

## Equivariant bundles and isotropy representations

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**Abstract.** We introduce a new construction, the *isotropy groupoid*, to organize the orbit data for split  $\Gamma$ -spaces. We show that equivariant principal  $G$ -bundles over split  $\Gamma$ -CW complexes  $X$  can be effectively classified by means of representations of their isotropy groupoids. For instance, if the quotient complex  $A = \Gamma \backslash X$  is a graph, with all edge stabilizers toral subgroups of  $\Gamma$ , we obtain a purely combinatorial classification of bundles with structural group  $G$  a compact connected Lie group. If  $G$  is abelian, our approach gives combinatorial and geometric descriptions of some results of Lashof–May–Segal [18] and Goresky–Kottwitz–MacPherson [10].

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### Introduction

In this paper we continue our study of equivariant principal bundles via isotropy representations (see [13]). If  $\Gamma$  and  $G$  are topological groups, then a  $\Gamma$ -equivariant principal  $G$ -bundle is a locally trivial, principal  $G$ -bundle  $p: E \rightarrow X$  such that  $E$  and  $X$  are left  $\Gamma$ -spaces. The projection map  $p$  is  $\Gamma$ -equivariant and  $\gamma(e \cdot g) = (\gamma e) \cdot g$ , where  $\gamma \in \Gamma$  and  $g \in G$  acts on  $e \in E$  by the principal action. Equivariant principal bundles, and their natural generalizations, were studied by T. E. Stewart [25], T. tom Dieck [26], [27], I(8.7), R. Lashof [15], [16] together with P. May [17] and G. Segal [18].

The *isotropy representation* at a point  $x \in X$  is the homomorphism  $\alpha_x: \Gamma_x \rightarrow G$  defined by the formula

$$\gamma \cdot \tilde{x} = \tilde{x} \cdot \alpha_x(\gamma)$$

where  $\tilde{x} \in p^{-1}(x)$ . The homomorphism  $\alpha_x$  is independent of the choice of  $\tilde{x}$  up to conjugation in  $G$ . Here  $\Gamma_x$  denotes the isotropy subgroup or stabilizer of  $x \in X$ .

The use of isotropy representations is particularly effective when the projection  $\pi: X \rightarrow \Gamma \backslash X \xrightarrow{\cong} A$  has a section  $\varphi: A \rightarrow X$ . We call the triple  $(X, \pi, \varphi)$  a

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split  $\Gamma$ -space over  $A$ . A natural source of examples is symplectic toric manifolds (see (4.8)), where  $A$  is the moment polytope. Under reasonable assumptions, a split  $\Gamma$ -space over  $A$  is uniquely determined by its *isotropy groupoid*

$$\mathcal{I} := \{(\gamma, a) \in \Gamma \times A \mid \gamma \in \Gamma_{\varphi(a)}\}$$

(see Proposition 2.1). A  $\Gamma$ -equivariant principal  $G$ -bundle  $\eta := (E \xrightarrow{p} X)$  is called *split* if the pull-back  $\varphi^*(\eta)$  is a trivial bundle. The isotropy representations of  $\eta$  then produce a continuous groupoid representation of  $\mathcal{I}$  in  $G$  which is well defined up to conjugation by  $\text{Map}(A, G)$ . We denote by

$$\text{Rep}^G(\mathcal{I}) = \text{Hom}(\mathcal{I}, G)/\text{Map}(A, G)$$

the space of conjugacy classes of such groupoid representations. The first part of this paper is devoted to proving the following general classification theorem.

**Theorem A.** *Suppose that  $\Gamma$  and  $G$  are compact Lie groups. Let  $X$  be split  $\Gamma$ -space over  $A$  with isotropy groupoid  $\mathcal{I}$ . Assume that  $A$  is locally compact, and that  $\mathcal{I}$  is locally maximal. Then the equivalence classes of split  $\Gamma$ -equivariant  $G$ -bundles over  $X$  are in bijection with  $\text{Rep}^G(\mathcal{I})$ .*

The relevant definitions are given in Section 3: see § 3A for equivariant bundles, § 3B for the notion of a locally maximal isotropy groupoid, and the proof of Theorem A is given in § 3C. In our applications we will assume that  $X$  is a  $\Gamma$ -CW-complex, equipped with a splitting  $\varphi: A \rightarrow X$  such that  $\Gamma_{\varphi(a)}$  is constant on each open cell of its quotient CW-complex  $A$ . This property doesn't always hold (see Remark 4.2), but it seems a natural assumption. We call the resulting isotropy groupoids *cellular* (see § 4A). A cellular groupoid is a combinatorial object and, when  $\Gamma$  is discrete, it is a particular case of a *developable simple complex of groups* as considered by M. Bridson and A. Haefliger [4]. Cellular groupoids whose stalks  $\mathcal{I}_a$  are compact Lie groups are called *proper* groupoids. They arise from proper actions of a Lie group  $\Gamma$ , as studied for example in [20] for  $\Gamma$  discrete. In Theorem 4.5 we extend Theorem A to the classification of split  $\Gamma$ -equivariant bundles over a proper groupoid.

In a second part of this paper, we describe some approaches to computing  $\text{Rep}^G(\mathcal{I})$  assuming  $\mathcal{I}$  is cellular. There is a corresponding notion of *cellular representations*, meaning those which are constant on the open cells of  $A$ , whose classes modulo conjugation by a fixed element of  $G$  form a set denoted by  $\text{Rep}_{\text{cell}}^G(\mathcal{I})$  (see § 5A). The cellular representations are also purely combinatorial, and for  $A$  a regular CW-complex,  $\text{Rep}_{\text{cell}}^G(\mathcal{I})$  is determined by restriction to the 0-skeleton of  $A$  (see Proposition 5.1). We consider Theorem A to be an effective method of classifying equivariant bundles whenever  $\text{Rep}^G(\mathcal{I})$  can be reduced to  $\text{Rep}_{\text{cell}}^G(\mathcal{I})$ . We therefore study the natural map  $\text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \text{Rep}^G(\mathcal{I})$ , which turns out to be injective (Proposition 5.18) but not surjective in general (see (5.21)). It is however bijective when  $G$  is compact abelian (Proposition 5.3), or when  $A$  is a tree (Proposition 5.20).

We next consider the case where  $A$  is a graph. Here it is useful to define a related object  $\overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$ , by allowing conjugation of cellular homomorphisms over each cell of  $A$  independently. It turns out that there exists a natural map  $\iota: \text{Rep}^G(\mathcal{I}) \rightarrow \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$ . In Theorem 5.11, we study this map for  $G$  a compact connected Lie group. A sample application of Theorem 5.11 is given by the following:

**Theorem B.** *Let  $\Gamma$  and  $G$  be a compact Lie groups, with  $G$  connected, and suppose that  $A$  is a graph. If  $\mathcal{I}$  is a cellular groupoid with all edge stabilizers torus subgroups of  $\Gamma$ , then the map  $\iota: \text{Rep}^G(\mathcal{I}) \rightarrow \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  is a bijection.*

Recall that there exists a classifying space  $B(\Gamma, G)$  for  $\Gamma$ -equivariant principal  $G$ -bundle [26], so the classification of equivariant bundles in particular cases can also be approached by studying the  $\Gamma$ -equivariant homotopy type of  $B(\Gamma, G)$ . If the structural group  $G$  of the bundle is *abelian*, then the main result of Lashof, May and Segal [18] states that equivariant bundles over a  $\Gamma$ -space  $X$  are classified by the ordinary homotopy classes of maps  $[X \times_{\Gamma} E\Gamma, BG]$ . For non-abelian structural groups, it appears that the natural map  $\theta: [X, B(\Gamma, G)]_{\Gamma} \rightarrow [X \times_{\Gamma} E\Gamma, BG]$  misses a lot of information, and our results could be interpreted as studying  $\theta^{-1}(\bullet)$ .

Our isotropy groupoid  $\mathcal{I}$  has a classifying space  $B\mathcal{I}$  constructed by Haefliger [12], p. 140. We observe that  $B\mathcal{I} \simeq X \times_{\Gamma} E\Gamma$  when  $\mathcal{I}$  is cellular. In our language, the result of [18] implies that the natural map  $B: \text{Rep}^G(\mathcal{I}) \rightarrow [B\mathcal{I}, BG]$  is injective for  $G$  compact abelian. More generally, we show in Corollary 6.4:

**Theorem C.** *Let  $G$  and  $\Gamma$  be compact Lie groups, with  $G$  abelian. Let  $X$  be a split  $\Gamma$ -space over  $A$  with cellular isotropy groupoid  $\mathcal{I}$ . Suppose that  $H^1(A; \pi_0(G)) = H^2(A; \mathbb{Z}) = 0$ . Then the map  $B: \text{Rep}^G(\mathcal{I}) \rightarrow [B\mathcal{I}, BG]$  is a bijection.*

In 6.5 we point out the connection between our classification results and equivariant K-theory. Finally, in 6.7, we compare our results with some classical theorems in equivariant cohomology, due to Chang–Skjelbred [5] and Goresky–Kottwitz–MacPherson [10].

## 1. Preliminaries

Most of this section contains folklore facts about compact Lie groups. Let  $K$  and  $G$  be topological groups. We denote by  $\text{Hom}(K, G)$  the space of continuous homomorphisms from  $K$  to  $G$ , endowed with the compact-open topology. Two homomorphisms  $\alpha_1, \alpha_2 \in \text{Hom}(K, G)$  are called *conjugate* if there exists  $g \in G$  such that  $\alpha_2(\gamma) = g^{-1}\alpha_1(\gamma)g$  for all  $\gamma \in K$ . We denote by  $\overline{\text{Hom}}(K, G)$  the space of conjugacy classes, endowed with the quotient topology.

**Lemma 1.1.** *Let  $K$  and  $G$  be compact Lie groups. Then the space  $\overline{\text{Hom}}(K, G)$  is totally disconnected.*

*Proof.* A bi-invariant Riemannian metric on  $G$  gives rise to a bi-invariant distance  $d$  on  $G$ . The uniform convergence distance on  $\text{Hom}(K, G)$ ,

$$d(\alpha, \beta) = \max_{k \in K} d(\alpha(k), \beta(k)),$$

induces the compact-open topology. Let us denote by  $\bar{\alpha}, \bar{\beta} \in \overline{\text{Hom}}(K, G)$  the conjugacy classes of  $\alpha$  and  $\beta$ . One checks that the formula

$$\bar{d}(\bar{\alpha}, \bar{\beta}) = \min_{g \in G} d(g\alpha g^{-1}, \beta) = \min_{g, h \in G} d(g\alpha g^{-1}, h\beta h^{-1})$$

defines a distance on  $\overline{\text{Hom}}(K, G)$ . Because  $\bar{d}(\bar{\alpha}, \bar{\beta}) \leq d(\alpha, \beta)$ , the projection  $p: (\text{Hom}(K, G), d) \rightarrow (\overline{\text{Hom}}(K, G), \bar{d})$  is continuous, so the quotient topology on  $\overline{\text{Hom}}(K, G)$  is finer than the metric topology (one can check that they are equal, but we shall not use that).

The space  $\overline{\text{Hom}}(K, G)$  has at most countably many points [2], Prop. 10.14. Therefore, the set  $\mathcal{D} = \{d(a, b) \mid a, b \in \overline{\text{Hom}}(K, G)\}$  is at most countable. Let  $a, b \in \overline{\text{Hom}}(K, G)$  with  $a \neq b$ . There exists  $\lambda \in \mathbb{R}$  with  $0 < \lambda < \bar{d}(a, b)$  and  $\lambda \notin \mathcal{D}$ . The space  $\overline{\text{Hom}}(K, G)$  is then the disjoint union of  $\{x \mid \bar{d}(a, x) < \lambda\}$  and  $\{x \mid \bar{d}(a, x) > \lambda\}$ . These are non-empty open sets for the topology induced by  $\bar{d}$  and then for the quotient topology. This proves that any subspace of  $\overline{\text{Hom}}(K, G)$  containing more than one point is not connected.  $\square$

**Lemma 1.2.** *Let  $K$  and  $G$  be compact Lie groups. Let  $B$  be a space homeomorphic to a compact disk and let  $b \in B$ . Let  $x \mapsto \beta_x$  be a continuous map from  $B$  to  $\text{Hom}(K, G)$ . Then there is a continuous  $x \mapsto g_x$  from  $B$  to  $G$  with  $g_b = 1$  such that  $\beta_x(\gamma) = g_x^{-1}\beta_b(\gamma)g_x$  for all  $\gamma \in K$  and all  $x \in B$ .*

*Proof.* If  $B$  is of dimension  $n$ , then, by a pointed homeomorphism, one can replace the pair  $(B, b)$  by  $([0, 1]^n, 0)$  if  $b$  lies in the boundary of  $B$ , or by  $([-1, 1]^n, 0)$  otherwise. By Lemma 1.1,  $\beta_x$  stays for all  $x$  in the same conjugacy class  $\mathcal{O}$  of  $\text{Hom}(K, G)$ . As seen in the proof of Lemma 1.1, the space  $\text{Hom}(K, G)$  is metric, therefore Hausdorff. Therefore,  $\mathcal{O}$  is compact. As  $G$  is compact, the map  $p: G \rightarrow \mathcal{O}$  given by  $g \mapsto g\beta_0 g^{-1}$  can then be identified with a principal bundle whose structure group is the centraliser  $\mathcal{Z}(\beta_0(K))$ , which is a closed subgroup of  $G$ . Lemma 1.2 then follows from the a recursive use of the homotopy lifting property.  $\square$

**Lemma 1.3.** *Let  $K$  be a compact abelian Lie group. Denote by  $K_1$  the connected component of the unit element  $1 \in K$ . Then there is a bicontinuous isomorphism  $K_1 \times \pi_0(K) \xrightarrow{\cong} K$ .*

*Proof.* As  $K$  is abelian, it suffices to construct a homomorphic section of the projection  $K \rightarrow \pi_0(K)$ . As  $\pi_0(K)$  is finite, one can reduce to the case where  $\pi_0(K) = C$  is a cyclic group of order  $m$ . Let  $c \in C$  be a generator and choose  $\tilde{c} \in K$  representing  $c$ . Then  $\tilde{c}^m$  is in  $K_1$  and there exists  $\gamma \in K_1$  such that  $\gamma^m = \tilde{c}^m$ . The map  $\sigma: C \rightarrow G$  defined by  $\sigma(c^k) = \tilde{c}^k \gamma^{-k}$  is a homomorphic section of the projection  $K \rightarrow C$ .  $\square$

**1.4. Spaces over  $A$ .** Let  $f: X \rightarrow A$  be a continuous map between topological spaces. This enables us to consider  $X$  as a “space over  $A$ ”. For  $a \in A$ , the *stalk* over  $a$  is  $X_a = f^{-1}(a)$ . Any subspace  $Y$  of  $B \times A$  is seen as a space over  $A$  via the projection onto  $A$  restricted to  $Y$ .

## 2. Split $\Gamma$ -spaces

Let  $A$  be a topological space and  $\Gamma$  be a topological group. A  $\Gamma$ -*space* is a topological space equipped with a continuous left action of  $\Gamma$ . If  $X$  is a  $\Gamma$ -space and  $x \in X$ , we denote by  $\Gamma_x$  the stabiliser of  $x$ .

A *split  $\Gamma$ -space over  $A$*  is a triple  $(X, \pi, \varphi)$  where

- (i)  $X$  is a  $\Gamma$ -space.
- (ii)  $\pi: X \rightarrow A$  is a continuous surjective map and, for each  $a \in A$ , the preimage  $\pi^{-1}(a)$  is a single orbit.
- (iii)  $\varphi: A \rightarrow X$  is a continuous section of  $\pi$

The maps  $\pi$  and  $\varphi$  may be omitted from the notation and we might speak of a split  $\Gamma$ -space  $X$  over  $A$ . By (ii), the map  $\pi$  descends to  $\bar{\pi}: \Gamma \backslash X \rightarrow A$  which is a homeomorphism (its continuous inverse is provided by the section  $\varphi$ ). The map  $\pi$  may thus be identified with the projection of  $X$  onto the orbit space  $\Gamma \backslash X$ .

A  $(\Gamma, A)$ -*groupoid* is a subspace  $\mathcal{I} \subset \Gamma \times A$  such that, for each  $a \in A$ , the space  $\mathcal{I}_a = \mathcal{I} \cap (\Gamma \times \{a\})$  is of the form  $\tilde{\mathcal{I}}_a \times \{a\}$ , where  $\tilde{\mathcal{I}}_a$  is a closed subgroup of  $\Gamma$ . We consider  $\mathcal{I}_a$  as a topological group, naturally isomorphic to the closed subgroup  $\tilde{\mathcal{I}}_a$  of  $\Gamma$ . We will often identify these two groups, and write, for instance,  $\mathcal{I}_a = \mathcal{I}_b$  when we mean  $\tilde{\mathcal{I}}_a = \tilde{\mathcal{I}}_b$ . The space  $\mathcal{I}$  is regarded as a topological groupoid whose space of objects is  $A$ : if  $a, b \in A$ , the space of morphisms from  $a$  to  $b$  is empty when  $a \neq b$  and is equal to  $\mathcal{I}_a$  otherwise.

Let  $\mathcal{I}$  be a  $(\Gamma, A)$ -groupoid. A (*left*) *action* of  $\mathcal{I}$  on a topological space  $W$  is a continuous map  $\beta: \mathcal{I} \times W \rightarrow W$  such that, for each  $a \in A$ , the restriction of  $\beta$  to  $\mathcal{I}_a \times W$  is an action of  $\mathcal{I}_a$  on  $W$ . The notation  $\zeta \cdot w$  is used for  $\beta(\zeta, w)$ . A *right* action is defined accordingly, as a continuous map from  $W \times \mathcal{I}$  to  $W$ . When  $\mathcal{I}$  acts on the right on a space  $V$  and on the left on a space  $W$ , we define the quotient space

$$V \times_{\mathcal{I}} W = (V \times W) / \sim,$$

where “ $\sim$ ” is the smallest equivalence relation such that  $(v \cdot \zeta, w) \sim (v, \zeta \cdot w)$  for all  $\zeta \in \mathcal{I}$ . If  $W$  is a space over  $A$ , then  $V \times_{\mathcal{I}} W$  is a space over  $A$  as well. The stalk over  $a$  is then  $V \times_{\mathcal{I}_a} W_a$ .

Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space over  $A$ . Its *isotropy groupoid* is the  $(\Gamma, A)$ -groupoid defined by

$$\mathcal{I}(X) = \mathcal{I}(X, \pi, \varphi) := \{(\gamma, a) \in \Gamma \times A \mid \gamma \in \Gamma_{\varphi(a)}\}.$$

A  $(\Gamma, A)$ -groupoid  $\mathcal{I}$  is called *weakly locally maximal* if each point  $a \in A$  admits a neighbourhood  $U$  such that  $\mathcal{I}_u$  is a subgroup of  $\mathcal{I}_a$  for all  $u \in U$ . A space  $X$  is called *locally compact* if it is Hausdorff and if every point of  $X$  admits a compact neighbourhood. The main result of this section is the following proposition.

**Proposition 2.1** (Reconstruction). *Let  $\Gamma$  be a compact topological group and  $A$  be a locally compact space. Let  $\mathcal{I}$  be a weakly locally maximal  $(\Gamma, A)$ -groupoid. Then the following properties hold.*

- (i) *There is a split  $\Gamma$ -space  $(Y_{\mathcal{I}}, \pi, \phi)$  over  $A$  with isotropy groupoid  $\mathcal{I}$ ; the space  $Y_{\mathcal{I}}$  is locally compact.*
- (ii) *Let  $(X, \pi, \varphi)$  and  $(X', \pi', \varphi')$  be two split  $\Gamma$ -spaces over  $A$  with isotropy groupoid  $\mathcal{I}$ . Suppose that  $X$  and  $X'$  are locally compact. Then there is a unique  $\Gamma$ -equivariant homeomorphism  $F: X \rightarrow X'$  such that  $\varphi' = F \circ \varphi$ .*

Proposition 2.1 permits us to speak about *the* split  $\Gamma$ -space over  $A$  with isotropy groupoid  $\mathcal{I}$  (as we speak about *the* real number field instead of *a* real number field). The triple  $(Y_{\mathcal{I}}, \pi, \phi)$  constructed for the proof of (i) is an explicit model for this space, but other models also occur naturally, as will be seen in examples.

