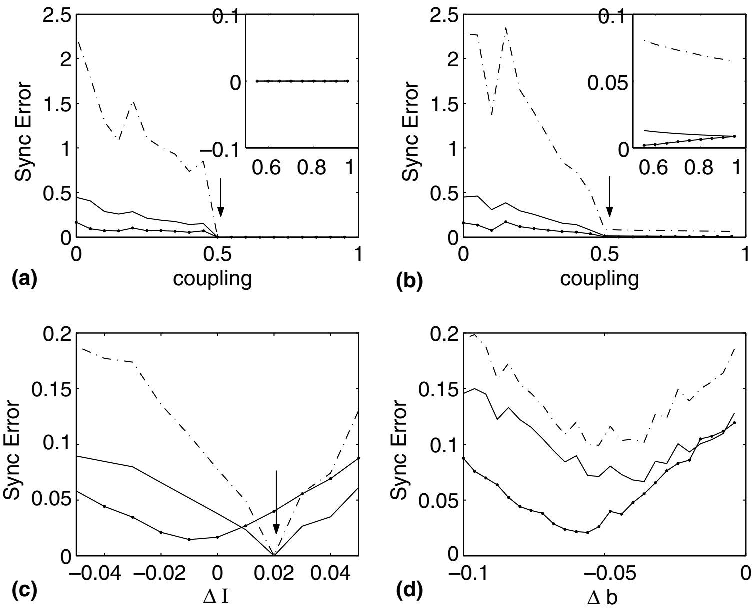

Fig. 4. Synchronization errors versus diffusive coupling strength ε (top panels) and versus parameter mismatching (bottom panels). As in Fig. 1, we show synchronization errors between x_1 and x_2 (solid line), y_1 and y_2 (dot-dash line) and z_1 and z_2 (dotted-solid line). In panels (a–c), the parameter χ is mismatched ($\chi_1 = 1.56$, $\chi_2 = 1.57$) while in panel (d) it is d that is mismatched ($d_1 = 5.05$, $d_2 = 4.95$). In panels (c) and (d), the noise intensity is $D^x = 2.5$. Panel (c) reveals a critical mismatching of applied current I ($\Delta I = 0.02$), i.e., ($I_1 = 2.98$, $I_2 = 3.02$) at which the synchronization error is minimized.

For other models, or for other parameter combinations in the HR model, if similar dual parameter mismatched GS is observed, it will not necessarily be possible to derive simple parameter relationships analytically to explain the dynamics. However, the LE spectrum and unstable period distribution can always be computed, as we did in Fig. 3, and this may signal the existence of such a parameter relationship.

We investigated a variety of other parameter mismatchings for two coupled HR neurons, and in each case we attempted to find an additional mismatching that "mends" the pair of neurons so that they can oscillate synchronously (by which we mean GS, as for the case of I and c above). We found that "mending" of parameter mismatching can be achieved in the presence of diffusive coupling, not only when common noise is applied to the system.

It is useful to recall that for two identical HR neurons coupled diffusively without noise, CS occurs for sufficiently high coupling strength ε (Fig. 4a); the form of the coupling used here is to add a term $\varepsilon(x_{3-i} - x_i)$ to $\dot{x}_i$. Moreover, approximate CS occurs for sufficiently high coupling strength, even if the parameter χ is mismatched between the two neurons [20] (Fig. 4b). In fact, the approximate CS observed in this case is perhaps more accurately described as approximate GS, since there is a large difference in the magnitudes of the average synchronization errors of x, y and z (inset panel of Fig. 4b), similar to what is seen in Fig. 1b and c. We remark that, in the spirit of our above discussion, it is possible to view this approximate GS as induced by "mending" of parameter mismatching, i.e., the coupling strength acts as a parameter that can mend the mismatching of χ, leading to approximate GS. Note that, just as c can be transferred to I via the variable change $w = y - c$, χ can also be transferred to I by the variable change $w = z + S\chi$ (i.e., the parameter mismatch in the equation for $\dot{z}$ is equivalent to a parameter mismatch in the equation for $\dot{x}$ where coupling occurs).

Returning to the uncoupled situation, if we begin with two uncoupled HR neurons with mismatched χ ($\chi_1 = -1.56$, $\chi_2 = -1.57$) and noise intensity $D^x = 2.5$, then GS can be achieved by introducing parameter mismatching of I (Fig. 4c). Exact GS is achieved only for the mismatch $|I_1 - I_2| = 0.04$, in which case synchronization errors in x and y are zero while z maintains a constant difference $|z_1 - z_2| = 0.04$ between the two neurons. Note that using the variable change $w = z + S\chi$, it can easily be seen that a given mismatch in χ can be mended by a mismatch of magnitude $S(\chi_1 - \chi_2)$ in I.

Finally, we considered parameter mismatching of d ($d_1 = 5.05$, $d_2 = 4.95$), again with noise intensity $D^x = 2.5$. No mismatching of parameter b can mend the system and yield exact GS (Fig. 4d). However, the synchronization errors can be reduced to half their original values if we impose a particular mismatching of b.

3. An electronic model neuron and a map model

We briefly mention two other models for which exact mending of parameter mismatching can be achieved by an additional imposed parameter mismatching.

