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EXAMPLES OF FREE ACTIONS ON PRODUCTS OF SPHERES

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Abstract

We construct a non-abelian extension Γ of S^1 by $\mathbf{Z}/3 \times \mathbf{Z}/3$, and prove that Γ acts freely and smoothly on $S^5 \times S^5$. This gives new actions on $S^5 \times S^5$ for an infinite family $\mathcal{P}$ of finite 3-groups. We also show that any finite odd-order subgroup of the exceptional Lie group G_2 admits a free smooth action on $S^{11} \times S^{11}$. This gives new actions on $S^{11} \times S^{11}$ for an infinite family $\mathcal{E}$ of finite groups. We explain the significance of these families $\mathcal{P}$, $\mathcal{E}$ for the general existence problem, and correct some mistakes in the literature.

Introduction

In this paper, we construct some new examples of smooth, free, finite group actions on a product of two spheres of the same dimension. A necessary condition discovered by Conner [13] is that G has rank at most two, meaning that G does not contain an elementary abelian subgroup of order p^3, for any prime p.

QUESTION *What group theoretic conditions characterize the rank two finite groups which can act freely and smoothly on $S^n \times S^n$, for some $n \geqq 1$?*

It was shown by Oliver [20] that the alternating group A_4 of order 12 has rank 2, but does not admit such an action, so the rank 2 condition is not sufficient. It was also observed by Adem–Smith [2, p. 423] that A_4 is a subgroup of every rank 2 non-abelian simple group, so all these are ruled out too.

In order to answer this question, it is useful to have more examples. In this note, we present two new infinite families of such actions. Let Γ be the Lie group given by the following presentation

$$\Gamma = \langle a, b, z \mid z \in S^1, a^3 = b^3 = [a, z] = [b, z] = 1, [a, b] = \omega \rangle,$$

where $[x, y] = x^{-1}y^{-1}xy$ and $\omega = e^{2\pi i/3}$ in $S^1 \subseteq \mathbb{C}$. We make an explicit equivariant glueing construction to prove our first result.

THEOREM A *The group Γ acts freely and smoothly on $S^5 \times S^5$.*

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For a positive integer $k \geq 3$, let $P(k)$ be the group of order 3^k given by the following presentation

$$P(k) = \left\langle a, b, c \mid a^3 = b^3 = c^{3^{k-2}} = [a, c] = [b, c] = 1, [a, b] = c^{3^{k-3}} \right\rangle.$$

We will write

$$\mathcal{P} = \{P(k) \mid k \geq 3\}$$

and note that $\mathcal{P}$ is a collection of subgroups of Γ (take $c = \mathrm{e}^{2\pi i/3^{k-2}} \in S^1$). Therefore, Theorem A constructs free smooth $P(k)$-actions on $S^5 \times S^5$ for all $k \geq 3$. Note that $P(3) \cong (\mathbf{Z}/3 \times \mathbf{Z}/3) \rtimes \mathbf{Z}/3$ is the extraspecial 3-group of order 27 and exponent 3.

We prove our second result by using equivariant surgery theory to modify a construction based on the exceptional Lie group G_2 of dimension 14.

Theorem B *All odd-order finite subgroups of G_2 act freely and smoothly on $S^{11} \times S^{11}$.*

Information about the finite subgroups of G_2 can be found in [**12**]. Here is a specific family of examples. For a prime number p, let $E(p)$ be the group of order $3p^2$ given by the following presentation

$$E(p) = \left\langle u, v, w \mid u^p = v^p = w^3 = [u, v] = 1, [u, w] = u^{-2}v^{-1}, [v, w] = uv^{-1} \right\rangle.$$

We will write

$$\mathcal{E} = \{E(p) \mid p \text{ is an odd prime}\}.$$

The group $E(2)$ is isomorphic to the alternating group A_4 of order 12, and the group $E(3)$ is another presentation for the extraspecial group $P(3)$. An explicit isomorphism $P(3) \cong E(3)$ is given by the map

$$a \mapsto w, \quad b \mapsto vu \quad \text{and} \quad c \mapsto v^{-1}u.$$

The groups $E(p)$ are all subgroups of $\mathrm{SU}(3)$, and hence contained in the exceptional Lie group G_2. For $p = 3$, let $\omega = \mathrm{e}^{2\pi i/3}$ and consider the representation of $P(3)$ as follows:

$$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix}.$$

For $p \neq 3$, define $\alpha = \mathrm{e}^{2\pi i/p}$ and $\beta = \mathrm{e}^{2\pi i(p-2)/p}$ and consider a representation of $E(p)$ as follows:

$$u = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad v = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Therefore, Theorem B proves the existence of free smooth $E(p)$-actions on $S^{11} \times S^{11}$, for all odd primes p.

We introduce one more family of 3-groups

$$B(k,\epsilon) = \left\langle a, b, c \mid a^3 = b^3 = c^{3^{k-2}} = [b,c] = 1, [a,c] = b, [a,b] = c^{\epsilon 3^{k-3}} \right\rangle,$$

where $k \geq 4$, and ϵ is 1 or -1. One can check that $B(k,\epsilon)$ is not a subgroup of SU(3) for $k > 4$ or $\epsilon = 1$. However, the group $B(4,-1)$ is a subgroup of SU(3) by the following representation

$$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma^3 & 0 \\ 0 & 0 & \gamma^6 \end{bmatrix}, \quad c = \begin{bmatrix} \gamma^5 & 0 & 0 \\ 0 & \gamma^8 & 0 \\ 0 & 0 & \gamma^5 \end{bmatrix}$$

where $\gamma = e^{2\pi i/9}$. Therefore, Theorem B shows that $B(4,-1)$ acts freely and smoothly on $S^{11} \times S^{11}$.