*Proof of Proposition 2.1.* The groupoid  $\mathcal{I}$  acts by multiplication on the right on  $\Gamma$ . We let it act trivially on the left on  $A$  and form the space

$$Y_{\mathcal{I}} = \Gamma \times_{\mathcal{I}} A.$$

The projection  $\Gamma \times A \rightarrow A$  descends to a continuous surjective map  $\pi: Y_{\mathcal{I}} \rightarrow A$ . The section  $\phi: A \rightarrow Y_{\mathcal{I}}$  is defined by  $\phi(a) = [1, a]$ , where 1 is the unit element in  $\Gamma$ . The  $\Gamma$ -action  $\beta \cdot (\gamma, x) = (\beta\gamma, x)$  on  $\Gamma \times A$  descends to a  $\Gamma$ -action on  $Y_{\mathcal{I}}$ . The stalk  $\pi^{-1}(a)$  is the orbit through  $\phi(a)$  and  $\Gamma_{\phi(a)} = \mathcal{I}_a$ . Thus,  $(Y_{\mathcal{I}}, \pi, \phi)$  is a split  $\Gamma$ -space over  $A$  with isotropy groupoid  $\mathcal{I}$ .

To prove that  $Y_{\mathcal{I}}$  is Hausdorff, let  $x$  and  $y$  be two distinct points in  $Y_{\mathcal{I}}$ . Let us show that they admit disjoint neighbourhoods. If  $\pi(x) \neq \pi(y)$ , this is obvious since  $A$  is Hausdorff. In the case  $\pi(x) = \pi(y) = a$ , let  $U$  be a neighbourhood of  $a$  for which  $\mathcal{I}_b$  is a subgroup of  $\mathcal{I}_a$  for all  $b \in U$ . Then  $\pi^{-1}(U)$  is a neighbourhood of  $\{x, y\}$  and there is a continuous map  $f: \pi^{-1}(U) \rightarrow (\Gamma_{\phi(a)} \backslash \Gamma) \times U$  such that  $f(x) \neq f(y)$ . As  $\Gamma_{\phi(a)} = \mathcal{I}_a$  is a closed subgroup of  $\Gamma$ , the space  $(\Gamma_{\phi(a)} \backslash \Gamma) \times U$  is Hausdorff and then  $x$  and  $y$  admit disjoint neighbourhoods. Observe that the proof that  $Y_{\mathcal{I}}$  is Hausdorff uses only that  $\Gamma$  and  $A$  are Hausdorff and that  $\mathcal{I}$  is weakly locally maximal.

Let  $x \in Y_{\mathcal{I}}$ . As  $A$  is locally compact,  $\pi(x)$  admits a compact neighbourhood  $V$  in  $A$ . Then  $\pi^{-1}(V)$  is a neighbourhood of  $x$  and is the continuous image of  $\Gamma \times V$  under the natural projection  $\Gamma \times A \rightarrow Y_{\mathcal{I}}$ . As  $\Gamma \times V$  is compact and  $Y_{\mathcal{I}}$  is Hausdorff,  $\pi^{-1}(V)$  is compact. This shows that  $Y_{\mathcal{I}}$  is locally compact and finishes the proof of (i).

Let us prove (ii), starting with uniqueness. Let  $F_1, F_2: X \rightarrow X'$  be two  $\Gamma$ -equivariant isomorphisms satisfying  $F_1 \circ \varphi = F_2 \circ \varphi$ . Then  $F_1 = F_2$  by equivariance,

since  $\varphi(A)$  is a fundamental domain for the  $\Gamma$ -action. Now, let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space with isotropy groupoid  $\mathcal{I}$ . The map  $\tilde{F}: \Gamma \times A \rightarrow X$  defined by  $\tilde{F}(\gamma, a) = \gamma \cdot \varphi(a)$  is continuous,  $\Gamma$ -equivariant and surjective. It descends to a  $\Gamma$ -equivariant continuous bijection  $F: Y_{\mathcal{I}} \rightarrow X$ . Observe that  $\tilde{F}$  is proper, that is  $\tilde{F}^{-1}(K)$  is compact for all compact subsets  $K$  in  $X$ . Indeed,  $\tilde{F}^{-1}(K)$  is a closed subset of  $\Gamma \times \pi(K)$  which is compact. Since  $Y_{\mathcal{I}}$  is Hausdorff, the map  $F$  is proper as well. A proper continuous bijection between locally compact spaces is a homeomorphism, which proves (b).  $\square$

**Remark 2.2.** If in Proposition 2.1 we only assume that  $\Gamma$  is locally compact, the space  $Y_{\mathcal{I}}$  constructed for the proof of (i) might not be locally compact. As an example, take  $A = [0, 1]$ ,  $\mathcal{I}_a$  trivial for  $a < 1$  and  $\mathcal{I}_1 = \Gamma$ . Then  $Y_{\mathcal{I}}$  is the cone on  $\Gamma$ , which is not locally compact if  $\Gamma$  is not compact. Moreover, the uniqueness also fails in this case. Let  $\Gamma = \mathbb{Z}$ . Then the cone  $V$  on the real integers in the complex plane, with vertex  $i$  say, and the induced metric, is a split  $\Gamma$ -space with isotropy groupoid  $\mathcal{I}$ . The proof of Proposition 2.1 provides a  $\Gamma$ -equivariant continuous bijection from the  $Y_{\mathcal{I}}$  onto  $V$  but it is not a homeomorphism (a set containing one point in the interior of each segment would be closed in  $Y_{\mathcal{I}}$ , even if it contains a subsequence converging to the vertex  $i$ ). A version of Proposition 2.1 with  $\Gamma$  non-compact is given in Proposition 4.3.

A stronger local maximality condition will play a role in Section 3. A  $(\Gamma, A)$ -groupoid  $\mathcal{I}$  is called *locally maximal* if, for each point  $a \in A$  and each neighbourhood  $B$  of  $a$ , there exists an open set  $U$  of  $A$ , with  $a \in U \subset B$ , and a homotopy  $\rho_t: U \rightarrow U$  ( $t \in [0, 1]$ ) satisfying  $\rho_0(u) = u$ ,  $\rho_1(u) = a$  and  $\mathcal{I}_u \subset \mathcal{I}_{\rho_t(u)}$  for all  $u \in U$  and  $t \in [0, 1]$ . The neighbourhood  $U$  is then contractible. Locally maximal implies weakly locally maximal.

**Lemma 2.3.** *Let  $\Gamma$  be a compact topological group and  $A$  be a locally compact space. Let  $\mathcal{I}$  be a locally maximal  $(\Gamma, A)$ -groupoid. Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space over  $A$  with isotropy groupoid  $\mathcal{I}$  and let  $x \in X$ . Then there is a  $\Gamma$ -equivariant open neighbourhood  $\hat{U}$  of the orbit  $\Gamma x$  and a  $\Gamma$ -equivariant  $\hat{\rho}_t: \hat{U} \rightarrow \hat{U}$  such that  $\hat{\rho}_0 = \text{id}$  and  $\hat{\rho}_1(\hat{U}) = \Gamma \cdot x$ . Moreover,  $\hat{\rho}_t$  satisfies  $\pi \circ \hat{\rho}_t = \rho_t \circ \pi$  and  $\varphi \circ \rho_t = \hat{\rho}_t \circ \varphi$ .*

*Proof.* By Proposition 2.1, one may suppose that  $(X, \pi, \varphi) = (Y_{\mathcal{I}}, \pi, \varphi)$ . Let  $a = \pi(x)$  and let  $\rho_t: U \rightarrow U$  be a homotopy from an open neighbourhood  $U$  of  $A$  to itself, such that  $\rho_0(u) = u$ ,  $\rho_1(u) = a$  and  $\mathcal{I}_u \subset \mathcal{I}_{\rho_t(u)}$  for all  $u \in U$ . Let  $\hat{U} = \pi^{-1}(U)$ . We check that the required homotopy  $\hat{\rho}_t$  can be defined by  $\hat{\rho}_t([\gamma, u]) := [\gamma, \rho_t(u)]$ .  $\square$

Let  $\mathcal{I}$  be a  $(\Gamma, A)$ -groupoid and let  $\mathcal{I}'$  be a  $(\Gamma', A')$ -groupoid. A *morphism of*

groupoids from  $\mathcal{I}$  to  $\mathcal{I}'$  is a commutative diagram

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{f} & \mathcal{I}' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\bar{f}} & A' \end{array}$$

where  $f$  and  $\bar{f}$  are continuous maps such that, for each  $a \in A$ , the restriction of  $f$  to  $\mathcal{I}_a$  is a homomorphism  $f_a: \mathcal{I}_a \rightarrow \mathcal{I}'_{\bar{f}(a)}$ . The map  $\bar{f}$  is not mentioned when it is obvious, like an inclusion or a constant map.

### 3. Split equivariant principal bundles

**§ 3A. Definitions.** Let  $G$  be a topological group and  $X$  be a topological space. By a  $G$ -principal bundle  $\eta$  over  $X$ , we mean, as usual, a continuous surjection  $p: E \rightarrow X$  from a space  $E = E(\eta)$  and a free right action  $E \times G \rightarrow E$  so that  $p(z \cdot g) = p(z)$ , with the standard local triviality condition. Two  $G$ -principal bundles  $\eta: E \xrightarrow{p} X$  and  $\eta': E' \xrightarrow{p'} X$  over  $X$  are *isomorphic* if there exists a  $G$ -homeomorphism  $f: E \rightarrow E'$  such that  $p' \circ f = p$ . Isomorphism classes of  $G$ -principal bundles over  $X$  are denoted by  $\text{Bun}^G(X)$ .

Let  $X$  be a  $\Gamma$ -space for a topological group  $\Gamma$ . A  $G$ -principal bundle  $\eta: E \xrightarrow{p} X$  is called a  $\Gamma$ -equivariant principal  $G$ -bundle if it is given a left action  $\Gamma \times E \rightarrow E$  commuting with the free right action of  $G$  and such that the projection  $p$  is  $\Gamma$ -equivariant. Two  $\Gamma$ -equivariant principal  $G$ -bundles  $\eta$  and  $\eta'$  are called  $\Gamma$ -isomorphic (or just *isomorphic*) if there exists a  $G$ -homeomorphism from  $E(\eta)$  to  $E(\eta')$  over the identity of  $X$  which is  $\Gamma$ -equivariant. The set of  $\Gamma$ -isomorphism classes of  $\Gamma$ -equivariant  $G$ -principal bundles over  $X$  is denoted by  $\text{Bun}_\Gamma^G(X)$ . There is a forgetful map  $\text{Bun}_\Gamma^G(X) \rightarrow \text{Bun}^G(X)$ .

Let  $(X, \pi, \varphi)$  is a split  $\Gamma$ -space over  $A$  with isotropy groupoid  $\mathcal{I}$ . Let  $\xi$  be a  $\Gamma$ -equivariant principal  $G$ -bundle over  $X$ . We say that  $\xi$  is *split* if the induced bundle  $\varphi^*\xi$  is trivial. For instance, any  $\Gamma$ -equivariant principal  $G$ -bundle is split when  $A$  is contractible and paracompact, which is the case in many examples of Section § 4B. Two split  $\Gamma$ -equivariant principal  $G$ -bundles over  $(X, \pi, \varphi)$  are *isomorphic* if they are isomorphic just as  $\Gamma$ -equivariant principal  $G$ -bundles over  $X$ . The set of isomorphism classes of  $\Gamma$ -equivariant split  $G$ -principal bundles over  $(X, \pi, \varphi)$  is denoted by  $\text{SBun}_\Gamma^G(X, \pi, \varphi)$  or simply by  $\text{SBun}_\Gamma^G(X)$ . It is a subset of  $\text{Bun}_\Gamma^G(X)$ .

**§ 3B. The isotropy representation.** Let  $\mathcal{I}$  be a  $(\Gamma, A)$ -groupoid and  $G$  be a topological group. A *continuous representation* of  $\mathcal{I}$  to  $G$  is a continuous map  $\alpha: \mathcal{I} \rightarrow G$  such that, for all  $a \in A$ , the restriction  $\alpha_a$  of  $\alpha$  to  $\mathcal{I}_a$  is a homomorphism (it is thus



a morphism of groupoids between  $\mathcal{I}$  and the  $(pt, G)$ -groupoid  $G \rightarrow pt$ . Two continuous representations  $\alpha_1$  and  $\alpha_2$  are called *conjugate* if there exists a continuous map  $\psi: A \rightarrow G$  such that  $\alpha_2(\zeta) = \psi(\pi_2(\zeta))^{-1}\alpha_1(\zeta)\psi(\pi_2(\zeta))$  for all  $\zeta \in \mathcal{I}$ , where  $\pi_2: \mathcal{I} \rightarrow A$  is the second factor projection.

A continuous representation of  $\alpha: \mathcal{I} \rightarrow G$  is called *locally maximal* if each point  $a \in A$  admits a neighbourhood  $U$  such that  $\mathcal{I}_u$  is a subgroup of  $\mathcal{I}_a$  for all  $u \in U$ , together with a continuous map  $g: U \rightarrow G$  such that  $\alpha_u(\gamma) = g(u)\alpha_a(\gamma)g(u)^{-1}$  for all  $u \in U$  and all  $\gamma \in \mathcal{I}_u$ . This implies that  $\mathcal{I}$  is weakly locally maximal. It is easy to see that, if  $\alpha, \beta: \mathcal{I} \rightarrow G$  are two conjugate representations of  $\mathcal{I}$ , then  $\beta$  is locally maximal if and only if  $\alpha$  is weakly locally maximal. We denote by  $\text{Rep}^G(\mathcal{I})$  the set of conjugacy classes of locally maximal continuous representations of  $\mathcal{I}$ .

Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space over  $A$  with isotropy groupoid  $\mathcal{I}$ . Let  $\eta: E \xrightarrow{p} X$  be a split  $\Gamma$ -equivariant  $G$ -principal bundle over  $X$ . As  $\varphi^*\eta$  is trivial, there exists a continuous lifting  $\tilde{\varphi}: A \rightarrow E$  of  $\varphi$ . The equation

$$\gamma \cdot \tilde{\varphi}(a) = \tilde{\varphi}(a) \tilde{\alpha}_a(\gamma), \quad (1)$$

valid for  $a \in A$  and  $\gamma \in \mathcal{I}_a$ , determines a continuous representation  $\alpha_{\eta, \tilde{\varphi}}: \mathcal{I} \rightarrow G$ .

**Lemma 3.1.** *Suppose that  $\Gamma$  and  $G$  are compact Lie groups and that  $A$  is locally compact. If  $\mathcal{I}$  is locally maximal, then the continuous representation  $\alpha_{\eta, \tilde{\varphi}}$  is weakly locally maximal.*

*Proof.* Let  $a \in A$  and let  $B$  be a compact neighbourhood of  $a$ . Since  $\mathcal{I}$  is locally maximal, there exists an open set  $U$ , with  $a \in U \subset B$  and a homotopy  $\rho_t: U \rightarrow U$  such that  $\rho_0(u) = u$ ,  $\rho_1(u) = a$  and  $\mathcal{I}_u \subset \mathcal{I}_{\rho_t(u)}$  for all  $u \in U$ . If  $Z \subset A$ , we denote  $\hat{Z} = \pi^{-1}(Z)$ ; if  $Y$  is a  $\Gamma$ -invariant subspace of  $X$ , we denote  $E_Y = p^{-1}(Y)$ . The latter is the total space of a split  $\Gamma$ -equivariant  $G$ -principal bundle  $\eta_Y$  over  $Y$ .

By Proposition 2.1 and its proof, the space  $\hat{B}$  is compact. Then  $E_{\hat{B}}$  is compact and therefore totally regular. By [27], Proposition 8.10, the bundle  $\eta_{\hat{B}}$  is then a locally trivial numerable  $\Gamma$ -equivariant  $G$ -principal bundle in the sense of [27], p. 58. The same then holds for its restriction  $\eta_{\hat{U}}$ .

By Lemma 2.3 and its proof, the homotopy  $\rho_t: U \rightarrow U$  is covered by a  $\Gamma$ -equivariant homotopy  $\hat{\rho}_t: \hat{U} \rightarrow \hat{U}$  such that  $\hat{\rho}_0 = \text{id}$  and  $\hat{\rho}_1(\hat{U}) = \pi^{-1}(a)$ . By [27], Theorem 8.15, the induced bundle  $\hat{\rho}_1^*\eta_{\pi^{-1}(a)}$  is then isomorphic to  $\eta_{\hat{U}}$ . More precisely, let

$$E_1 := \{(x, z) \in \hat{U} \times E_{\pi^{-1}(a)} \mid \hat{\rho}_1(x) = p(z)\}$$

be the total space of  $\hat{\rho}_1^*\eta_{\pi^{-1}(a)}$ . Then there is a  $(\Gamma \times G)$ -equivariant homeomorphism  $\mu: E_1 \rightarrow E_{\hat{U}}$  which commutes with the projections onto  $\hat{U}$ . By Lemma 2.3 one has  $\varphi \circ \rho_t = \hat{\rho}_t \circ \varphi$ ; therefore,  $(\varphi(u), \varphi(a)) \in E_1$  for all  $u \in U$ . This enables to define  $\tilde{\varphi}': U \rightarrow E$  by  $\tilde{\varphi}'(u) = \mu(\varphi(u), \tilde{\varphi}(a))$ . For  $\gamma \in \mathcal{I}_u \subset \mathcal{I}_a$ , we have

$$\gamma \cdot \tilde{\varphi}'(u) = \mu(\varphi(u), \gamma \tilde{\varphi}(a)) = \mu(\varphi(u), \tilde{\varphi}(a) \alpha_a(\gamma)) = \mu(\varphi(u), \tilde{\varphi}(a)) \cdot \alpha_a(\gamma). \quad (2)$$

On the other hand,  $\tilde{\varphi}$  and  $\tilde{\varphi}'$  are two liftings of  $\varphi$  over  $U$ . Hence, there exists a continuous map  $g: U \rightarrow G$  such that  $\tilde{\varphi}'(u) = \tilde{\varphi}(u) \cdot g(u)$  for all  $u \in U$ . Therefore

$$\gamma \cdot \tilde{\varphi}'(u) = \gamma \tilde{\varphi}(u) g(u) = \tilde{\varphi}(u) \alpha_u(\gamma) g(u) = \tilde{\varphi}'(u) \cdot (g(u)^{-1} \alpha_u(\gamma) g(u)). \quad (3)$$

Comparing equations (2) with (3), we get that  $\alpha_u(\gamma) = g(u) \alpha_a(\gamma) g(u)^{-1}$ , which proves Lemma 3.1.  $\square$

By Lemma 3.1,  $\alpha_{\eta, \tilde{\varphi}}$  determines a class  $\alpha_\eta \in \text{Rep}^G(\mathcal{I})$ . We check that  $\alpha_\eta$  does not depend on the choice of  $\tilde{\varphi}$  and depends only on the  $\Gamma$ -equivariant isomorphism class of  $\eta$ ; details are as in the proof of [13], Lemma 3.2. This defines a map

$$\Phi: \text{SBun}_\Gamma^G(X) \rightarrow \text{Rep}^G(\mathcal{I})$$

called the *isotropy representation*.

**§3C. The classification theorem.** The following theorem corresponds to Theorem A of the introduction.

**Theorem 3.2** (Classification). *Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space over  $A$  with isotropy groupoid  $\mathcal{I}$ . Suppose that  $A$  is locally compact, that  $\mathcal{I}$  is locally maximal and that  $\Gamma$  is a compact Lie group. Then, for any compact Lie group  $G$ , the isotropy representation  $\Phi: \text{SBun}_\Gamma^G(X) \rightarrow \text{Rep}^G(\mathcal{I})$  is a bijection.*

*Proof.* We first prove the surjectivity of  $\Phi$ . By Proposition 2.1, we may assume that  $(X, \pi, \varphi) = (Y_{\mathcal{I}}, \pi, \phi)$ . Recall that  $Y_{\mathcal{I}} = \Gamma \times_{\mathcal{I}} A$ .