The first example is an electronic model neuron (EN), an analog circuit implementation of a four-dimensional neuron model. This model, which is an accurate representation of a real experimental apparatus [24–26], has the form

$$\begin{aligned}
\dot{x} &= y + 3x^2 - x^3 - z + J_{dc} + J(t) \\
\dot{y} &= 1 - 5x^2 - y - gw \\
\dot{z} &= \mu[-z + 4(x + h)] \\
\dot{w} &= v[-w + 3(y + l)]
\end{aligned} \tag{4}$$

where g, h, l, μ, v and J_{dc} are parameters and $J(t)$ is an external current. Note that the nonlinear terms in this model involve only the variable x (as in the HR model). As a means of investigating information transfer, in Ref. [25], parameter mismatching on J_{dc} was studied both theoretically and experimentally for two otherwise identical neurons. In fact, parameter mismatchings on h and l can be eliminated by controlling J_{dc} in the model, and this control should also be possible in experiments.

The second example is a minimal formal map model of excitable and bursting cells [27], which can be written

$$\begin{aligned}
x(t+1) &= \tanh\left(\frac{x(t) - Ky(t) + z(t) + I(t)}{T}\right) \\
y(t+1) &= x(t) \\
z(t+1) &= (1 - \delta)z(t) - \lambda(x(t) - x_R)
\end{aligned} \tag{5}$$

where K, T, δ, λ and x_R are parameters and $I(t)$ is an external current. Parameter mismatching of x_R can be mended by introducing a constant difference in the applied current $I(t)$, yielding identical dynamics for the two neurons except for a constant difference in the variable z, i.e., mending the parameter mismatching with a controllable parameter yields GS.

4. Conclusions

We have shown that two nonidentically parameterized dynamical systems that are coupled either diffusively or by noise can sometimes be "mended" (i.e., manipulated so as to display generalized synchronization) by imposing an additional mismatching of another parameter. This possibility will be most important in cases where parameters whose values differ between the two systems cannot be changed in the real system that is being modelled, but where imposed changes in controllable parameters can mend the dynamics.

We have given examples, focussing on Hindmarsh–Rose neurons, where a simple variable transformation explains why certain parameters can be manipulated to mend mismatches of another parameter. More generally, such explanatory transformations may not be possible to derive, but mending of parameter mismatching can still be investigated numerically, as we did in Fig. 3.

In previous work [28,29], it has been shown that increasing parameter mismatching between driver and receiver oscillators can dramatically decrease the critical coupling strength for generalized synchronization between two directionally coupled Rössler oscillators (because the parameter mismatching effectively enhances the coupling between driver and receiver. In contrast, the present paper provides an alternative mechanism for GS, based on multiple parameter mismatching (rather than increasing mismatching of a single parameter).

Acknowledgments

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References

[1] Pecora LM, Carroll TL. Phys Rev Lett 1990;64:821.
[2] Pecora LM, Carroll TL. Phys Rev A 1991;44:2374.
[3] Cuomo KM, Oppenheim AV. Phys Rev Lett 1993;71:65.
[4] Rinzel J, Sherman A, Stokes CL. Channels, coupling, and synchronized rhythmic bursting activity. In: Edckman FH, editor. Analysis and modeling of neural systems. Dordrecht: Kluwer Academic Publishers; 1989. p. 29.
[5] Earn DJD, Rohani P, Grenfell BT. Proc R Soc Lond B 1998;265:7.
[6] Earn DJD, Levin SA, Rohani P. Science 2000;290:1360.
[7] Rulkov NF, Afraimovich VS, Lewis CT, Chazottes J, Cordonet A. Phys Rev E 2001;64:016217.
[8] Boccaletti S, Kurths J, Osipov G, Valladares DL, Zhou CS. Phys Rep 2002;366:1.
[9] Choi M, Volodchenko KV, Rim S, Kye WH, Kim CM, Park YJ, Kim GU. Opt Lett 2003;28:1013.
[10] Pikovsky AS. Phys Lett A 1992;165:33.
[11] Mritan A, Banavar JR. Phys Rev Lett 1994;72:1451.
[12] Lai CH, Zhou CS. Europhys Lett 1998;43:376.
[13] Zhou CS, Kurths J. Phys Rev Lett 2002;88:230602.
[14] Hindmarsh JL, Rose RM. Nature (Lond) 1982;296:162.
[15] Hindmarsh JL, Rose RM. Proc R Soc London B 1984;221:87.
[16] Huerta R, Rabinovich MI. Phys Rev E 1997;55:R2108.
[17] He D, Shi P, Stone L. Phys Rev E 2003;67:027201.
[18] Hodgkin AL, Huxley AF. J Physiol (Lond) 1952;166:500.
[19] Zhou CS, Kurths J. Chaos 2003;13:401.
[20] Shuai JW, Durand DM. Phys Lett A 1999;264:289.
[21] Neiman AB, Russell DF. Phys Rev Lett 2002;88:138103.
[22] Rinzel J, Lee YS. J Math Biol 1987;25:653.
[23] Eccles JC. The understanding of the brain. New York: McGraw-Hill; 1973.
[24] Pinto RD, Varona P, Volkovskii AR, Szücs A, Abarabanel HDI, Rabinovich MI. Phys Rev E 2000;62:2644.
[25] Eguia MC, Rabinovich MI, Abarbanel HDI. Phys Rev E 2000;62:7111.
[26] Szucs A, Elson RC, Rabinovich MI, Abarbanel HDI, Selverston AI. J Neurophysiol 2001;85:1623.
[27] Kuva SM, Lima GF, Kinouchi O, Tragtenberg MHR, Roque AC. Neurocomputing 2001;38–40:255.
[28] Zheng Z, Hu G. Phys Rev E 2000;62:7882.
[29] He D, Stone L, Zheng ZG. Phys Rev E 2002;66:036208.