Let  $\beta: \mathcal{I} \rightarrow G$  be a continuous representation. Then  $\mathcal{I}$  acts on the left on  $A \times G$  by  $\zeta \cdot (a, g) = (a, \beta(\zeta)g)$ . Form the space  $E_\beta = \Gamma \times_{\mathcal{I}} (A \times G)$ . The continuous map  $p: E_\beta \rightarrow Y_{\mathcal{I}}$  given  $p([\gamma, (a, g)]) = [\gamma, a]$  coincides with the projection  $E_\beta \rightarrow E_\beta/G$  of  $E_\beta$  to its orbit space for the obvious free right  $G$ -action on  $E_\beta$ . A lifting  $\tilde{\phi}: A \rightarrow E_\beta$  of  $\phi$  is given by  $\tilde{\phi}(a) = [1, (a, 1)]$ , where 1 denotes the unit elements. For  $a \in A$  and  $\gamma \in \mathcal{I}_a$ , one has

$$\gamma \cdot \tilde{\phi}(a) = \gamma \cdot [1, (a, 1)] = [\gamma, (a, 1)] = [1, (a, \beta_a(\gamma))] = \tilde{\phi}(a) \cdot \beta_a(\gamma).$$

We now prove that  $p$  admits local trivializations when  $\beta$  is locally maximal. Let  $a \in A$ . Choose an open neighbourhood  $U_a$  of  $a$  such that  $\mathcal{I}_u$  is a subgroup of  $\mathcal{I}_a$  for all  $u \in U_a$ , together with a continuous map  $g_a: U_a \rightarrow G$  such that  $\beta_u(\gamma) = g_a(u) \beta_a(\gamma) g_a(u)^{-1}$  for all  $u \in U_a$  and all  $\gamma \in \mathcal{I}_u$ . This gives an open cover  $\mathcal{U} = \{U_a \mid a \in A\}$  of  $A$ . Setting  $\hat{U}_a = \pi^{-1}(U_a)$  gives rise to an open cover  $\hat{\mathcal{U}} = \{\hat{U}_a \mid a \in A\}$  of  $X$ , indexed by  $A$ . Define  $\tilde{f}_a: \Gamma \times U_a \times G \rightarrow \Gamma \times \{a\} \times G$  by  $\tilde{f}_a(\gamma, u, g) := (\gamma, a, g_a(u)g)$ . If  $\delta \in \mathcal{I}_u$ , we have

$$\tilde{f}_a(\gamma \delta, u, g) = (\gamma \delta, a, g_a(u)g) = (\gamma, a, \beta_a(\delta) g_a(u)g)$$

and

$$\tilde{f}_a(\gamma, u, \beta_u(\delta)g) = (\gamma, a, g_a(u)\beta_u(\delta)g).$$

Since  $\beta_a(\delta)g_a(u) = g_a(u)\beta_u(\delta)$ , this shows that  $\tilde{f}_a$  descends to a continuous  $G$ -equivariant map  $f_a: p^{-1}(\hat{U}_a) \rightarrow p^{-1}(\pi^{-1}(a))$ . Passing to the quotient by  $G$  gives rise to a commutative diagram

$$\begin{array}{ccccc} p^{-1}(\hat{U}_a) & \xrightarrow{f_a} & p^{-1}(\pi^{-1}(a)) & \xleftarrow{\approx} & \Gamma \times_{\mathcal{I}_a} G \\ p \downarrow & & \downarrow p & & \downarrow \\ \hat{U}_a & \xrightarrow{\tilde{f}_a} & \pi^{-1}(a) & \xleftarrow{\approx} & \Gamma/\mathcal{I}_a. \end{array}$$

Since  $\Gamma$  is a Lie group and  $\mathcal{I}_a$  a closed subgroup, the projection  $q_a: \Gamma \rightarrow \Gamma/\mathcal{I}_a$  admits local sections  $\sigma_V: V \rightarrow \Gamma$  for each  $V$  in some open covering  $\mathcal{V}_a$  of  $\Gamma/\mathcal{I}_a$  (see, e.g., [24], §7.5). We check that the formula

$$\zeta_V(\gamma, g) = \beta_a(\sigma(q(\gamma))^{-1}\gamma) g$$

defines a  $G$ -equivariant continuous map  $\zeta_V: p^{-1}(V) \rightarrow G$ , which gives rise to a trivialization over  $V$  of  $p: \Gamma \times_{\mathcal{I}_a} G \rightarrow \Gamma/\mathcal{I}_a$ . Therefore,  $\zeta_V \circ f_a: p^{-1}(\tilde{f}_a^{-1}(V)) \rightarrow G$  is a  $G$ -equivariant continuous map giving rise to a trivialization over  $\tilde{f}_a^{-1}(V)$  of  $p: p^{-1}(\hat{U}_a) \rightarrow \hat{U}_a$ . This gives rise to a trivializing open cover  $\mathcal{W} = \{\tilde{f}_a^{-1}(V) \mid (a, V) \in \mathcal{A}\}$  of  $X$ , indexed by  $\mathcal{A} = \{(b, V) \mid b \in A \text{ and } V \in \mathcal{V}_b\}$ . We have proved thus the surjectivity of  $\Phi$ .

We now prove the injectivity of  $\Phi$ . Let  $\eta: (E \xrightarrow{\bar{\varphi}} X)$  be a split  $\Gamma$ -equivariant bundle with  $\Phi(\eta) = [\beta]$ . A lifting  $\bar{\varphi}: A \rightarrow E$  of  $\varphi$  then produces a continuous representation  $\bar{\beta} = \alpha_{\eta, \bar{\varphi}}$  with  $[\bar{\beta}] = [\beta]$ . There exists then a continuous map  $\psi: A \rightarrow G$  such that  $\beta(\zeta) = \psi(q(\zeta))^{-1}\bar{\beta}(\zeta)\psi(q(\zeta))$ . The map  $\tilde{\varphi}: A \rightarrow E$  given by  $\tilde{\varphi}(a) = \bar{\varphi}(a) \cdot \psi(a)$  is then another lifting of  $\varphi$  such that  $\alpha_{\eta, \tilde{\varphi}} = \beta$ . One checks that the correspondence  $[\gamma, (a, g)] \mapsto \gamma \cdot \varphi(a) \cdot g$  defines a  $(\Gamma \times G)$ -equivariant continuous bijection  $\tilde{F}: E_\beta \rightarrow E$ , covering the unique  $\Gamma$ -equivariant homeomorphism  $F: Y_{\mathcal{I}} \rightarrow X$  such that  $F \circ \varphi = \phi$ , obtained in Proposition 2.1. Since  $F$  is a homeomorphism, so is  $\tilde{F}$ . Indeed, choose an open set  $Z$  in  $Y_{\mathcal{I}}$  such that  $\xi_\beta$  is trivial over  $Z$  and  $\xi$  is trivial over  $F(Z)$ . Using trivializations, we can write  $\tilde{F}(z, g) = (F(z), \mu(z)g)$ , where  $\mu: Z \rightarrow G$  is a continuous map. Then  $\tilde{F}^{-1}$  has, over  $F(Z)$ , the form  $\tilde{F}^{-1}(y, h) = (F^{-1}(y), \mu(F^{-1}(y))^{-1}h)$ , which is continuous. We have thus proven that two split  $\Gamma$ -equivariant principal  $G$ -bundles  $\eta$  and  $\eta'$  with  $\Phi(\eta) = \Phi(\eta')$  are  $\Gamma$ -equivariantly isomorphic.  $\square$

**Remark 3.3.** Recall that an open cover of a space is *numerable* if it admits a refinement by a locally finite partition of unity. In the proof of Theorem 3.2, the covers  $\mathcal{V}_a$  are numerable, since  $\Gamma/\mathcal{I}_a$  are manifolds. Hence we can check that the trivializing cover  $\mathcal{W}$  of  $X$  is numerable if  $\mathcal{U}$  is numerable. This observation will be used in Theorem 4.5.

**3.4. Non-split bundles and abelian structure group.** Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space over  $A$ . One has the map  $\varphi^*: \text{Bun}_\Gamma^G(X) \rightarrow \text{Bun}^G(A)$ , sending  $\xi$  to  $\varphi^*\xi$ . This map is surjective: if  $\eta \in \text{Bun}^G(A)$ , then  $\pi^*\eta$  admits a natural  $\Gamma$ -action, since  $\pi$  is  $\Gamma$ -invariant, so  $\pi^*$  is a section of  $\varphi^*$ . Theorem 3.2 computes the pre-image of the trivial bundle, which is  $\text{SBun}_\Gamma^G(X)$ .

Let us now assume that  $G$  is abelian. Recall that there is then a composition law “ $\otimes$ ” on  $\text{Bun}_\Gamma^G(X)$  which makes the latter an abelian group. If  $\xi_i: E_i \xrightarrow{p_i} X$  ( $i = 1, 2$ ) are  $\Gamma$ -equivariant principal  $G$ -bundles, one defines  $\xi_1 \otimes \xi_2: E \xrightarrow{p_i} X$  by first forming the pull-back

$$\begin{array}{ccc} E_1 \hat{\times} E_2 & \longrightarrow & E_1 \\ \downarrow & & \downarrow p_1 \\ E_2 & \xrightarrow{p_2} & X \end{array}$$

where the map  $E_1 \hat{\times} E_2 \rightarrow X$  is a principal  $G \times G$ -bundle. Set  $E = E_1 \hat{\times}_G E_2$  (as  $G$  is abelian, it acts on the left or on the right on  $E_i$ ) and check that  $\xi_1 \otimes \xi_2$  is a principal  $G$ -bundle over  $X$ . The diagonal  $\Gamma$ -action on  $E_1 \hat{\times} E_2$  descends to a  $\Gamma$ -action on  $E$ , making  $\xi_1 \otimes \xi_2$  a  $\Gamma$ -equivariant principal  $G$ -bundle. When  $G = S^1$ , we can think of  $\xi_i$  as  $\Gamma$ -equivariant complex line bundles over  $X$ , thus “ $\otimes$ ” becomes the standard tensor product. The map  $\varphi^*: \text{Bun}_\Gamma^G(X) \rightarrow \text{Bun}^G(A)$  is a group-homomorphism.

Another special feature of the case  $G$  abelian is that the isotropy representation is defined on  $\text{Bun}_\Gamma^G(X)$ : in equation (1), one can just use a local section  $\tilde{\varphi}$  around  $a \in A$ , whose choice is irrelevant if  $G$  is abelian. The set  $\text{Rep}^G(\mathcal{I})$  is an abelian group, by multiplication of the images, and  $\Phi: \text{Bun}_\Gamma^G(X) \rightarrow \text{Rep}^G(\mathcal{I})$  is a group homomorphism. Using Theorem 3.2, we get

**Proposition 3.5.** *Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space over  $A$  with isotropy groupoid  $\mathcal{I}$ . Suppose that  $A$  is locally compact, that  $\mathcal{I}$  is locally maximal and that  $\Gamma$  is a compact Lie group. Then, for any compact abelian Lie group  $G$ , one has an isomorphism of abelian groups*

$$(\Phi, \varphi^*): \text{Bun}_\Gamma^G(X) \xrightarrow{\cong} \text{Rep}^G(\mathcal{I}) \times \text{Bun}^G(A).$$

**3.6. Functorial properties.** Theorem 3.2 enjoys functorial properties which are contravariant in  $(\Gamma, A)$  and covariant in  $G$ . For the contravariant ones, let  $f: A' \rightarrow A$  be a continuous map between locally compact spaces and  $h: \Gamma' \rightarrow \Gamma$  be a continuous homomorphism between compact Lie groups. Let  $\mathcal{I}$  be a  $(\Gamma, A)$ -groupoid. Then

$$\mathcal{I}' := (h, f)^* \mathcal{I} := \{(\gamma', a') \in \Gamma' \times A' \mid h(\gamma') \in \mathcal{I}_{f(a')}\}$$

is a  $(\Gamma', A')$ -groupoid, with  $\mathcal{I}'_{a'} = h^{-1}(\tilde{\mathcal{I}}_{f(a')}) \times \{a'\}$ . One has the continuous map

$$\Gamma' \times_{\mathcal{I}'} A' \xrightarrow{(h, f)} \Gamma \times_{\mathcal{I}} A. \tag{4}$$

Therefore, if  $\mathcal{I}$  and  $\mathcal{I}'$  are locally maximal, Proposition 2.1 together with equation (4) implies the following: if  $(X, \pi, \varphi)$  and  $(X', \pi', \varphi')$  are the split spaces with isotropy groupoids  $\mathcal{I}$  and  $\mathcal{I}'$ , there is a unique map  $F = F_{h,f}: X' \rightarrow X$  such that  $F(\gamma'x') = h(\gamma')F(x')$ ,  $\pi \circ F = f \circ \pi'$  and  $\varphi \circ f = F \circ \varphi'$ . Let  $\eta$  be a split  $\Gamma$ -equivariant principal  $G$ -bundle over  $X$ . Using Theorem 3.2, one checks that  $\eta' := F^*\eta$  is a split  $\Gamma'$ -equivariant principal  $G$ -bundle over  $X'$  and that the isotropy representations  $\alpha' \in \text{Rep}^G(\mathcal{I}')$  and  $\alpha \in \text{Rep}^G(\mathcal{I})$  satisfy  $\alpha' = h^*\alpha$ , where  $h^*\alpha = \alpha \circ h$ . Therefore, one gets a commutative diagram

$$\begin{array}{ccc} \text{SBun}_{\Gamma}^G(X) & \xrightarrow{F^*} & \text{SBun}_{\Gamma'}^G(X') \\ \Phi \downarrow \approx & & \Phi \downarrow \approx \\ \text{Rep}^G(\mathcal{I}) & \xrightarrow{h^*} & \text{Rep}^G(\mathcal{I}'). \end{array} \quad (5)$$

As for the covariant functoriality in  $G$ , let  $\mu: G \rightarrow G'$  be a continuous homomorphism between compact Lie groups. If  $\eta: (E \rightarrow X)$  is a split  $\Gamma$ -equivariant principal  $G$ -bundle over  $X$ , one checks that  $\mu_*\eta: (E \times_{\mu} G' \rightarrow X)$  is a split  $\Gamma$ -equivariant principal  $G'$ -bundle with isotropy representation  $\mu_*\alpha = \mu \circ \alpha$ . One gets a commutative diagram

$$\begin{array}{ccc} \text{SBun}_{\Gamma}^G(X) & \xrightarrow{\mu_*} & \text{SBun}_{\Gamma}^{G'}(X) \\ \Phi \downarrow \approx & & \Phi \downarrow \approx \\ \text{Rep}^G(\mathcal{I}) & \xrightarrow{\mu_*} & \text{Rep}^{G'}(\mathcal{I}). \end{array} \quad (6)$$

In particular, let  $G = G' \times G''$  and let  $p'$  and  $p''$  be the two projections. Diagram (6) becomes

$$\begin{array}{ccc} \text{SBun}_{\Gamma}^G(X) & \xrightarrow{(\mu'_*, \mu''_*)} & \text{SBun}_{\Gamma}^{G'}(X) \times \text{SBun}_{\Gamma}^{G''}(X) \\ \Phi \downarrow \approx & & \Phi \times \Phi \downarrow \approx \\ \text{Rep}^G(\mathcal{I}) & \xrightarrow[\approx]{(\mu'_*, \mu''_*)} & \text{Rep}^{G'}(\mathcal{I}) \times \text{Rep}^{G''}(\mathcal{I}). \end{array} \quad (7)$$

Diagram (7) then shows that the map

$$(\mu'_*, \mu''_*): \text{SBun}_{\Gamma}^{G' \times G''}(X) \xrightarrow{\cong} \text{SBun}_{\Gamma}^{G'}(X) \times \text{SBun}_{\Gamma}^{G''}(X) \quad (8)$$

is a bijection.

#### 4. Cellular groupoids – examples

**§ 4A. Cellular groupoids.** Let  $A$  be a CW-complex filtered by its skeleta  $A^{(n)}$ . We denote by  $\Lambda = \Lambda(A)$  the set of cells of  $A$ . The dimension of a cell  $e \in \Lambda$  is denoted

by  $d(e)$  and we set  $\Lambda_n = \{e \in \Lambda \mid d(e) = n\}$ . For each  $e \in \Lambda$ , there exists a characteristic map  $\sigma_e: (\mathbb{D}^{d(e)}, \mathbb{S}^{d(e)-1}) \rightarrow (A^{(d(e))}, A^{(d(e)-1)})$ , and  $\sigma_e$  restricted to the interior of  $\mathbb{D}^{d(e)}$  is an embedding whose image is denoted by  $|e|$ . For  $a \in A$ , we denote by  $e(a) \in \Lambda$  the cell  $e$  of smallest dimension such that  $a \in \sigma(e)$ . The set  $\Lambda$  is partially ordered:  $f' \leq f$  if  $f'$  is a *face* of  $f$ , which means that there exists  $x \in \mathbb{S}^{d(f)-1}$  such that  $e(\sigma_{f'}(x)) = f$ .

Let  $\Gamma$  be a topological group and  $A$  be a CW-complex. A  $(\Gamma, A)$ -groupoid  $\mathcal{I}$  is called *cellular* if it is locally maximal and if  $\tilde{\mathcal{I}}_a = \tilde{\mathcal{I}}_b$  when  $e(a) = e(b)$ . We write  $\mathcal{I}^{(n)}$  for the restriction of  $\mathcal{I}$  over  $A^{(n)}$ . Recall that  $\tilde{\mathcal{I}}_a = \tilde{\mathcal{I}}_a \times \{a\}$  where  $\tilde{\mathcal{I}}_a \in \text{Gr}(\Gamma)$ , the poset of closed subgroups of  $\Gamma$ . One can then define a map  $\tilde{\mathcal{I}}: \Lambda(A) \rightarrow \text{Gr}(\Gamma)$  by  $\tilde{\mathcal{I}}(e) = \tilde{\mathcal{I}}_a$  for  $a$  with  $e(a) = e$ . The local maximality of  $\mathcal{I}$  implies that  $\tilde{\mathcal{I}}(e) \subset \tilde{\mathcal{I}}(f)$  when  $f \leq e$ . Thus,  $\tilde{\mathcal{I}}$  is a contravariant functor from the poset  $\Lambda(A)$  to the poset  $\text{Gr}(\Gamma)$ . A cellular groupoid is a combinatorial construction.

**Lemma 4.1.** *The correspondence  $\mathcal{I} \rightarrow \tilde{\mathcal{I}}$  is a bijection between the set of cellular groupoids whose object-space is  $A$  and the set of contravariant functors from  $\Lambda(A)$  to  $\text{Gr}(\Gamma)$ .*

*Proof.* The correspondence is clearly injective. For the surjectivity, let  $\mathcal{F}: e \mapsto \mathcal{F}_e$  be a contravariant functor from  $\Lambda(A)$  to  $\text{Gr}(\Gamma)$ . By induction on  $n$ , we shall construct  $\mathcal{I}^{(n)}$ , giving rise to a  $(\Gamma, A)$ -groupoid  $\mathcal{I}$ , with  $\tilde{\mathcal{I}} = \mathcal{F}$  and then check that  $\mathcal{I}$  is locally maximal. Define  $\mathcal{I}^{(0)} = \coprod_{v \in \Lambda_0(A)} \mathcal{F}_v \times \{v\}$ . Suppose that  $\mathcal{I}^{(n-1)}$  is constructed. The  $n$ -skeleton  $A^{(n)}$  of  $A$  is obtained as the quotient space

$$A^{(n)} = \left( \coprod_{e \in \Lambda_n(A)} \mathbb{D}_e \right) \coprod A^{(n-1)} / \{x \sim \sigma_e(x) \mid x \in \mathbb{S}_e\},$$

where  $(\mathbb{D}_e, \mathbb{S}_e)$  is a copy of  $(\mathbb{D}^n, \mathbb{S}^{n-1})$  and  $\sigma_e: \mathbb{D}_e \rightarrow A$  is a characteristic map for the cell  $e$ . We then define

$$\mathcal{I}^{(n)} = \left( \coprod_{e \in \Lambda_n(A)} (\mathcal{F}_e \times \mathbb{D}_e) \right) \coprod \mathcal{I}^{(n-1)} / \{(\gamma, x) \sim (\gamma, \sigma_e(x)) \mid x \in \mathbb{S}_e\}.$$

The equivalence relation  $\sim$  makes sense since, for  $x \in \mathbb{S}_e$ , one has  $\mathcal{F}_e \subset \tilde{\mathcal{I}}_{\mu_e(x)}^{(n-1)}$ . Clearly,  $\tilde{\mathcal{I}} = \mathcal{F}$ . Now each  $a \in A$  admits a fundamental system of open neighbourhoods  $U$  of  $a$  such that  $e(a) \leq e(u)$  for all  $u \in U$ . One can also require that  $U$  admits a homotopy  $\rho_t: U \rightarrow U$  such that  $\rho_0 = \text{id}$ ,  $\rho_1(U) = \{a\}$  and  $e(\rho_t(u)) = e(u)$  for  $t < 1$  (see [21], Theorem 6.1 and its proof, or proof of Lemma 4.4 below). Therefore,  $\mathcal{I}$  is locally maximal.  $\square$

The notation  $\tilde{\mathcal{I}}$  was introduced in order to state and prove Lemma 4.1 properly. In future occurrences we shall write  $\mathcal{I}(e)$  instead of  $\tilde{\mathcal{I}}(e)$ .

**Remark 4.2.** Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space over a CW-complex  $A$ , with a cellular isotropy groupoid  $\mathcal{I}$ . Then  $X$  is provided with a  $\Gamma$ -equivariant CW-complex structure

(see, e.g., [27], Chapter 2) with  $\Gamma$ -cells indexed by  $\Lambda(A)$ . If  $\sigma_e: \mathbb{D}^{d(e)} \rightarrow A$  is a characteristic map for  $e \in \Lambda(A)$ , then  $\tilde{\sigma}_e: \Gamma/\mathcal{I}(e) \times \mathbb{D}^{d(e)} \rightarrow X$ , defined by  $\tilde{\sigma}(\gamma, a) = \gamma \varphi(a)$ , is a characteristic map for the  $\Gamma$ -cell of  $X$  corresponding to  $e$ .

On the other hand, let  $X$  be a  $\Gamma$ -CW-complex and  $A = \Gamma/X$  be its orbit space with the induced CW-structure. Suppose that there exists a section  $\varphi: A \rightarrow X$  of the projection  $\pi: X \rightarrow A$ , so that the isotropy groupoid  $\mathcal{I}$  is weakly locally maximal. Then  $\mathcal{I}$  is cellular, since  $\mathcal{I}_a$  is constant on the interior of each cell. We call  $X$  a *split  $\Gamma$ -CW-complex* over  $A$ .

However, one has the following example of a split  $\Gamma$ -space over a CW-complex admitting no splitting for which the isotropy groupoid is cellular. Let  $X = ([0, 1] \times S^2)/\{(1, x) \sim (0, -x)\}$ , the mapping cylinder of the antipodal map of  $S^2$ , endowed with the natural action of  $\Gamma = \text{SO}(3)$ . Then  $A = [0, 1]/\{0 \sim 1\} \approx S^1$ . Any splitting is of the form  $\varphi(t) = (t, f(t))$  with  $\lim_{t \rightarrow 0} f(t) = -\lim_{t \rightarrow 1} f(t)$ . Thus,  $f(t)$  is not constant and  $\mathcal{I}$  is not weakly locally maximal. Observe that  $X$  is a smooth closed 3-manifold and that the  $\text{SO}(3)$ -action is smooth with cohomogeneity one.

For cellular groupoids we have a stronger version of Proposition 2.1, which applies to any topological group  $\Gamma$ .

**Proposition 4.3** (Reconstruction II). *Let  $\Gamma$  be a topological group, and  $A$  be a CW-complex. Given a cellular  $(\Gamma, A)$ -groupoid  $\mathcal{I}$ , there is a unique split  $\Gamma$ -CW-complex over  $A$  with isotropy groupoid  $\mathcal{I}$ .*

*Proof.* The space  $Y_{\mathcal{I}}$  is a split  $\Gamma$ -CW-complex over  $A$ . Suppose that  $(X, \pi, \varphi)$  is another split  $\Gamma$ -CW-complex with isotropy groupoid  $\mathcal{I}$ . As in the proof of Proposition 2.1, the map  $\tilde{F}: \Gamma \times A \rightarrow X$  defined by  $\tilde{F}(\gamma, a) = \gamma \cdot \varphi(a)$  descends to give a continuous  $\Gamma$ -equivariant bijection  $F: Y_{\mathcal{I}} \rightarrow X$ . For each cell  $e \in \Lambda_n(Y)$  with characteristic map  $\sigma_e: \mathbb{D}^n \rightarrow A$ , there is a commutative diagram

$$\begin{array}{ccc} \Gamma \times_{\mathcal{I}(e)} \mathbb{D}^n & \xrightarrow{\text{id}} & \Gamma \times_{\mathcal{I}(e)} \mathbb{D}^n \\ \sigma_e^{Y_{\mathcal{I}}} \downarrow & & \downarrow \sigma_e^Y \\ X & \xrightarrow{F} & Y. \end{array}$$

Therefore  $F$  is an open map and hence a homeomorphism. □

Let  $\Gamma$  be a topological group and  $A$  be a CW-complex. A  $(\Gamma, A)$ -groupoid  $\mathcal{I}$  such that  $\mathcal{I}_a$  is a compact Lie group for all  $a \in A$  is called *proper*. When  $\Gamma$  is itself a Lie group, this is equivalent to saying that the  $\Gamma$ -action on the corresponding split  $\Gamma$ -CW-complex with isotropy groupoid  $\mathcal{I}$  is proper; see [19], Theorem 1.23. We have a classification theorem for equivariant bundles over split  $\Gamma$ -spaces with proper isotropy groupoids in Theorem 4.5. First we give a version of Lemma 3.1.

**Lemma 4.4.** *Let  $\Gamma$  be a topological group, and  $A$  be a CW-complex. Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid. Then any continuous representation of  $\mathcal{I}$  to a compact Lie group  $G$  is weakly locally maximal.*

*Proof.* Let  $\alpha: \mathcal{I} \rightarrow G$  be a continuous representation. Let  $a \in A$ . We shall construct a pair  $(U, g)$ , where  $U$  is an open set of  $A$ , such that  $\mathcal{I}_u$  is a subgroup of  $\mathcal{I}_a$  for each  $u \in U$ , and  $g: U \rightarrow G$  is a continuous map satisfying  $\alpha_u(\gamma) = g(u)\alpha_a(\gamma)g(u)^{-1}$  for all  $u \in U$  and all  $\gamma \in \mathcal{I}_u$ . Call the pair  $(U, g)$  an  $a$ -straightening of  $\alpha$  in  $A$ . The final open set  $U$  will be a neighbourhood of  $a$ , but the definition of an  $a$ -straightening does not use that  $a \in U$ , just that the element  $g(a) \in G$  is defined. An  $a$ -straightening is equivalent to the data of a sequence  $(U^{(d)}, g_d)$  of  $a$ -straightenings of  $\alpha$  in  $A^{(d)}$ , such that  $U^{(d+1)} \cap A^{(d)} = U^{(d)}$  and  $g_{d+1}|_{U^{(d)}} = g_d$ .

We construct  $(U^{(d)}, g_d)$  by induction on  $d$ , setting  $U_d = \emptyset$  if  $d < d(e(a))$ . If  $d(e(a)) = 0$ , then we set  $U^{(0)} = \{a\}$  and  $g_0(a) = 1$ . If  $d(e(a)) > 0$ , then there exists a neighbourhood  $U^{d(e(a))}$  of  $a$  in  $e(a)$  with a pointed homeomorphism  $(U^{d(e(a))}, a) \xrightarrow{\cong} ([-1, 1]^{d(e(a))}, 0)$ . The existence of  $g_{d(e(a))}$  is guaranteed by Lemma 1.2. Suppose that an  $a$ -straightening  $(U^{(d)}, g_d)$  of  $\alpha$  in  $A^{(d)}$  is constructed, with  $d \geq d(e(a))$  and  $a \in U_d$ . For  $e \in \Lambda_{d+1}(A)$ , let  $\sigma_e: (\mathbb{D}_e^{d+1}, \mathbb{S}_e^d) \rightarrow (A^{(d+1)}, A^{(d)})$  be a characteristic map for the cell  $e$ . Let  $V_e$  be the open set of  $\mathbb{S}_e^d$  defined by  $V_e = \sigma_e^{-1}(U^{(d)})$ . Let  $W_e$  be the open set of  $\mathbb{D}_e^{d+1}$  defined by  $W_e = \{tx \mid x \in V_e \text{ and } t \in (0, 1]\}$ . The correspondence  $u \mapsto \alpha_{\sigma_e(u)} \in \text{Hom}(\mathcal{I}(e), G)$  is a continuous representation  $\alpha_e$  of the  $(\Gamma, \mathbb{D}_e^{d+1})$ -groupoid  $\mathcal{I}(e) \times \mathbb{D}_e^{d+1}$ . The pair  $(V_e, g_d \circ \sigma_e)$  is a  $a$ -straightening of  $\alpha_e$  in  $\mathbb{S}_e^d$ . As  $W_e$  is homeomorphic to  $V_e \times [0, 1]$ , this  $a$ -straightening extends to a  $a$ -straightening  $(W_e, g_e)$  of  $\alpha_e$  in  $\mathbb{D}_e^{d+1}$ . The family  $g_e$  defines a map  $g_{d+1}: U^{(n+1)} \rightarrow G$ , where  $U^{(n+1)} = \bigcup_{e \in \Lambda_{n+1}(A)} W_e$ , giving rise to the  $a$ -straightening of  $\alpha$  in  $A^{(n+1)}$ .  $\square$

By Lemma 4.4, the isotropy representation  $\Phi: \text{SBun}_\Gamma^G(X) \rightarrow \text{Rep}^G(\mathcal{I})$  is defined, as in Section §3B. The classification theorem for split bundles over a split  $\Gamma$ -CW-complex with proper isotropy groupoid takes the following form.

**Theorem 4.5** (Classification II). *Let  $\Gamma$  be a Lie group, and  $A$  be a CW-complex. Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid. Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -CW-complex over  $A$  with isotropy groupoid  $\mathcal{I}$ . Then, for any compact Lie group  $G$ , the isotropy representation  $\Phi: \text{SBun}_\Gamma^G(X) \rightarrow \text{Rep}^G(\mathcal{I})$  is a bijection. Moreover, any split bundle over  $X$  is numerable.*

*Proof.* The proof of Theorem 4.5 is the same as that of Theorem 3.2, using Lemma 4.4 instead of Lemma 3.1 and Proposition 4.3 instead of Proposition 2.1. Being a CW-complex,  $A$  is paracompact, so each open cover is numerable. The last assertion of Theorem 4.5 comes from Remark 3.3.  $\square$

**Remark 4.6.** The assumption that  $\Gamma$  is a Lie group is only used to ensure that the quotient projection  $q_a: \Gamma \rightarrow \Gamma/\mathcal{I}_a$  is a (numerable) principal bundle. If we do



not care about numerability, the existence of local cross-sections of  $q_a$  holds more generally (see [22] and [23]).

As in Proposition 3.5, Theorem 4.5 extends to a classification of all  $\Gamma$ -equivariant  $G$ -bundles if  $G$  is abelian. More precisely:

**Proposition 4.7.** *Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid for a Lie group  $\Gamma$ . Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -CW-complex over  $A$  with isotropy groupoid  $\mathcal{I}$ . Then, for any compact abelian Lie group  $G$ , one has an isomorphism of abelian groups*

$$(\Phi, \varphi^*): \text{Bun}_\Gamma^G(X) \xrightarrow{\cong} \text{Rep}^G(\mathcal{I}) \times \text{Bun}^G(A).$$

Moreover, any principal  $\Gamma$ -equivariant  $G$ -bundle over  $X$  is numerable.

*Proof.* The proof Proposition 4.7 is the same as that of Proposition 3.5, using Theorem 4.5 instead of Theorem 3.2. For the numerability, observe that the inverse of the bijection  $(\Phi, \varphi^*)$  is given by  $(\Phi, \varphi^*)^{-1}(\xi, \eta) = \xi \otimes \pi^*\eta$ . By Theorem 3.2,  $\xi$  is numerable. Since  $A$  is a CW-complex,  $\eta$  is numerable and thus  $(\Phi, \varphi^*)^{-1}(\xi, \eta)$  is numerable. Hence, any  $\Gamma$ -equivariant principal  $G$ -bundle over  $X$  is numerable.  $\square$

## § 4B. Examples

**4.8.** Generalised toric manifolds of real dimension  $2m$ , in the sense of [6], are split  $\mathbb{T}$ -spaces where  $\mathbb{T}$  is an  $m$ -dimensional torus. The orbit space  $A$  is a simple polytope and the section  $\varphi$  is given in [6], Lemma 1.4. This includes symplectic toric manifolds, see, e.g., [11], where  $\pi: X \rightarrow A$  is the moment map and  $A \subset \text{Lie}(\mathbb{T})^*$  the moment polytope. Our reconstruction Proposition 2.1 is the topological content of Delzant's theorem [11], Theorem 1.8, or [6], Proposition 1.7.

**4.9.** When  $\Gamma$  is discrete, the “strata preserving actions with strict fundamental domain” of [4], Chapter II.12, are generalizations of split  $\Gamma$ -spaces with a cellular isotropy groupoid. Several examples are given in [4], Chapter II.12.9.

Several of the examples below involve  $(\Gamma, A)$ -groupoids where  $A = \Delta^m$  is the standard  $m$ -simplex in  $\mathbb{R}^{m+1}$ :

$$\Delta^m = \{(t_0, \dots, t_m) \in \mathbb{R}^{m+1} \mid t_i \geq 0 \text{ and } \sum_{i=0}^m t_i = 1\}.$$

We use the standard simplicial structure on  $\Delta^m$ , with  $\Lambda_k(\Delta^m)$  being the set of all subsets of  $\{0, \dots, m\}$  containing  $k + 1$  elements. When  $m = 1, 2$ , we use special notations illustrated by the following pictures.



**4.10.** Let  $\mathbb{T} = (S^1)^{m+1}$ . We define a cellular  $(\mathbb{T}, A)$ -groupoid  $\mathcal{I}$  with  $A = \Delta^m$  by

$$\mathcal{I}(e) = \{(\gamma_0, \dots, \gamma_m) \mid \gamma_i = 1 \text{ if } i \in e\}.$$

A model  $(X, \pi, \varphi)$  for the split  $\mathbb{T}$ -space with isotropy groupoid  $\mathcal{I}$  is given by  $X = S^{2m+1} \subset \mathbb{C}^{m+1}$  with the  $\mathbb{T}$ -action  $(\gamma_0, \dots, \gamma_m) \cdot (z_0, \dots, z_m) = (\gamma_0 z_0, \dots, \gamma_m z_m)$ . The map  $\pi$  and  $\varphi$  may be chosen as

$$\begin{aligned} \pi(z_0, \dots, z_m) &= (|z_0|^2, \dots, |z_m|^2), \\ \varphi(t_0, \dots, t_m) &= (\sqrt{t_0}, \dots, \sqrt{t_m}). \end{aligned} \tag{9}$$

More generally, let  $\mathbb{T}$  be any torus and let  $\chi_0, \dots, \chi_m \in \text{Hom}(\mathbb{T}, S^1)$ . Define a cellular  $(\mathbb{T}, A)$ -groupoid  $\mathcal{I}$  with  $A = \Delta^m$  by  $\mathcal{I}(e) = \bigcap_{j \in e} \ker \chi_j$ . A model for the split  $\mathbb{T}$ -space with isotropy groupoid  $\mathcal{I}$  is again given by  $(S^{2m+1}, \pi, \varphi)$ , where  $\pi$  and  $\varphi$  are defined by equations (9) and where the  $\mathbb{T}$ -action on  $S^{2m+1}$  is

$$\gamma \cdot (z_0, \dots, z_m) = (\chi_0(\gamma)z_0, \dots, \chi_m(\gamma)z_m).$$

**4.11.** Let  $\mathcal{I}$  be a  $(\Gamma, A)$ -groupoid and let  $\Gamma_0$  be a closed subgroup of  $\Gamma$ . A  $(\Gamma, A)$ -groupoid  $\mathcal{I}_0$  is then defined on  $A$  by  $\mathcal{I}_0(e)$  be the subgroup generated by  $\mathcal{I}(e) \cup \Gamma_0$ . If  $(X, \pi, \varphi)$  is the split  $\Gamma$ -space over  $A$  with isotropy groupoid  $\mathcal{I}$ , then that with isotropy groupoid  $\mathcal{I}(\Gamma_0)$  is  $(\Gamma_0 \backslash X, \pi^0, \varphi^0)$ , where  $\pi^0$  is induced by  $\pi$  and  $\varphi^0$  is  $\varphi$  composed with the projection  $X \rightarrow \Gamma_0 \backslash X$ . For instance, if we take  $\Gamma_0$  to be the diagonal  $S^1$  in Example 4.10, we get a split  $(S^1)^{m+1}$ -structures on the complex projective space  $\mathbb{C}P^m$ .

**4.12.** Let  $\Gamma = \text{SO}(n+1)$ . We see  $\text{SO}(n)$  as the subgroup of  $\Gamma$  leaving the last coordinate fixed. Consider the  $(\Gamma, A)$ -groupoid with  $A = [-1, 1]$ , defined by  $\mathcal{I}_{\pm 1} = \Gamma$  and  $\mathcal{I}_{(-1,1)} = \text{SO}(n)$ . The split  $\Gamma$ -space with isotropy groupoid  $\mathcal{I}$  is  $(S^n, \pi, \varphi)$  with  $\pi(x_1, \dots, x_{n+1}) = x_{n+1}$  ( $\varphi$  may be defined using a meridian). The classification of split  $\Gamma$ -equivariant  $G$ -bundles over  $S^n$  has been studied in [13].

**4.13.** Let  $X$  be a  $\Gamma$ -CW-complex  $X$  so that the orbit space, with its induced CW-structure, is a segment (say  $\Delta^1$ ). This is one type of *cohomogeneity one action*. There are then subgroups  $\Gamma_0, \Gamma_1, \Gamma_{01}$  of  $\Gamma$  so that  $X$  is  $\Gamma$ -equivariantly homeomorphic to  $\Gamma/\Gamma_{01} \times [0, 1]$  glued to  $\Gamma/\Gamma_0 \times \{0\}$  and  $\Gamma/\Gamma_1 \times \{1\}$  by equivariant maps. Sending  $t$  to  $([e], t) \in \Gamma/\Gamma_{01} \times [0, 1]$  produces a splitting  $\varphi$  with a cellular isotropy groupoid  $\mathcal{I}$ ,

satisfying  $\mathcal{I}_{01} = \Gamma_{01}$ ,  $\mathcal{I}_0 = \Gamma_0$  and  $\mathcal{I}_1 = \Gamma_1$ . The space  $X$  is then a split  $\Gamma$ -space with isotropy groupoid  $\mathcal{I}$ . If  $\Gamma$  is a compact Lie group, one checks that  $X$  has a natural smooth manifold structure for which the action is smooth. For more details and references on cohomogeneity one action, see [13], §8, where  $\Gamma$ -equivariant  $G$ -bundles over such  $\Gamma$ -spaces are classified (they are all split).

**4.14.** Let  $\mathbb{T}$  be any torus and let  $\chi$  be a non-trivial element in  $\text{Hom}(\mathbb{T}, S^1)$ . Define a cellular  $(\mathbb{T}, A)$ -groupoid  $\mathcal{I}$  with  $A = \Delta^1$  by  $\mathcal{I}_0 = \mathcal{I}_1 = \mathbb{T}$  and  $\mathcal{I}_{01} = \ker \chi$ . The split  $\mathbb{T}$ -space with isotropy groupoid  $\mathcal{I}$  is  $(\mathbb{C}P^1, \pi, \varphi)$ , where  $\pi([z_0: z_1]) = (|z_0|^2, |z_1|^2)$ ,  $\varphi(t_0, t_1) = [\sqrt{t_0}: \sqrt{t_1}]$  and the  $\mathbb{T}$ -action is given by  $\gamma[x_0: x_1] = [\chi(\gamma)x_0: x_1]$ . We denote this split  $\mathbb{T}$ -space by  $\mathbb{C}P^1(\chi)$ .

### 5. Cellular representations – computations of $\text{Rep}^G(\mathcal{I})$

**§ 5A. Cellular representations.** Let  $\mathcal{I}$  be a  $(\Gamma, A)$ -groupoid. A representation  $\beta: \mathcal{I} \rightarrow G$  is called *cellular* if  $\beta_a = \beta_b$  when  $e(a) = e(b)$ . For each  $e \in \Lambda(A)$ , this thus defines  $\beta_e \in \text{Hom}(\mathcal{I}(e), G)$ , with the face compatibility conditions  $\beta_e = \beta_f|_{\mathcal{I}(e)}$  whenever  $f \leq e$ . Two cellular representations  $\alpha$  and  $\beta$  are called *conjugate* if there exists  $g \in G$  such that  $\beta(\gamma) = g^{-1}\alpha(\gamma)g$  for all  $\gamma \in \mathcal{I}_a$  and all  $a \in A$ . Denote by  $\text{Rep}_{\text{cell}}^G(\mathcal{I})$  the set of conjugacy classes of cellular representations of  $\mathcal{I}$  into  $G$ .

To a cellular representation  $\alpha: \mathcal{I} \rightarrow G$  and a cell  $e$  of  $A$ , one can associate its conjugacy class  $[\alpha_e] \in \overline{\text{Hom}}(\mathcal{I}(e), G)$ . This gives rise to a map

$$\kappa: \text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \prod_{e \in \Lambda(A)} \overline{\text{Hom}}(\mathcal{I}(e), G).$$

If an element  $(b_e)$  of this product is in the image of  $\kappa$ , it must satisfy the face compatibility conditions, that is the equation  $b_e = b_f|_{\mathcal{I}(e)}$  holds in  $\overline{\text{Hom}}(\mathcal{I}(e), G)$  whenever  $f \leq e$ . We then define

$$\overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I}) = \{(b_e) \in \prod_{e \in \Lambda(A)} \overline{\text{Hom}}(\mathcal{I}(e), G) \mid b_e = b_f|_{\mathcal{I}(e)} \text{ if } f \leq e\}$$

and see  $\kappa$  as a map  $\kappa: \text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$ . When  $\mathcal{I}$  is a proper  $(\Gamma, A)$ -groupoid, the map  $\kappa$  sits in a commutative diagram

$$\begin{array}{ccc} \text{Rep}_{\text{cell}}^G(\mathcal{I}) & \xrightarrow{j} & \text{Rep}^G(\mathcal{I}) \\ & \searrow \kappa & \swarrow \iota \\ & \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I}). & \end{array} \tag{10}$$

The map  $j$  is obvious, since a cellular representation is a representation which is clearly weakly locally maximal. To define  $\iota(\beta)_e$  for  $e \in \Lambda(A)$ , we choose  $a \in A$

with  $e(a) = e$  and set  $\iota(\beta)_e = [\beta_a]$ . Since cells are connected,  $\iota$  is well defined by Lemma 1.1. Although none of these maps is either surjective or injective in general, diagram (10) is the source of all our information about  $\text{Rep}^G(\mathcal{I})$  so far.

One useful method for computing  $\text{Rep}_{\text{cell}}^G(\mathcal{I})$  and  $\overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  is to restrict representations of  $\mathcal{I}$  to skeleta of  $A$ . This yields restriction maps  $\text{res}_k: \text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \text{Rep}_{\text{cell}}^G(\mathcal{I}^{(k)})$  and  $\overline{\text{res}}_k: \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I}) \rightarrow \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I}^{(k)})$ . Recall that a CW-complex  $A$  is *regular* if each cell  $e$  admits a characteristic map  $\sigma_e: \mathbb{D}^{d(e)} \rightarrow A$  that is an embedding, sending  $\mathbb{S}^{d(e)-1}$  onto a subcomplex of  $A^{d(e)-1}$ . We set  $\|e\| = \sigma_e(\mathbb{D}^{d(e)})$ , the closure of  $|e|$ . To simplify the notations, we write  $\partial e$  instead of  $\partial\|e\|$  for the boundary of  $\|e\|$ .

**Proposition 5.1.** *Let  $\mathcal{I}$  be a  $(\Gamma, A)$ -groupoid for  $\Gamma$  a topological group. Assume that  $A$  is a regular CW-complex. Then, for any topological group  $G$ , one has*

- (a)  $\text{res}_0: \text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \text{Rep}_{\text{cell}}^G(\mathcal{I}^{(0)})$  and  $\overline{\text{res}}_0: \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I}) \rightarrow \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I}^{(0)})$  are injective;
- (b)  $\text{res}_1: \text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \text{Rep}_{\text{cell}}^G(\mathcal{I}^{(1)})$  and  $\overline{\text{res}}_1: \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I}) \rightarrow \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I}^{(1)})$  are bijective.

*Proof.* Let  $\alpha \in \text{Rep}_{\text{cell}}^G(\mathcal{I})$  (the proof for  $\overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  is the same). As  $A$  is regular, each cell of  $A$  has a face which is a vertex. Therefore,  $\text{res}_0(\alpha)$  determines  $\alpha$  which proves (a) and the injectivity part of (b).

For the surjectivity in (b), it is enough to prove that the restriction map  $\text{Rep}_{\text{cell}}^G(\mathcal{I}^{(n)}) \rightarrow \text{Rep}_{\text{cell}}^G(\mathcal{I}^{(n-1)})$  is onto when  $n \geq 2$ . Let  $\beta: \mathcal{I}^{(n-1)} \rightarrow G$  be a cellular representation. We must extend  $\beta$  to  $\hat{\beta} = \beta \cup \{\beta_e\} \in \text{Rep}^G(\mathcal{I})$ , which may be done for each  $n$ -cell independently. For each  $e \in \Lambda_n(A)$ , choose  $f \in \Lambda_{n-1}(A)$  with  $f \leq e$  and define  $\beta_e = \beta_f | \mathcal{I}(e)$ . We must check that  $\beta_e$  does not depend on the choice of  $f$ . Let  $f'$  be another choice. As  $n \geq 2$ , there exists a continuous path  $c(t)$  in the frontier of  $|e|$  joining  $a \in |f|$  to  $a \in |f'|$ . By the face compatibility condition,  $\beta_a | \mathcal{I}_{c(t)}$  is constant, thus  $\beta_f | \mathcal{I}(e) = \beta_{f'} | \mathcal{I}(e)$ .  $\square$

**Corollary 5.2.** *Suppose that the hypotheses of Proposition 5.1 hold true. Let  $b \in \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$ . If  $\overline{\text{res}}_1(b) \in \kappa(\text{Rep}_{\text{cell}}^G(\mathcal{I}^{(1)}))$ , then  $b \in \kappa(\text{Rep}_{\text{cell}}^G(\mathcal{I}))$ .*

*Proof.* This is a consequence of the commutative diagram

$$\begin{array}{ccc}
 \text{Rep}_{\text{cell}}^G(\mathcal{I}) & \xrightarrow[\approx]{\text{res}_1} & \text{Rep}_{\text{cell}}^G(\mathcal{I}^{(1)}) \\
 \downarrow \kappa & & \downarrow \kappa \\
 \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I}) & \xrightarrow[\approx]{\overline{\text{res}}_1} & \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I}^{(1)}),
 \end{array} \tag{11}$$

the bijectivity of the horizontal arrows coming from Proposition 5.1.  $\square$

### § 5B. Case where $G$ is abelian

**Proposition 5.3.** *Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid for a topological group  $\Gamma$ . Let  $G$  be a compact abelian Lie group. Then the three maps  $\iota, j, \kappa$  of diagram (10) are bijective.*

*Proof.* The map  $\kappa$  is bijective since conjugation has no effect if  $G$  is abelian. It is then enough to prove that  $j$  is surjective. Let  $\beta \in \text{Rep}^G(\mathcal{I})$ . As in the construction of  $\iota$ , one shows that  $\beta(\zeta) = \beta(\zeta')$  if  $e(\pi_2(\zeta)) = e(\pi_2(\zeta'))$ , which is equivalent to  $\beta$  being in the image of  $j$ .  $\square$

If  $\Gamma$  is a Lie group, Proposition 5.3 together with the classification Theorem 4.5 gives a bijection  $\text{SBun}_\Gamma^G(X) \approx \text{Rep}_{\text{cell}}^G(\mathcal{I})$ . Using Proposition 5.1 and the functorial property in diagram (5) (which holds true in the framework of Theorem 4.5), this also shows that, for  $G$  abelian, the restriction maps  $\text{SBun}_\Gamma^G(X) \rightarrow \text{SBun}_\Gamma^G(X^{(0)})$  and  $\text{SBun}_\Gamma^G(X) \rightarrow \text{SBun}_\Gamma^G(X^{(1)})$  are respectively injective and bijective, when  $A$  is a regular CW-complex.

By Lemma 1.3, one has  $G \xrightarrow{\approx} G_0 \times \pi_0(G)$ , where  $G_0$  is the identity component of the unit element. Therefore,  $\text{Rep}_{\text{cell}}^G(\mathcal{I}) \approx \text{Rep}_{\text{cell}}^{\pi_0(G)}(\mathcal{I}) \times \text{Rep}_{\text{cell}}^{G_0}(\mathcal{I})$  (the same decomposition holds for  $\text{SBun}_\Gamma^G(X)$  by equation (8), again true in the context of Theorem 4.5). The group  $G_0$  is isomorphic to a product of circles, so  $\text{Rep}_{\text{cell}}^{G_0}(\mathcal{I})$  is a product of copies of  $\text{Rep}_{\text{cell}}^S(\mathcal{I})$ . We shall now study the latter.

**§ 5C.  $\text{Rep}_{\text{cell}}^G s(\mathcal{I})$  for  $\mathcal{I}$  a toric groupoid.** Let  $\mathbb{T}$  be a torus. A cellular  $(\mathbb{T}, A)$ -groupoid  $\mathcal{I}$  is called *0-toric* if  $\mathcal{I}_v = \mathbb{T}$  for all  $v \in \Lambda_0 = \Lambda_0(A)$ . It is called *1-toric* if it is 0-toric and if, for each  $e \in \Lambda_1 = \Lambda_1(A)$ ,  $\mathcal{I}(e)$  is a codimension 1 subtorus of  $\mathbb{T}$ . There is then  $\chi_e \in \text{Hom}(\mathbb{T}, S^1)$  with  $\ker \chi_e = \mathcal{I}(e)$ . The part of  $X$  above the closure  $\|e\|$  of  $|e|$  is a  $\mathbb{T}$ -space isomorphic to  $\mathbb{C}P^1(\chi_e)$  of Example 4.14. The split  $\mathbb{T}$ -space with isotropy groupoid  $\mathcal{I}^{(1)}$  is then a graph of  $\mathbb{C}P^1(\chi)$ 's. Such a space  $\mathbb{T}$ -space  $X$  is also called a *GKM-space*, as this property was first studied by M. Goresky, R. Kottwitz and R. MacPherson in [10].

Let  $\mathfrak{t}$  be the Lie algebra of  $\mathbb{T}$  and let  $\mathfrak{l} = \ker(\exp: \mathfrak{t} \rightarrow \mathbb{T})$ . Moreover, let  $\mathfrak{l}^* = \{w \in \mathfrak{t}^* \mid w(\mathfrak{l}) \subset \mathbb{Z}\}$  (the dual lattice). Consider  $S^1$  as  $\mathbb{R}/\mathbb{Z}$ . The correspondence which assigns to  $\alpha \in \text{Hom}(\mathbb{T}, S^1)$  its differential at the unit element of  $\mathbb{T}$  (the *weight* of  $\alpha$ ) produces an isomorphism between  $\text{Hom}(\mathbb{T}, S^1)$  and the additive group  $\mathfrak{l}^*$ . We shall thus identify  $\text{Hom}(\mathbb{T}, S^1)$  with  $\mathfrak{l}^*$ .

Let  $\mathcal{I}$  be a 1-toric cellular  $(\mathbb{T}, A)$ -groupoid with  $A$  a regular complex. By Proposition 5.1,  $\text{Rep}_{\text{cell}}^G s(\mathcal{I})$  injects into  $\text{Rep}_{\text{cell}}^G s(\mathcal{I}^{(0)})$ , which is the direct product of character groups

$$\text{Rep}_{\text{cell}}^G s(\mathcal{I}) \subset \prod_{v \in \Lambda_0} \text{Hom}(\mathbb{T}, S^1) = \prod_{v \in \Lambda_0} \mathfrak{l}^*. \quad (12)$$

Let us orient each edge  $e$ ; this determines an ordering  $\partial_-e, \partial_+e$  of the two vertices of  $e$ . The character  $\chi_e$  will also be seen in  $\mathfrak{l}^*$ . A family  $(a_v)_{v \in \Lambda_0}$  is said to satisfy

the *GKM-conditions* if, for each  $e \in \Lambda_1$ , the difference  $a_{\partial_+e} - a_{\partial_-e}$  is a multiple of  $\chi_e$ . These conditions, considered in [10], are also discussed in Proposition 6.9 and Remark 6.10.

**Proposition 5.4.** *Let  $\mathcal{I}$  be a 1-toric cellular  $(\mathbb{T}, A)$ -groupoid. Then the image of  $\text{Rep}_{\text{cell}}^G s(\mathcal{I})$  in  $\prod_{v \in \Lambda_0} \mathbb{I}^*$  is the set of families  $(a_v)_{v \in \Lambda_0}$  satisfying the GKM-condition.*

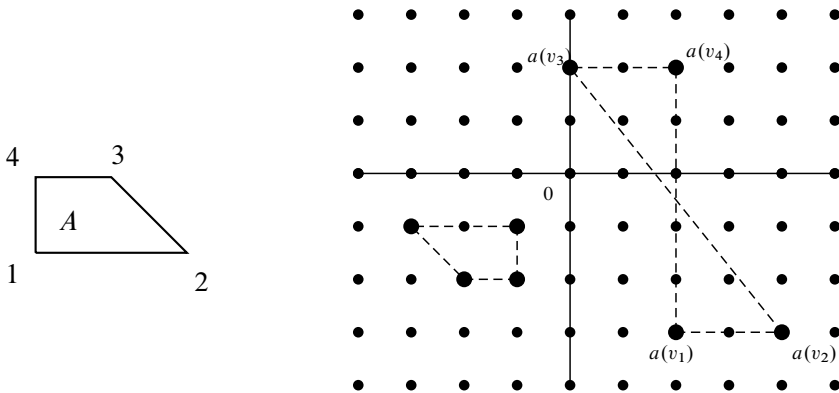
*Proof.* By Proposition 5.1, it is enough to show that this condition characterises the image of  $\text{Rep}^G(\mathcal{I}^{(1)})$  in  $\text{Rep}^G(\mathcal{I}^{(0)})$ . Denote by  $\alpha_v \in \text{Hom}(\mathbb{T}, S^1)$  the element with weight  $a_v \in \mathbb{I}^*$ . For  $e \in \Lambda_1$  the three following conditions are equivalent:

- (a) the difference  $a_{\partial_+e} - a_{\partial_-e}$  is a multiple of  $\chi_e$ .
- (b)  $\mathcal{I}(e) \subset \ker \alpha_{\partial_+e} \alpha_{\partial_-e}^{-1}$ .
- (c)  $\alpha_{\partial_+e}|_{\mathcal{I}(e)} = \alpha_{\partial_-e}|_{\mathcal{I}(e)}$ .

The equivalence between (a) and (b) comes from  $\mathcal{I}(e)$  being of codimension 1 in  $\mathbb{T}$ . This proves Proposition 5.4. □

**Example 5.5.** Let  $X$  be a symplectic toric manifold of dimension  $2n$ . It is a split  $\mathbb{T}^n$ -space, with  $\pi: X \rightarrow A \subset \mathfrak{t}^*$  being the moment map, and the isotropy groupoid  $\mathcal{I}$  is 1-toric. The moment polytope  $A$  is a  $n$  dimensional convex polytope of  $\mathfrak{t}^*$ . It is known that each edge  $e$  of  $A$  is parallel to  $\chi_e$  (see, e.g., [3], §4.2.4). By Proposition 5.4,  $\text{Rep}_{\text{cell}}^G s(\mathcal{I})$  may be visualised as the set of affine maps  $\alpha: A \rightarrow \mathfrak{t}^*$  such that  $\alpha(\lambda_0(A)) \subset \mathbb{I}^*$  and  $\alpha(|e|)$  parallel to  $\chi_e$  for each  $e \in \lambda_1(A)$ .

The left figure below shows a 2-dimensional moment polytope for a toric manifold, a Hirzebruch surface diffeomorphic to  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ . The torus  $\mathbb{T}$  is  $S^1 \times S^1$ ,  $\mathcal{I}_{12} = \mathcal{I}_{34} = \{1\} \times S^1$ ,  $\mathcal{I}_{14} = S^1 \times \{1\}$ ,  $\mathcal{I}_{23}$  is the diagonal subgroup and the isotropy group for the 2-cell is trivial. The right figure visualises two elements of  $\text{Rep}^G(\mathcal{I})$ .



Let  $\alpha \in \text{Rep}_{\text{cell}}^G s(\mathcal{I})$ . Let  $X$  be the split  $\mathbb{T}$ -space with isotropy groupoid  $\mathcal{I}$ . Let  $\eta$  be a split  $\mathbb{T}$ -equivariant  $S^1$ -principal bundle over  $X$ , with isotropy representation  $\alpha$ .

Let  $e \in \Lambda_1$ . In Proposition 5.4, the integer  $k_e \in \mathbb{Z}$  such that  $a_{\partial_+ e} - a_{\partial_- e} = k_e \chi_e$  is related to the Euler number of  $\eta$  restricted to  $X_e$ , the part of  $X$  above the closure  $\|e\|$  of  $|e|$ , which is homeomorphic to  $\mathbb{C}P^1$ . Choose a generator  $[X_e]$  of  $H_2(X_e; \mathbb{Z})$ . Let  $\varepsilon \in H^2(X_e; \mathbb{Z})$  be the Euler class of  $\eta$  restricted to  $X_e$ .

**Proposition 5.6.**  $\varepsilon([X_e]) = \pm k_e$ .

*Proof.* It is enough to consider the case where  $X = X_e = \mathbb{C}P^1(\chi)$  for  $\chi \in \text{Hom}(\mathbb{T}, S^1)$ . The quotient space  $A$  is then a segment, with two 0-cells 0 and 1 and a 1-cell  $e$ , and we identify  $A$  with  $[0, 1]$ . One has  $\mathcal{I}_0 = \mathcal{I}_1 = \mathbb{T}$  and  $\mathcal{I}(e) = \ker \chi$ . The elements  $\alpha_0, \alpha_1 \in \text{Hom}(\mathbb{T}; S^1)$  have weights  $a_0, a_1 \in \mathfrak{I}^*$ . The bundle  $\eta$  may then be identified with the bundle  $E_\alpha \xrightarrow{\pi} Y_{\mathcal{I}}$  of the proof of Theorem 3.2.

Let  $U_0 = A - \{1\}$  and  $U_1 = A - \{0\}$  and call  $\mathcal{W}_0$  and  $\mathcal{W}_1$  the open sets of  $X$  above  $U_0$  and  $U_1$ . One has local sections  $\sigma_i: \mathcal{W}_i \rightarrow E_\alpha$  of  $\pi$  defined by  $\sigma_i([\gamma, u]) = [\gamma, u, \alpha_i(\gamma)]$ . Let  $s \in \text{Hom}(S^1, \mathbb{T})$  such that  $\chi \circ s: S^1 \rightarrow \mathbb{T}/\mathcal{I}_e$  is surjective. Define  $\hat{s}: S^1 \rightarrow Y_{\mathcal{I}}$  by  $\hat{s}(\delta) = [s(\delta), 1/2]$ . One has

$$\sigma_1(\hat{s}(\delta)) = \sigma_0(\hat{s}(\delta)) \alpha_0(s(\delta))^{-1} \alpha_1(s(\delta)) = \sigma_0(\hat{s}(\delta)) \cdot \chi(s(\delta))^{\pm k_e}.$$

By the classification of  $S^1$ -principal bundles over a 2-sphere, this proves Proposition 5.6. □

**§ 5D. Smooth circle bundles.** Let  $\mathcal{I}$  be a cellular  $(\mathbb{T}, A)$ -groupoid with  $A$  a regular complex. Let  $(X, \pi, \varphi)$  be a split  $\mathbb{T}$ -space with isotropy groupoid  $\mathcal{I}$ . Suppose that  $X$  is (closed) smooth manifold and that the  $\mathbb{T}$ -action is smooth. In this subsection, we relate the isotropy representation  $\Phi: \text{SBun}_{\mathbb{T}}^G(X) \rightarrow \text{Rep}^G(\mathcal{I}) \approx \prod_{v \in \Lambda_0} \mathfrak{I}^*$  with some “moment map”  $\Phi: X \rightarrow \mathfrak{t}^*$ . The material of this section is inspired by [14].

Let  $\eta = (E \xrightarrow{p} X)$  be a smooth  $\mathbb{T}$ -equivariant split principal  $S^1$ -bundle over  $X$ . Choose  $\theta \in \Omega^1(E)$  be an  $\mathbb{T}$ -invariant connection of the bundle  $\eta$  (we see  $S^1 = \mathbb{R}/\mathbb{Z}$ , so  $\text{Lie}(S^1) = \mathbb{R}$ ). This gives rise to a “moment map”  $\Phi: X \rightarrow \mathfrak{t}^*$  determined as follows. For  $\xi \in \mathfrak{t}$ , denote by  $\xi_E$  the vector field on  $E$  induced by the action of  $\mathbb{T}$ . The map  $\Phi$  is defined by the equation

$$\langle \Phi(x), \xi \rangle = \theta(\xi_E(y)),$$

for any  $x \in X$  and  $z \in p^{-1}(x)$ . As  $\theta$  is  $\mathbb{T}$ -invariant, the map  $\Phi$  descends to a continuous map  $\bar{\Phi}: A \rightarrow \mathfrak{t}^*$ .

Let  $\alpha \in \text{Rep}^{S^1}(\mathcal{I})$  be the isotropy representation of  $\eta$ . For each  $v \in \Lambda_0(A)$ , the homomorphism  $\alpha_v \in \text{Hom}(\mathbb{T}, S^1)$  is determined by its weight  $a_v \in \mathfrak{I}^*$ .

**Proposition 5.7.** *Suppose that  $\mathcal{I}$  is 0-toric. Then, for each  $v \in \Lambda_0(A)$ , one has  $\bar{\Phi}(v) = a_v$ .*

*Proof.* Let  $\xi \in \mathfrak{t}$ . Let  $v \in \Lambda_0(A)$  and  $z \in E$  with  $p(z) = \varphi(v)$ . As  $\varphi(v)$  is a fixed point, the vector  $\xi_E(z)$  is tangent to the  $S^1$ -orbit  $z \cdot S^1$ . If we identify the latter with  $S^1$ , then  $\theta(\xi_E(z))$  is the derivative of  $\alpha_v$ , that is,  $a_v$ .  $\square$

**Remark 5.8.** The figure of Example 5.5 suggests a possible relationship with the “twisted polytopes” of [14] which remains to be investigated.

**§ 5E. Case where  $A$  is a graph.** In this section we shall determine  $\text{Rep}^G(\mathcal{I})$  for a  $(\Gamma, A)$ -groupoid  $\mathcal{I}$  when  $A$  is a graph, generalising the case treated in [13] where  $A$  is a segment. One may suppose that the graph  $A$  is regular. Indeed, the subdivision of an edge  $e$ , by adding a vertex  $\hat{e} \in |e|$  and setting  $\mathcal{I}_{\hat{e}} = \mathcal{I}(e)$ , changes neither  $\text{Rep}^G(\mathcal{I})$  nor  $\overline{\text{Rep}}^G_{\text{cell}}(\mathcal{I})$ . Observe also that if  $G$  is connected, any  $G$ -principal bundle over  $A$  is trivial, so for a split  $\Gamma$ -space  $X$  over  $A$  one has  $\text{SBun}^G_\Gamma(X) = \text{Bun}^G_\Gamma(X)$ . We start with some preliminary material.

**Lemma 5.9.** *Let  $\Gamma$  and  $G$  be topological groups. Let  $\mathcal{I}$  be a  $(\Gamma, A)$ -groupoid, where  $A$  is a tree. Then  $\kappa: \text{Rep}^G_{\text{cell}}(\mathcal{I}) \rightarrow \overline{\text{Rep}}^G_{\text{cell}}(\mathcal{I})$  is surjective.*

*Proof.* The lemma is true for  $A = \emptyset$  since then both  $\text{Rep}^G_{\text{cell}}(\mathcal{I})$  and  $\overline{\text{Rep}}^G_{\text{cell}}(\mathcal{I})$  are empty. Otherwise, let  $b \in \overline{\text{Rep}}^G_{\text{cell}}(\mathcal{I})$  and let  $v$  be a vertex of  $A$ . Choose  $\beta_v \in \text{Hom}(\mathcal{I}_v, G)$  representing  $b_v$ . For an edge  $e$  between  $v$  and  $v'$ , define  $\beta_e \in \text{Hom}(\mathcal{I}(e), G)$  by  $\beta_e = \beta_v|_{\mathcal{I}(e)}$ . Since  $b \in \overline{\text{Rep}}^G_{\text{cell}}(\mathcal{I})$ , one can choose  $\beta_{v'} \in \text{Hom}(\mathcal{I}_{v'}, G)$  which represents  $b_{v'}$  such that  $\beta_{v'}|_{\mathcal{I}(e)} = \beta_e$ . This constructs a cellular representation  $\beta^1$  over the tree  $A(v, 1)$  of points of distance  $\leq 1$  from  $v$  (for the distance where each edge has length 1). The same methods will propagate  $\beta^1$  to  $\beta^2$ , over  $A(v, 2)$  and then to  $A(v, n)$  for all  $n$ . This defines  $\beta \in \text{Rep}^G_{\text{cell}}(\mathcal{I})$  with  $\kappa(\beta) = b$ .  $\square$

**Lemma 5.10.** *Let  $\mathcal{I}$  be a  $(\Gamma, A)$ -groupoid, where  $\Gamma$  is a topological group and  $A$  is a graph. Let  $G$  be a path-connected topological group. Then  $\iota: \text{Rep}^G(\mathcal{I}) \rightarrow \overline{\text{Rep}}^G_{\text{cell}}(\mathcal{I})$  is surjective.*

*Proof.* We may suppose that  $A$  is connected: otherwise, both  $\text{Rep}^G(\mathcal{I})$  and  $\overline{\text{Rep}}^G_{\text{cell}}(\mathcal{I})$  simply decompose into disjoint unions over components of  $A$ . Let  $A_0$  be a maximal tree of  $A$  and let  $\mathcal{I}_0$  be the restriction of  $\mathcal{I}$  over  $A_0$ . Let  $b \in \overline{\text{Rep}}^G_{\text{cell}}(\mathcal{I})$ . By Lemma 5.9, there exists a cellular representation  $\beta: \mathcal{I}_0 \rightarrow G$  such that  $\kappa(\beta) = b|_{\mathcal{I}_0}$ . We want to extend  $\beta$  to  $\hat{\beta}: \mathcal{I} \rightarrow G$ . This can be done by defining  $\hat{\beta}$  over  $\|e\|$  for each edge  $e$  of  $A \setminus A_0$ . Let  $v, v' \in \Lambda_0(A_0)$  be the vertices of  $e$ . As  $b \in \overline{\text{Rep}}^G_{\text{cell}}(\mathcal{I})$ , there is  $g \in G$  with  $g^{-1}\beta_v(\gamma)g = \beta_{v'}(\gamma)$  for all  $\gamma \in \mathcal{I}(e)$ . Since  $G$  is path-connected, there exists a continuous map  $a \mapsto g_a$ , from  $\|e\|$  to  $G$  with  $g_v = 1$  and  $g_{v'} = g$ . For  $a \in \|e\|$ , we then define  $\hat{\beta}_a: \mathcal{I}(e) \rightarrow G$  by  $\hat{\beta}_a(\gamma) = g(a)^{-1}\beta_v(\gamma)g(a)$ .  $\square$

We now introduce some material in order to describe the preimage  $\iota^{-1}(\alpha)$  of  $\alpha \in \overline{\text{Rep}}^G_{\text{cell}}(\mathcal{I})$ . Let  $K$  be a topological group and let  $\tilde{\alpha} \in \text{Hom}(K, G)$ . Define  $\mathcal{C}(\tilde{\alpha})$



to be the centraliser of  $\tilde{\alpha}(K)$  in  $G$ . Let  $\tilde{\alpha}' \in \text{Hom}(K, G)$  be such that  $[\tilde{\alpha}] = [\tilde{\alpha}']$  in  $\overline{\text{Hom}}(K, G)$ . Choose  $b \in G$  such that  $\tilde{\alpha}'(\gamma) = b\tilde{\alpha}(\gamma)b^{-1}$ . Sending  $z \in \mathcal{C}(\tilde{\alpha})$  to  $bzb^{-1}$  produces a continuous isomorphism  $r_{\tilde{\alpha}', \tilde{\alpha}}: \mathcal{C}(\tilde{\alpha}) \rightarrow \mathcal{C}(\tilde{\alpha}')$  which does not depend on the choice of  $b$ . Moreover, one has  $r_{\tilde{\alpha}'', \tilde{\alpha}} \circ r_{\tilde{\alpha}', \tilde{\alpha}'} = r_{\tilde{\alpha}'', \tilde{\alpha}}$ . Therefore, a topological group  $\mathcal{C}(\alpha)$  is defined for  $\alpha \in \overline{\text{Hom}}(K, G)$ : take the disjoint union of  $\mathcal{C}(\tilde{\alpha})$  for all representatives  $\tilde{\alpha}$  of  $\alpha$  and identify  $z \in \mathcal{C}(\tilde{\alpha})$  with  $r_{\tilde{\alpha}', \tilde{\alpha}}(z) \in \mathcal{C}(\tilde{\alpha}')$ . If  $K'$  is a subgroup of  $K$ , one checks that  $\mathcal{C}(\alpha)$  is a subgroup of  $\mathcal{C}(\alpha|_{K'})$ .

Let  $\mathcal{I}$  be a  $(\Gamma, A)$ -groupoid and  $\alpha \in \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$ . Let  $\dot{A}$  be the first barycentric subdivision of  $A$ . We assume that  $A$  is regular, so  $\Lambda_1(\dot{A})$  is the set of pairs  $(v, e) \in \Lambda_0(A) \times \Lambda_1(A)$  with  $v < e$ ; the edge corresponding to  $(v, e)$  joins  $v$  to the barycentre  $\hat{e}$  of  $e$ . Form the group  $X(\alpha) = \prod_{(v, e) \in \Lambda_1(\dot{A})} \pi_0(\mathcal{C}(\alpha_e))$ . Let  $J^0: \prod_{v \in \Lambda_0(A)} \pi_0(\mathcal{C}(\alpha_v)) \rightarrow X(\alpha)$  be the homomorphism sending  $(x_v)$  to  $(z_{(w, e)})$  with  $z_{(w, e)} = j_{w, e}(x_w)$ , where  $j_{w, e}: \pi_0(\mathcal{C}(\alpha_w)) \rightarrow \pi_0(\mathcal{C}(\alpha_e))$  is the homomorphism induced by the inclusion. Consider also the homomorphism  $J^1: \prod_{e \in \Lambda_1(A)} \pi_0(\mathcal{C}(\alpha_e)) \rightarrow X(\alpha)$  sending  $(y_e)$  to  $(z_{(w, f)})$ , where  $z_{(w, f)} = y_f$ . Set  $Y^0(\alpha)$  and  $Y^1(\alpha)$  to be the images of  $J^0$  and  $J^1$  and consider the double coset family  $Z(\alpha) = Y^0(\alpha) \backslash X(\alpha) / Y^1(\alpha)$ .

**Theorem 5.11.** *Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid, with  $\Gamma$  a topological group and  $A$  a graph. Let  $G$  be a compact connected Lie group. Then  $\iota: \text{Rep}^G(\mathcal{I}) \rightarrow \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  is surjective and the preimage  $\iota^{-1}(\alpha)$  of  $\alpha \in \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  is in bijection with  $Z(\alpha)$ .*

Before proving Theorem 5.11, we state some of its corollaries, in which we assume the hypotheses of Theorem 5.11 and mention only the additional hypotheses.

**Corollary 5.12.** *Let  $A$  be a finite graph. Then the preimages of  $\iota: \text{Rep}^G(\mathcal{I}) \rightarrow \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  are finite.*

*Proof.* As  $\mathcal{C}(\alpha_e)$  is a closed subgroup in  $G$ ,  $\pi_0(\mathcal{C}(\alpha_e))$  is finite for each edge  $e$  of  $A$ . Therefore  $Z(\alpha)$  is finite.  $\square$

The next corollary corresponds to [13], Theorem B and 8.12.

**Corollary 5.13.** *Suppose that  $\mathcal{I}$  is a proper  $(\Gamma, \Delta^1)$ -groupoid. Then the preimage  $\iota^{-1}(\alpha)$  of  $\alpha \in \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  is in bijection with the set of double cosets  $\pi_0(\mathcal{C}(\alpha_0)) \backslash \pi_0(\mathcal{C}(\alpha_{01})) / \pi_0(\mathcal{C}(\alpha_1))$ .*

*Proof.* The group  $X(\alpha)$  is isomorphic to  $\pi_0(\mathcal{C}(\alpha_{01})) \times \pi_0(\mathcal{C}(\alpha_{01}))$  with  $Y^1(\alpha) \approx \pi_0(\mathcal{C}(\alpha_{01}))$  being the diagonal subgroup. The group  $Y^0(\alpha)$  is  $\pi_0(\mathcal{C}(\alpha_0)) \times \pi_0(\mathcal{C}(\alpha_1))$ . Therefore, the map  $X(\alpha) \rightarrow \pi_0(\mathcal{C}(\alpha_{01}))$  given by  $(z_0, z_1) \mapsto z_0 z_1^{-1}$  descends to a bijection from  $Z(\alpha)$  to  $\pi_0(\mathcal{C}(\alpha_0)) \backslash \pi_0(\mathcal{C}(\alpha_{01})) / \pi_0(\mathcal{C}(\alpha_1))$  (see [13], Section 8).  $\square$

**Corollary 5.14.** *Suppose that  $\mathcal{I}(e)$  is a torus for all edges  $e$  of  $A$ . Then it follows that  $\iota: \text{Rep}^G(\mathcal{I}) \rightarrow \widetilde{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  is a bijection.*

*Proof.* Let  $\alpha \in \widetilde{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  and let  $\tilde{\alpha}: \mathcal{I} \rightarrow G$  be a continuous representation with  $\iota(\tilde{\alpha}) = \alpha$ . Our hypotheses imply that  $\tilde{\alpha}_a(\mathcal{I}_e)$  is a torus for all  $e \in \Lambda_1(A)$  and all  $a \in |e|$ . As  $G$  is connected, the group  $\mathcal{C}(\alpha_e)$  is then connected (see, e.g., [8], Theorem 3.3.1). Therefore,  $Z(\alpha)$  reduces to a single element.  $\square$

*Proof of Theorem 5.11.* The surjectivity of  $\iota$  is established in Lemma 5.10. Let  $\alpha \in \widetilde{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$ . The strategy is to construct a transitive action of  $X(\alpha)$  on  $\iota^{-1}(\alpha)$  and study the stabilisers.

Let  $\tilde{\alpha}^0: \mathcal{I}^{(0)} \rightarrow G$  be a representative of  $\alpha^{(0)}$ . Let  $\widetilde{\text{Rep}}^G(\mathcal{I}, \tilde{\alpha}^0)$  be the set of continuous representations from  $\mathcal{I}$  to  $G$  which restrict to  $\tilde{\alpha}^0$  on  $\alpha^{(0)}$ . As  $G$  is connected, any map from  $A^{(0)}$  to  $G$  extends to  $A$ , which implies that each class in  $\iota^{-1}(\alpha)$  has a representative in  $\widetilde{\text{Rep}}^G(\mathcal{I}, \tilde{\alpha}^0)$ . Also, if  $\tilde{\alpha} \in \widetilde{\text{Rep}}^G(\mathcal{I}, \tilde{\alpha}^0)$ , then  $\iota(\tilde{\alpha}) = \alpha$  by Proposition 5.1. Thus, the map  $\tilde{\alpha} \mapsto [\tilde{\alpha}] \in \text{Rep}^G(\mathcal{I})$  produces a surjection  $\widetilde{\text{Rep}}^G(\mathcal{I}, \tilde{\alpha}^0) \twoheadrightarrow \iota^{-1}(\alpha)$ .

Form the group  $\tilde{X}(\tilde{\alpha}^0) = \prod_{(v,e) \in \Lambda_1(A)} \mathcal{C}(\tilde{\alpha}^0(\mathcal{I}(e)))$ . Let  $z = (z_{(v,e)}) \in \tilde{X}(\tilde{\alpha}^0)$  and  $\tilde{\alpha} \in \widetilde{\text{Rep}}^G(\mathcal{I}, \tilde{\alpha}^0)$ . For each edge  $e$  of  $A$  with  $\partial e = \{v, v'\}$ , choose, using that  $G$  is connected, a continuous map  $g_e: \|e\| \rightarrow G$  such that  $g_e(v) = z_{(v,e)}$  and  $g_e(v') = z_{(v',e)}$ . We call  $\{g_e\}$  a *connecting family* for  $z$ . Define  $z \cdot_{\{g_e\}} \tilde{\alpha} \in \widetilde{\text{Rep}}^G(\mathcal{I}, \tilde{\alpha}^0)$  by

$$z \cdot_{\{g_e\}} \tilde{\alpha}(\gamma) = \begin{cases} g_e(a)\tilde{\alpha}(\gamma)g_e(a)^{-1} & \text{if } a \in |e| \text{ and } \gamma \in \mathcal{I}_a, \\ \tilde{\alpha}_a(\gamma) & \text{otherwise.} \end{cases} \quad (13)$$

For two connecting families  $\{g_e\}$  and  $\{\bar{g}_e\}$  for  $z$ , we check that

$$z \cdot_{\{g_e\}} \tilde{\alpha}(\gamma) = h(a)(z \cdot_{\{\bar{g}_e\}} \tilde{\alpha}(\gamma))h(a)^{-1},$$

where  $h: A \rightarrow G$  is the (continuous) map defined by  $h(a) = g_e(a)\bar{g}_e(a)^{-1}$  if  $a \in \|e\|$ . This thus defines  $z \cdot \tilde{\alpha}$  in  $\iota^{-1}(\alpha)$  which does not depend on the choice of the connecting family  $\{g_e\}$ .

Now suppose that  $\tilde{\alpha}, \tilde{\alpha}' \in \widetilde{\text{Rep}}^G(\mathcal{I}, \tilde{\alpha}^0)$  represent the same element in  $\text{Rep}^G(\mathcal{I})$ . This means that there is a map  $h: A \rightarrow G$  such that  $\tilde{\alpha}'_a(\gamma) = h(a)\tilde{\alpha}_a(\gamma)h(a)^{-1}$ . Observe then that  $h(v) \in \mathcal{C}(\tilde{\alpha}^0(\mathcal{I}_v))$  for all  $v \in \Lambda_0(A)$  and hence

$$h(a)(z \cdot_{\{g_e\}} \tilde{\alpha})h(a)^{-1} = z \cdot_{\{h(a)g_e h(a)^{-1}\}} \tilde{\alpha}'.$$

We have thus defined an action of  $\tilde{X}(\tilde{\alpha}^0)$  on  $\iota^{-1}(\alpha)$ . We now prove that this action is transitive. Let  $\tilde{\alpha}, \tilde{\alpha}' \in \widetilde{\text{Rep}}^G(\mathcal{I}, \tilde{\alpha}^0)$ . Orient each edge  $e$  of  $A$ , getting then  $\partial e = \{\partial_- e, \partial_+ e\}$ . By Lemma 1.2, there exist  $s, s': \|e\| \rightarrow G$  such that  $\tilde{\alpha}(\gamma) = s(a)^{-1}\tilde{\alpha}_{\partial_- e}^0(\gamma)s(a)$  and  $\tilde{\alpha}'(\gamma) = s'(a)^{-1}\tilde{\alpha}_{\partial_- e}^0(\gamma)s'(a)$  for all  $\gamma \in \mathcal{I}(e)$  and all  $a \in \|e\|$ . This implies that  $s(\partial e)$  and  $s'(\partial_- e)$  are contained in  $\mathcal{C}(\tilde{\alpha}(\mathcal{I}(e)))$ . Hence one

has  $\tilde{\alpha}' = z_{\{g_e\}} \cdot \tilde{\alpha}$ , where  $z_{(\partial_-e,e)} = s'(\partial_-e)^{-1}s(\partial_-e)$ ,  $z_{(\partial_+e,e)} = s'(\partial_+e)^{-1}s(\partial_+e)$  and  $g_e(a) = s'(a)^{-1}s(a)$ . Hence the action of  $\tilde{X}(\tilde{\alpha}^0)$  on  $\iota^{-1}(\alpha)$  is transitive.

If  $z = (z_{(v,e)})$  is in the unit component of  $\tilde{X}(\tilde{\alpha}^0)$ , then  $z \cdot_{\{g_e\}} \tilde{\alpha} = \tilde{\alpha}$ , if the maps  $g_e$  are chosen so that  $g_e(\hat{e}) = 1$  and  $g_e(\|(v,e)\|) \subset \mathcal{C}(\tilde{\alpha}_v^0(\mathcal{I}(e)))$ . This implies that the action of  $\tilde{X}(\tilde{\alpha}^0)$  on  $\iota^{-1}(\alpha)$  descends to an action of the group  $\prod_{(v,e) \in \Lambda_1(A)} \pi_0(\mathcal{C}(\tilde{\alpha}_v^0(\mathcal{I}(e))))$  which is isomorphic to  $X(\alpha)$ .

Let  $f$  be an edge of  $A$ , with  $\partial f = \{v, v'\}$ . The representation  $\tilde{\alpha}^0: \mathcal{I}^{(0)} \rightarrow G$  can be chosen such that the restrictions to  $\mathcal{I}_e$  of  $\tilde{\alpha}_v^0$  and  $\tilde{\alpha}_{v'}^0$  coincide. For each  $\zeta \in \mathcal{C}(\tilde{\alpha}_v^0(\mathcal{I}(f))) = \mathcal{C}(\tilde{\alpha}_{v'}^0(\mathcal{I}(f)))$  we can then consider the element  $z(\zeta)$  of  $\tilde{X}(\tilde{\alpha}^0)$  satisfying  $z_{(v,f)}(\zeta) = z_{(v',f)}(\zeta) = \zeta$  and  $z_{(w,e)}(\zeta) = 1$  if  $e \neq f$ . Then  $z(\zeta) \cdot_{\{g_e\}} \tilde{\alpha} = \tilde{\alpha}$  if the  $g_e$  are constant maps. This may be done for each edge  $f$  of  $A$ , showing that the group  $Y^1(\alpha)$  acts trivially on  $\beta$  for all  $\beta \in \iota^{-1}(\alpha)$ .

Let  $y \in \prod_{v \in \Lambda_0(A)} (\mathcal{C}(\tilde{\alpha}_v^0(\mathcal{I}_v)))$  and  $z \in \tilde{X}(\tilde{\alpha}^0)$ . Consider the element  $yz \in \tilde{X}(\tilde{\alpha}^0)$  defined by  $(yz)_{(v,e)} = y_v z_{(v,e)}$ . Choose a connecting family  $g_e: \|e\| \rightarrow G$  for  $z$ . For each  $(v,e) \in \Lambda_1(A)$ , choose  $h_{(v,e)}: \|(v,e)\| \rightarrow G$  such that  $h_{(v,e)}(v) = y_v$  and  $h_{(v,e)}(\hat{e}) = 1$ . This defines a continuous map  $h: A \rightarrow G$ , by  $h(a) = h_{(v,e)}(a)$  if  $a \in \|(v,e)\|$ , which conjugates  $(yz) \cdot_{\{hg_e\}} \tilde{\alpha}$  with  $z \cdot_{\{g_e\}} \tilde{\alpha}$ . This shows that  $ux \cdot \beta = x \cdot \beta$  in  $\iota^{-1}(\alpha)$ , for all  $u \in Y^0(\alpha)$ ,  $x \in X(\alpha)$  and  $\beta \in \iota^{-1}(\alpha)$ .

Fix  $\beta \in \iota^{-1}(\alpha)$ , represented by  $\tilde{\beta} \in \widetilde{\text{Rep}}^G(\mathcal{I}, \tilde{\alpha}^0)$ . Consider the map  $\tilde{\Psi}: \tilde{X}(\tilde{\alpha}^0) \rightarrow \iota^{-1}(\alpha)$  given by  $\tilde{\Psi}(z) = [z \cdot \tilde{\beta}]$ . By the above, we have shown that  $\tilde{\Psi}$  descends to a surjection  $\Psi: Z(\alpha) \twoheadrightarrow \iota^{-1}(\alpha)$ . It remains to show that  $\Psi$  is injective. Let  $z' \in \tilde{X}(\tilde{\alpha}^0)$  with  $\Psi(z') = \Psi(z)$ . Choose connecting families  $\{g_e\}$  and  $\{g'_e\}$  for  $z$  and  $z'$ . If  $\Psi(z') = \Psi(z)$ , there exists a map  $h: A \rightarrow G$  with  $(z' \cdot_{\{g'_e\}} \tilde{\beta})(\gamma) = h(a)(z \cdot_{\{g_e\}} \tilde{\beta})(\gamma)h(a)^{-1}$ . Observe that  $h(v) \in \mathcal{C}(\tilde{\alpha}^0(\mathcal{I}_v))$  and therefore  $h^{(0)}: \mathcal{I}^{(0)} \rightarrow G$  defines an element  $y \in \prod_{v \in \Lambda_0(A)} (\mathcal{C}(\tilde{\alpha}_v^0(\mathcal{I}_v)))$  satisfying  $((yz) \cdot_{\{hg_e\}} \tilde{\beta})(\gamma) = h(a)(z \cdot_{\{g_e\}} \tilde{\beta})(\gamma)h(a)^{-1}$ . Let  $\bar{z} = yz$  and  $\bar{g}_e = hg_e$ . One has  $[\bar{z}] = u[z]$  in  $X(\alpha)$  with  $u \in Y^0(\alpha)$ . Thus,  $\bar{z}$  and  $z$  represent the same class in  $Y^0(\alpha) \setminus X(\alpha)$  and the equality  $z' \cdot_{\{g'_e\}} \tilde{\beta} = \bar{z} \cdot_{\{\bar{g}_e\}} \tilde{\beta}$  holds in  $\widetilde{\text{Rep}}^G(\mathcal{I}, \tilde{\alpha}^0)$ . Therefore,  $\bar{g}_e(a)^{-1}g'_e(a) \in \mathcal{C}(\beta_a(\mathcal{I}(e)))$  for all  $a \in \|e\|$ . This implies that  $z'$  and  $\bar{z}$  represent the same class in  $X(\alpha)/Y^1(\alpha)$ . Finally, we have shown that  $z$  and  $z'$  represent the same class in  $Z(\alpha)$ , proving the injectivity of  $\Psi$ .  $\square$

We now give some examples of the use of Theorem 5.11.

**5.15.** Let  $A$  be the 1-simplex  $\Delta^1$ . Let  $\Gamma = \text{SO}(n)$ , with  $n = 2k + 1 \geq 3$  and consider the  $(\Gamma, A)$ -groupoid  $\mathcal{I}$  with  $\mathcal{I}_0 = \mathcal{I}_1 = \Gamma$  and  $\mathcal{I}_{01} = \text{SO}(n-1)$  (the split  $\Gamma$ -space  $X$  with isotropy groupoid  $\mathcal{I}$  is  $S^n$  with the  $\text{SO}(n)$ -action fixing the north and the south pole). For  $G = \text{SO}(n)$ ,  $\widetilde{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  contains two elements, the trivial representation and the representation  $\alpha$  with  $\alpha_0 = \alpha_1 = \text{id}$ . The preimage by  $\iota$  of the trivial representation contains one element but  $\iota^{-1}(\alpha)$  contains two elements. For details and developments, see [13], Example 7.5.

**5.16.** If  $\pi_1(G) = \{1\}$ , Theorem 5.11 extends to a  $(\Gamma, A)$ -groupoid  $\mathcal{I}$  where  $A$  is of dimension 2, provided that  $\mathcal{I}(e) = \{1\}$  when  $e \in \Lambda_2(A)$ . Examples are given by toric manifolds of (real) dimension 4.

**5.17.** Let  $\mathcal{I}$  be the  $S^1$ -structure on the 1-simplex  $\Delta^1$  with  $\mathcal{I}_0 = \mathcal{I}_1 = S^1$  and  $\mathcal{I}_{01} = \{1\}$ . The split  $S^1$ -space with isotropy groupoid  $\mathcal{I}$  is  $S^2$  with  $S^1$  acting by rotation around an axis. By Theorem 5.11,  $\text{SBun}_{S^1}^G(S^2) \approx \overline{\text{Hom}}(\mathcal{I}_0, G) \times \overline{\text{Hom}}(\mathcal{I}_1, G)$ . Choosing a maximal torus  $\mathcal{T}$  in  $G$ , this yields  $\text{SBun}_{S^1}^G(S^2) \approx \text{Hom}(\mathcal{I}_0, \mathcal{T})/\mathcal{W} \times \text{Hom}(\mathcal{I}_1, \mathcal{T})/\mathcal{W}$  where  $\mathcal{W}$  is the Weyl group for  $\mathcal{T}$ . If  $G$  is of rank  $k$ , then  $\text{Hom}(\mathcal{I}_0, \mathcal{T})$  and  $\text{Hom}(\mathcal{I}_1, \mathcal{T})$  are both in bijection with  $\mathbb{Z}^k$ .

Let us specialise to  $G = \text{SO}(m)$  for  $m \geq 3$ . A maximal torus  $\mathcal{T}$  of  $\text{SO}(m)$  is formed by matrices containing 2-blocks concentrated around the diagonal, so isomorphic to  $\text{SO}(2)^k$ , and where  $k = [m/2]$ . The action of  $\mathcal{W}$  on  $\text{Hom}(S^1, \mathcal{T}) \approx \mathbb{Z}^k$  can be deduced from [1], p. 114. When  $m = 2k + 1$ , the action of  $\mathcal{W}$  on  $\mathbb{Z}^k$  is generated by the permutation of coordinates and sign changes in any of them. A fundamental domain  $\mathcal{D} \subset \mathbb{Z}^k$  is then

$$\mathcal{D} = \{(r_1, \dots, r_k) \in \mathbb{Z}^k \mid 0 \leq r_1 \leq \dots \leq r_k\}$$

and  $\text{SBun}_{S^1}^{\text{SO}(2k+1)}(S^2) \approx \mathcal{D} \times \mathcal{D}$ . When  $m = 2k$ , the sign changes must be even in number. A fundamental domain  $\mathcal{E} \subset \mathbb{Z}^k$  is then

$$\mathcal{E} = \{(r_1, \dots, r_k) \in \mathbb{Z}^k \mid 0 \leq r_1 \leq \dots \leq r_{k-1} \leq |r_k|\}$$

and  $\text{SBun}_{S^1}^{\text{SO}(2k)}(S^2) \approx \mathcal{E} \times \mathcal{E}$ .

This example was treated in our paper [13], Example 7.3, but the determination of  $\text{SBun}_{S^1}^{\text{SO}(m)}(S^2)$  is wrong there because, in the action of the Weyl group, the sign changes were forgotten. However, the computation in [13], Example 7.3, of the second Stiefel–Whitney number  $w_2(\xi)$  for  $\xi \in \text{SBun}_{S^1}^{\text{SO}(m)}(S^2)$ , being mod 2, is correct.

Here is an interesting consequence of the proof of Theorem 5.11.

**Proposition 5.18.** *Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid, with  $A$  a regular CW-complex and  $\Gamma$  a topological group. Let  $G$  be a compact connected Lie group. Then  $J: \text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \text{Rep}^G(\mathcal{I})$  is injective.*

*Proof.* As in the proof of Lemma 5.10, one may assume that  $A$  is connected. Let  $A_0$  be a maximal tree of  $A$  and let  $\mathcal{I}_0$  be the restriction of  $\mathcal{I}$  over  $A_0$ . As  $A$  is connected,  $A_0$  contains all the vertices of  $A$  and then the restriction map  $\text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \text{Rep}_{\text{cell}}^G(\mathcal{I}_0)$  is injective by Proposition 5.1. Therefore, it is enough to prove Proposition 5.18 when  $A$  is a tree.

Let  $\beta, \beta': \mathcal{I} \rightarrow G$  be cellular representations with  $J(\beta) = J(\beta')$ . Let  $v$  be a vertex of the tree  $A$ . By conjugation of  $\beta$  with a constant element of  $G$ , one may

assume that  $\beta_v = \beta'_v$ . Let  $e$  be an edge between  $v$  and  $v'$ ; one has  $\beta_e = \beta'_e$ . Suppose that  $\beta_{v'} \neq \beta'_{v'}$ . Then  $\beta'_v(\gamma) = z\beta(\gamma)z^{-1}$  with  $z \in \mathcal{C}(\beta_e(\mathcal{I}(e)))$  and  $z \notin \mathcal{C}(\beta_{v'}(\mathcal{I}_{v'}))$ . Choose a continuous map  $g_e: \|e\| \rightarrow G$  with  $g_e(v) = 1$  and  $g_e(v') = z$ . Let  $\mathcal{I}_{\|e\|}$  be the restriction of  $\mathcal{I}$  over  $\|e\|$  and let  $\beta'': \mathcal{I}_{\|e\|} \rightarrow G$  be the (non-cellular) representation defined by  $\beta''(\gamma) = g_e(a)^{-1}\beta'(\gamma)g_e(a)$ . Using the notations of the proof of Theorem 5.11, this means that  $\beta, \beta'' \in \text{Rep}^G(\mathcal{I}_{\|e\|}, \beta^{(0)})$  and  $\beta'' = y \cdot_{g_e} \beta$ , where  $y \in \tilde{X}(\beta^{(0)})$  is defined by  $y_{(v,e)} = 1$  and  $y_{(v',e)} = z$ . The element  $y$  is non-trivial in  $Z(\alpha_{\|e\|})$  which, by Theorem 5.11, would contradict the assumption  $J(\beta) = J(\beta')$ . Therefore,  $\beta_{v'} = \beta'_{v'}$ . This argument may be done independently for all edges adjacent to  $v$  and then propagated to the whole tree  $A$ .  $\square$

When  $A$  is a tree, the map  $J: \text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \text{Rep}^G(\mathcal{I})$  is actually bijective. More precisely:

**Lemma 5.19.** *Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid, where  $A$  is a graph and  $\Gamma$  a topological group. Let  $A_0$  be a subtree of  $A$ . Let  $G$  be a compact Lie group. Then any  $\alpha \in \text{Rep}^G(\mathcal{I})$  has a representative which is cellular over  $A_0$ .*

*Proof.* Let  $v$  be a vertex of  $A_0$ . For an edge  $e$  of  $A_0$ , between  $v$  and  $v'$ , there exists, by Lemma 1.2, a map  $\psi_e: \|e\| \rightarrow G$  such that  $\psi_e(a)\alpha_a(\gamma)\psi_e(a)^{-1} = \alpha_v(\gamma)$  for each  $a \in \|e\|$  and  $\gamma \in \mathcal{I}_e$ . This defines a map  $\psi_1: A_0(v, 1) \rightarrow G$  (notations as in the proof of Lemma 5.9). As  $A_0(v, 1)$  is contractible, the homotopy extension property permits us to extend  $\psi_1$  to a continuous map  $\psi_1: A \rightarrow G$ . The maps  $\psi_1$  conjugates  $\alpha$  to  $\alpha_1$  which is cellular over  $A_0(v, 1)$ . The process propagates over  $A_0(v, n)$  for all  $n$ , giving rise to a map  $\psi: A \rightarrow G$  which conjugates  $\alpha$  to a representation which is cellular over  $A_0$ .  $\square$

Proposition 5.18 together with Lemma 5.19 implies the following

**Corollary 5.20.** *Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid with  $\Gamma$  a topological group and  $A$  a tree. Let  $G$  be a compact connected Lie group. Then  $J: \text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \text{Rep}^G(\mathcal{I})$  is bijective.*

**5.21.** In contrast with Theorem 5.11, the map  $J: \text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \text{Rep}^G(\mathcal{I})$  is not surjective when the graph  $A$  is not a tree. Using Lemma 5.9, it is enough to find an example where  $\kappa: \text{Rep}_{\text{cell}}^G(\mathcal{I}) \rightarrow \overline{\text{Rep}}^G(\mathcal{I})$  is not surjective. Let  $A$  be the 1-skeleton of the 2-simplex  $\Delta^2$  with  $\mathcal{I}_0 = \mathcal{I}_1 = \mathcal{I}_2 = \Gamma = S^1 \times S^1$ ,  $\mathcal{I}_{01} = 1 \times S^1$ ,  $\mathcal{I}_{02} = S^1 \times 1$  and  $\mathcal{I}_{12}$  is the diagonal  $S^1$ . The split  $\Gamma$ -space with this isotropy groupoid is  $\mathbb{C}P^2$  with the action  $(c_1, c_2) \cdot [z_0: z_1: z_2] = [c_0z_0: c_1z_1: z_2]$ . Take  $G = \text{SU}(2)$ ; the diagonal torus  $H$  has dimension 1 and its Weyl group  $\mathcal{W}$  acts by passing to the inverse. Then

$$\begin{aligned} \overline{\text{Hom}}(\Gamma, \text{SU}(2)) &\approx \text{Hom}(\Gamma, H)/\mathcal{W} \\ &\approx \hat{\Gamma}/\{\chi \sim -\chi\} \approx (\mathbb{Z} \times \mathbb{Z})/\{(p, q) \sim -(p, q)\}. \end{aligned}$$

We identify  $\overline{\text{Hom}}(\Gamma, \text{SU}(2))$  with the fundamental domain  $\mathcal{D}$  in  $\mathbb{Z} \times \mathbb{Z}$ :

$$\mathcal{D} := \{(p, q) \in \mathbb{Z} \times \mathbb{Z} \mid q \geq 0 \text{ and } (p \geq 0 \text{ if } q = 0)\}.$$

If  $\beta \in \text{Rep}_{\text{cell}}^G(\mathcal{I})$  is not trivial, it must be not trivial on at least one edge-isotropy groups (say  $\mathcal{I}_{01}$ ). Then  $\beta$  is conjugate to  $\beta'$  such that  $\beta'_{01}(\mathcal{I}_{01}) \subset H$ . As  $H$  is maximal abelian,  $\beta'$  is then an algebraic representation of  $\mathcal{I}$  in  $H$ . By Proposition 5.4, one has an identification of  $\text{Rep}^H(\mathcal{I})$  with the set of triples

$$((p_0, q_0), (p_1, q_1), (p_2, q_2)) \in (\mathbb{Z} \times \mathbb{Z})^3$$

such that

$$p_0 = p_2, \quad q_0 = q_1, \quad \text{and} \quad p_1 + q_1 = p_2 + q_2. \tag{14}$$

A class in  $\overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  is a triple

$$([p_0, q_0], [p_1, q_1], [p_2, q_2]) \in \mathcal{D} \times \mathcal{D} \times \mathcal{D}$$

such that  $|p_0| = |p_2|$ ,  $q_0 = q_1$  and  $|p_1 + q_1| = |p_2 + q_2|$ . The class  $\alpha \in \overline{\text{Rep}}_{\text{cell}}^G(\mathcal{I})$  corresponding to  $([-1, 2], [3, 2], [1, 4])$  is not in the image of  $\kappa$ . Indeed, none of the 8 triples in  $(\mathbb{Z} \times \mathbb{Z})^3$  above  $\alpha$  satisfies equations (14).

## 6. Comparison with the homotopy-theoretic approach

**§ 6A. Haefliger classifying spaces.** Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space over a space  $A$  with isotropy groupoid  $\mathcal{I}$ . Let  $B\mathcal{I}$  be the Haefliger classifying space for  $\mathcal{I}$  [12], p. 140. For a groupoid like  $B\mathcal{I}$  where morphisms go from one object to itself, we check that the construction of [12], p. 140, takes the following form: set

$$E\mathcal{I} = \{(v, a) \in E\Gamma \times A \mid v \in E\mathcal{I}_a\},$$

with the induced topology, and define  $B\mathcal{I}$  as the quotient space  $E\mathcal{I}/\mathcal{I}$ . The projection  $\bar{\pi}: B\mathcal{I} \rightarrow A$  makes  $B\mathcal{I}$  a space over  $A$  whose stalk over  $a$  is the Milnor classifying space  $B\mathcal{I}_a$ . There is a section  $j: A \rightarrow B\mathcal{I}$  of  $\bar{\pi}$  sending  $a \in A$  to the class of  $(v_0, a)$  where  $v_0 = (1e, 0, \dots) \in E\Gamma$ , expressed as the infinite join, with  $e$  the unit element of  $\Gamma$ . The inclusion  $\mathcal{I} \subset \Gamma \times A$  is a morphism of topological groupoids and therefore induces a continuous map  $E\mathcal{I} \rightarrow E\Gamma \times A$  which descends to a continuous map  $B\mathcal{I} \rightarrow B\Gamma \times A$ .

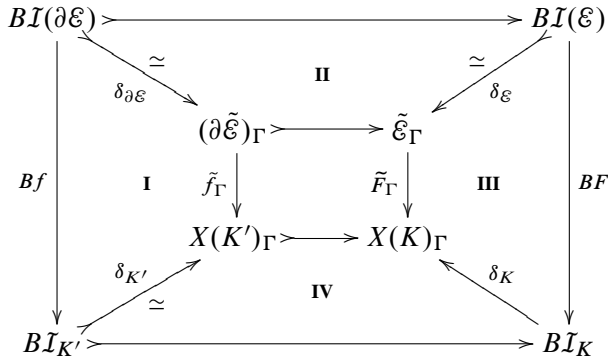
Recall that the *Borel construction* associates to  $X$  the space  $X_\Gamma = E\Gamma \times_\Gamma X$ . The map  $\pi: X \rightarrow A$  descends to a continuous and open surjective map  $\bar{\pi}: X_\Gamma \rightarrow A$ , with  $\bar{\pi}^{-1}(a) = E\Gamma \times_\Gamma \mathcal{I}_a \approx B\mathcal{I}_a$ . The composed map  $E\mathcal{I} \rightarrow E\Gamma \times A \xrightarrow{\text{id} \times \varphi} E\Gamma \times X$  descends to a continuous map  $\delta: B\mathcal{I} \rightarrow X_\Gamma$  over the identity of  $A$ . The restriction of  $\delta$  to each stalk is a weak homotopy equivalence. It would then be interesting to figure out, for instance in the spirit of Sections 2 and 3, under which hypotheses  $\delta$  is a weak homotopy equivalence. We will restrict ourselves to  $(\Gamma, A)$ -groupoids, where we get the following proposition.

**Proposition 6.1.** *Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid for a Lie group  $\Gamma$ . Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space over  $A$ , with isotropy groupoid  $\mathcal{I}$ . Then the map  $\delta: B\mathcal{I} \rightarrow X_\Gamma$  is a homotopy equivalence.*

*Proof.* By Proposition 4.3, we may suppose that  $(X, \pi, \varphi) = (Y_{\mathcal{I}}, \pi, \phi)$ . If  $K$  is a subspace of  $A$ , we denote by  $\mathcal{I}(K)$  the subgroupoid of  $\mathcal{I}$  formed by all the stalks over  $K$ , and we set  $X(K) = Y_{\mathcal{I}(K)}$ .

Observe first that Proposition 6.1 is true if  $\mathcal{I}_a$  is constant for all  $a \in A$ . Indeed, one then has  $X = \Gamma/\mathcal{I}_a \times A$ , so  $B\mathcal{I} \approx B\mathcal{I}_a \times A$  and  $X_\Gamma \approx E\Gamma/\mathcal{I}_a \times A$  and  $\delta$  is a homotopy equivalence. More generally, Proposition 6.1 remains true if  $\mathcal{I}_a$  is locally constant, meaning constant on each connected component of  $A$ .

Proposition 6.1 will be proved, by induction on  $n$ , for  $X(A^{(n)})$ , the split  $\Gamma$ -space over the  $n$ -skeleton  $A^{(n)}$  of  $A$ . It is true for  $n = 0$  since  $\mathcal{I}(A^{(0)})$  is locally constant. The induction step involves the subcomplexes  $K' = A^{(n-1)} \subset K = A^{(n)}$ , so  $K$  is obtained from  $K'$  by adjunction of  $\mathcal{E} = \coprod_{e \in \Lambda_n} D_e^n$ , via the attaching map  $f: \partial\mathcal{E} = \coprod_{e \in \Lambda_n} S_e^{n-1} \rightarrow K'$  ( $\Lambda_n = \Lambda_n(A)$ ). Then  $X(K)$  is obtained from  $X(K')$  by attaching the  $\Gamma$ -space  $\tilde{\mathcal{E}} = \coprod_{e \in \Lambda_n} (\Gamma/\mathcal{I}(e) \times D_e^n)$  via the  $\Gamma$ -equivariant map  $\tilde{f}: \partial\tilde{\mathcal{E}} = \coprod_{e \in \Lambda_n} (\Gamma/\mathcal{I}(e) \times S_e^{n-1}) \rightarrow X(K')$ . We denote by  $F: \mathcal{E} \rightarrow K$  and  $\tilde{F}: \tilde{\mathcal{E}} \rightarrow X(K)$  the characteristic maps, extending  $f$  and  $\tilde{f}$ . We see  $\tilde{\mathcal{E}}$  and  $\partial\tilde{\mathcal{E}}$  as split  $\Gamma$ -spaces over  $\mathcal{E}$  and  $\partial\mathcal{E}$  respectively with locally constant isotropy groupoids: if  $x \in D_e^n$ , then  $\mathcal{I}(\mathcal{E}) = \mathcal{I}(\partial\mathcal{E}) = \mathcal{I}(e)$ . Let us consider the following diagram:



The maps  $\delta_{\partial\mathcal{E}}$  and  $\delta_{\mathcal{E}}$  are homotopy equivalences since the isotropy groupoids are locally constant. The map  $\delta_{K'}$  is a homotopy equivalence by induction hypothesis. Restriction to any stalk shows that diagrams I–IV are commutative. As  $\Gamma$  is a Lie group, all the spaces under consideration have the homotopy type of CW-complexes. Therefore, the outer and inner square diagrams are homotopy push-out diagrams. By push-out properties, the map  $\delta_K$  is a homotopy equivalence.  $\square$

**§ 6B. Split bundles and classifying spaces.** Let  $\eta: (P \xrightarrow{p} X)$  be a  $\Gamma$ -equivariant principal  $G$ -bundle. The Borel construction  $E\Gamma \times_\Gamma P \rightarrow E\Gamma \times_\Gamma X$  yields a

principal  $G$ -bundle  $\eta_\Gamma$  over  $X_\Gamma$  with the same trivialising cover as  $\eta$ . If  $\eta$  is numerable, so is  $\eta_\Gamma$ . Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -CW-complex over  $A$ . By Theorem 4.5, any split  $\Gamma$ -equivariant principal bundle over  $X$  is numerable. Therefore we obtain a map  $\Psi: \text{Bun}_\Gamma^G(X) \rightarrow [X_\Gamma, BG]$ . The isotropy representation  $\Phi: \text{SBun}_\Gamma^G(X) \xrightarrow{\approx} \text{Rep}^G(\mathcal{I})$  is a bijection. Passing to the classifying spaces gives a map  $B: \text{Rep}^G(\mathcal{I}) \rightarrow [B\mathcal{I}, BG]$ . The map  $\delta: B\mathcal{I} \rightarrow X_\Gamma$  of Section § 6A gives rise to a map  $\delta^*: [X_\Gamma, BG] \rightarrow [B\mathcal{I}, BG]$ .

**Proposition 6.2.** *Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid for a Lie group  $\Gamma$ . Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -CW-complex over  $A$  with isotropy groupoid  $\mathcal{I}$ . Let  $G$  be a compact Lie group. Then the diagram*

$$\begin{array}{ccc} \text{SBun}_\Gamma^G(X) & \xrightarrow{\Psi} & [X_\Gamma, BG] \\ \approx \downarrow \Phi & & \downarrow \delta^* \\ \text{Rep}^G(\mathcal{I}) & \xrightarrow{B} & [B\mathcal{I}, BG] \end{array}$$

is commutative.

*Proof.* By Proposition 4.3, we may assume that  $(X, \pi, \varphi) = (Y_\mathcal{I}, \pi, \phi)$ . Let  $\varepsilon \in \text{SBun}_\Gamma^G(Y_\mathcal{I})$  and let  $\epsilon: \mathcal{I} \rightarrow G$  be a representative of  $\Phi(\varepsilon)$ . By Theorem 4.5 and its proof,  $\varepsilon$  has a representative  $\eta$  of the form

$$\Gamma \times_{\mathcal{I}} (A \times G) \rightarrow \Gamma \times_{\mathcal{I}} A = Y_\mathcal{I},$$

where  $\mathcal{I}$  acts on  $A \times G$  by  $\zeta \cdot (a, g) = (a, \epsilon(\zeta)g)$ . The bundle  $\eta_\Gamma$  takes the form

$$E\Gamma \times_{\mathcal{I}} (A \times G) \rightarrow E\Gamma \times_{\mathcal{I}} A = X_\Gamma.$$

Let  $q: L \rightarrow B\mathcal{I}$  be the induced bundle  $\delta^*\eta_\Gamma$ . To prove Proposition 6.2, it is enough to construct a  $G$ -equivariant map  $F: L \rightarrow EG$  making the diagram

$$\begin{array}{ccc} L & \xrightarrow{F} & EG \\ q \downarrow & & \downarrow \\ B\mathcal{I} & \xrightarrow{B\epsilon} & BG \end{array}$$

commutative. Restricted to the stalk over  $a$ , the bundle  $\delta^*\eta_\Gamma$  is of the form

$$E\mathcal{I}_a \times_{\mathcal{I}_a} (\{a\} \times G) \longrightarrow E\mathcal{I}_a \times_{\mathcal{I}_a} \{a\}.$$

Therefore, the required map  $F$  can be defined by

$$F(u, a, g) = E\epsilon(u) \cdot g. \quad \square$$



Proposition 6.2 allows us to study the map  $B: \text{Rep}^G(\mathcal{I}) \rightarrow [B\mathcal{I}, BG]$ , especially when  $G$  is abelian, in which case  $B$  is a homomorphism of abelian groups.

**Proposition 6.3.** *Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid for a Lie group  $\Gamma$ . Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -CW-complex over  $A$  with isotropy groupoid  $\mathcal{I}$ . Let  $G$  be a compact abelian Lie group. Then one has an isomorphism of split exact sequences of abelian groups:*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{SBun}_\Gamma^G(X) & \longrightarrow & \text{Bun}_\Gamma^G(X) & \begin{array}{c} \xrightarrow{\varphi^*} \\ \xleftarrow{\pi^*} \end{array} & \text{Bun}^G(A) & \longrightarrow & 0 \\
 & & \approx \downarrow \Phi & & \approx \downarrow \delta^* \circ \Psi & & \downarrow \approx & & (15) \\
 0 & \longrightarrow & \text{Rep}_{\text{cell}}^G(\mathcal{I}) & \xrightarrow{B} & [B\mathcal{I}, BG] & \begin{array}{c} \xrightarrow{j^*} \\ \xleftarrow{\bar{\pi}^*} \end{array} & [A, BG] & \longrightarrow & 0.
 \end{array}$$

*Proof.* The top split exact sequence of abelian groups comes from Proposition 4.7 and its proof. For the bottom one, one has at least a sequence

$$\text{Rep}^G(\mathcal{I}) \xrightarrow{B} [B\mathcal{I}, BG] \xrightarrow{j^*} [A, BG]$$

with  $j^* \circ B = 0$ . By Proposition 4.7, any principal  $\Gamma$ -equivariant  $G$ -bundle over  $X$  is numerable. Therefore, the map  $\delta^* \circ \Psi$  is defined and is a homomorphism of abelian groups. One checks that the left-hand square of the diagram (15) is commutative, as well as the right-hand square with  $\varphi^*$  and  $j^*$ . The map  $\delta^*$  is bijective by Proposition 6.1. As  $G$  is abelian, the map  $\Psi$  is a bijection by [18], Theorem A. Thus,  $\delta^* \circ \Psi$  is an isomorphism. This proves that the bottom sequence of diagram (15) is split exact.  $\square$

**Corollary 6.4.** *Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid for a Lie group  $\Gamma$ . Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space over  $A$  with isotropy groupoid  $\mathcal{I}$ . Let  $G$  be a compact abelian Lie group. Suppose that  $H^1(A; \pi_0(G)) = H^2(A; \mathbb{Z}) = 0$ . Then the map  $B: \text{Rep}^G(\mathcal{I}) \rightarrow [B\mathcal{I}, BG]$  is a bijection.*

*Proof.* The group  $G$  is a disjoint union of tori, so  $\pi_j(BG) = \pi_{j-1}(G) = 0$  for  $j > 2$ . One has  $\text{Hom}(\pi_1(A), \pi_1(BG)) = \text{Hom}(\pi_1(A), \pi_0(G)) \approx H^1(A; \pi_0(G)) = 0$ . A map  $f: A \rightarrow BG$  is then null-homotopic on the 1-skeleton and the obstruction theory to homotop it to a constant map is with constant coefficients. Our hypotheses implies that  $H^2(A; \pi_2(BG)) = 0$ , so one gets  $[A, BG] = 0$ . Corollary 6.4 then follows from Proposition 6.3.  $\square$

**6.5. Equivariant K-theory.** For vector bundles it is natural to stabilize, and then to study bundles via equivariant K-theory. For example, if  $G = U(n)$  we consider the stabilization maps

$$\text{SBun}_\Gamma^{U(n)}(X) \rightarrow \text{SBun}_\Gamma^{U(n+1)}(X)$$

and point out how stabilization is related to our classification results.

**Proposition 6.6.** *Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid for a Lie group  $\Gamma$ . Let  $(X, \pi, \varphi)$  be a split  $\Gamma$ -CW-complex over  $A$  with isotropy groupoid  $\mathcal{I}$ . Then there is a natural isomorphism*

$$\Phi: K_\Gamma(X, A) \cong \text{KRep}(\mathcal{I})$$

*of abelian groups induced by the isotropy representations.*

The group  $\text{KRep}(\mathcal{I})$  is the Grothendieck group of the abelian monoid obtained by stabilization from the system  $\{\text{Rep}^{U(n)}(\mathcal{I})\}$ .

**6.7. Equivariant cohomology.** Let  $\Gamma$  be a compact Lie group and  $X$  a  $\Gamma$ -CW-complex. By [18], Theorem A, one has isomorphisms

$$\text{Bun}_\Gamma^{S^1}(X) \xrightarrow{\cong} [X_\Gamma, BS^1] \approx H_\Gamma^2(X), \quad (16)$$

where  $H_\Gamma^*(X) = H_\Gamma^*(X; \mathbb{Z})$  denotes the equivariant cohomology. If  $(X, \pi, \varphi)$  is a split  $\Gamma$ -space over a CW-complex  $A$ , then the projection  $\pi$  descends to a map  $\bar{\pi}: X_\Gamma \rightarrow A$ . We denote by  $X^{(i)}$  the part of  $X$  above the  $i$ -skeleton of  $A$  and by  $r_i: H_\Gamma^*(X) \rightarrow H_\Gamma^*(X^{(i)})$  the restriction homomorphism induced by the inclusion  $X^{(i)} \subset X$ .

**Proposition 6.8.** *Let  $\Gamma$  be topological group and  $A$  be a CW-complex. Let  $\mathcal{I}$  be a proper  $(\Gamma, A)$ -groupoid and  $(X, \pi, \varphi)$  be a split  $\Gamma$ -space with isotropy groupoid  $\mathcal{I}$ . Then*

- (a) *the sequence  $0 \rightarrow H^2(A) \xrightarrow{\bar{\pi}^*} H_\Gamma^2(X) \xrightarrow{r_0} H_\Gamma^2(X^{(0)})$  is exact;*
- (b) *the two restriction homomorphisms*

$$r_0: H_\Gamma^2(X) \rightarrow H_\Gamma^2(X^{(0)}) \quad \text{and} \quad r_{10}: H_\Gamma^2(X^{(1)}) \rightarrow H_\Gamma^2(X^{(0)})$$

*have the same image.*

*Proof.* The map  $\bar{\pi}$  admits a section  $\bar{\varphi}: A \rightarrow X_\Gamma$  coming from  $\varphi$ . Therefore,  $\bar{\pi}^*: H^*(A) \rightarrow H_\Gamma^*(X)$  is injective.

One has  $H_\Gamma^2(X^{(0)}) \approx \text{SBun}^{S^1} X^{(0)} \approx \text{Rep}_{\text{cell } s}^G(\mathcal{I}^{(0)})$ . The composed homomorphism  $\text{Bun}^{S^1}(A) \approx H^2(A) \xrightarrow{\bar{\pi}^*} H_\Gamma^2(X) \xrightarrow{r_0} H_\Gamma^2(X^{(0)}) \approx \text{Rep}_{\text{cell } s}^G(\mathcal{I}^{(0)})$  sends an  $S^1$ -bundle  $\xi$  over  $A$  to the isotropy representation of  $\pi^*\xi$ , which is trivial. Thus,  $r_0 \circ \bar{\pi}^* = 0$ . Using Proposition 3.5, one has an isomorphism  $H_\Gamma^2(X) \approx \text{Rep}_{\text{cell } s}^G(\mathcal{I}) \times H^2(A)$ . The remainder of (a) and (b) follow from Proposition 5.1.  $\square$

We now specialise to  $\Gamma$  being a torus  $\mathbb{T}$ , with Lie algebra  $\mathfrak{t}$ , and use the definitions and notations of Section §5C. If  $\mathcal{I}$  is 0-toric, we have from equation (12), that

$$H_{\mathbb{T}}^2(X^{(0)}) \approx \text{Rep}_{\text{cell } s}^G(\mathcal{I}^{(0)}) \approx \prod_{v \in \Lambda_0} \mathfrak{t}^*$$

by using Proposition 6.8 and its proof, together with Proposition 5.4.

**Proposition 6.9.** *Let  $\mathcal{I}$  be a 1-toric cellular  $(\mathbb{T}, A)$ -groupoid and let  $(X, \pi, \varphi)$  be a split  $\mathbb{T}$ -space with isotropy groupoid  $\mathcal{I}$ . The image of  $r : H_{\mathbb{T}}^2(X) \rightarrow \prod_{v \in \Lambda_0} \mathfrak{l}^*$  is the set of elements  $(a_v)_{v \in \Lambda_0}$  satisfying the GKM-condition.*

**Remark 6.10.** Let  $X$  be as in Proposition 6.9. Suppose that  $X$  is *equivariantly formal*, i.e. the homomorphism  $H_{\mathbb{T}}^*(X) \rightarrow H^*(X)$  induced by the inclusion  $X \subset X_{\mathbb{T}}$  is surjective. In this case, the homomorphism  $H_{\mathbb{T}}^2(X) \xrightarrow{r_0} H_{\mathbb{T}}^2(X^{(0)})$  is injective and Part (b) of Proposition 6.8 holds, see [9], Theorem 1. The injectivity of  $r_0$  is considered as a “localisation theorem”, see e.g., [10], Theorem 6.3, and Part (b) of Proposition 6.8 is referred to as the “Chang–Skjelbred principle” (historically it occurred in [5], Lemma 2.3, for rational coefficients). But, by Part (a) of Proposition 6.8,  $X$  is equivariantly formal only if  $H^2(A) = 0$ , so our context is different. For complex coefficients, Proposition 6.9 was proven in [10], Theorem 7.2. There  $X$  need not to be split but again must be equivariantly formal.

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