In section 3, we make some concluding remarks about finite 3-groups and the role of the families $\mathcal{P}$ and $\mathcal{E}$ in the general existence problem.

1. An explicit construction

The idea of the construction is to start with a non-free action of Γ on $S^5 \times S^5$ and do an equivariant 'cut-and-paste' operation on it to get rid of the fixed points. This is an equivariant surgery construction, but none of the theory of equivariant surgery is needed: the proof of Theorem A just involves checking some explicit formulas.

For the initial action on $S^5 \times S^5$, the singular set is contained in a Γ-invariant disjoint union U of six copies of $S^1 \times D^4 \times S^5$. We replace this part by a new free action on U, which is Γ-equivariantly diffeomorphic to the original one on its boundary. We will use the following four representations of Γ in our construction.

(1) An irreducible representation $\varphi : \Gamma \to U(3)$:

$$a \longmapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad b \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad z \longmapsto \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix}.$$

(2) Three representations that pullback from representations of Γ/S^1:

(a) $\psi_0 : \Gamma \to U(3)$ given by:

$$a \longmapsto \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix}, \quad b \longmapsto \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad z \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) $\psi_1 : \Gamma \to U(3)$ given by:

$$a \longmapsto \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix}, \quad b \longmapsto \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad z \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) $\psi_2 : \Gamma \to U(3)$ given by:

$$a \longmapsto \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix}, \quad b \longmapsto \begin{bmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad z \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These representations give an action $\Phi : \Gamma \times Y \to Y$ on $Y = S^5$ given by

$$\Phi(g, \mathbf{z}) = \varphi(g)\mathbf{z},$$

where $\mathbf{z} = (z_1, z_2, z_3) \in S^5$, with $z_i \in \mathbb{C}$ and $\|\mathbf{z}\| = 1$.

Definition 1.1 (Model actions on $S^5 \times S^5$) For $i = 0, 1$ or 2, we obtain an action $\Phi_i : \Gamma \times X_i \to X_i$ on $X_i = S^5 \times S^5$ given by:

$$\Phi_i(g, (\mathbf{z}, \mathbf{w})) = (\varphi(g)\mathbf{z}, \psi_i(g)\mathbf{w}),$$

where $\mathbf{z}, \mathbf{w} \in S^5$.

To simplify our notations, we let $\Phi(g, \mathbf{z}) = g \cdot \mathbf{z}$ and $\Phi_i(g, (\mathbf{z}, \mathbf{w})) = g \cdot (\mathbf{z}, \mathbf{w})$, for any $\mathbf{z} \in Y$ and $(\mathbf{z}, \mathbf{w}) \in X_i$.

Remark 1.2 We will modify the initial action (X_0, Φ_0) by 'equivariant Dehn surgery' to obtain a free Γ-action on $S^5 \times S^5$, with replacement pieces coming from (X_1, Φ_1) and (X_2, Φ_2).

For $i = 0, 1$ or 2, we define a Γ-equivariant map

$$p_i : X_i \to Y \text{ given by } p_i(\mathbf{z}, \mathbf{w}) = \mathbf{z}.$$

Note that p_i is in fact a Γ-equivariant sphere bundle map. Fix $0 < \varepsilon < 1/9$, and define three subspaces V_1, V_2 and V_0 of Y as follows:

$$V_1 = \{a^k \cdot \mathbf{z} \in Y \mid 0 \le k \le 2, |z_2|^2 + |z_3|^2 \le \varepsilon\}, \quad V_2 = PV_1,$$

where

$$P = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \omega & 1 \\ 1 & 1 & \omega \\ \omega & 1 & 1 \end{bmatrix} \in U(3).$$

Note that $P\varphi(a)P^{-1} = \varphi(a)$ and $P\varphi(b)P^{-1} = \varphi(a^2 b)$, and let V_0 be the closure of $Y - V_1 \cup V_2$.

Lemma 1.3 $V_1 \cap V_2 = \emptyset$.

Proof. Suppose $\mathbf{z} \in V_1 \cap V_2$. Then there exists $\mathbf{z}' \in V_1$ such that $\mathbf{z} = P\mathbf{z}'$, since $\mathbf{z} \in V_2$. So there exists $i \neq j \in \{1, 2, 3\}$ such that $|z'_i|^2 + |z'_j|^2 \le \varepsilon$, since $\mathbf{z}' \in V_1$. Let $\{k\} = \{1, 2, 3\} - \{i, j\}$. Then for any q in $\{1, 2, 3\}$, we have $|z_q|^2 \ge 1/3(|z'_k|^2 - |z'_i|^2 - |z'_j|^2) \ge (1/3) - \varepsilon$. Therefore, any sum $|z_q|^2 + |z_r|^2 \ge (2/3) - 2\varepsilon > \varepsilon$, in contradiction to the condition $\mathbf{z} \in V_1$.

LEMMA 1.4 *The inclusions $t_i : V_i \to Y$ give Γ-equivariant subspaces of Y.*

Proof. Assume $1 \le i \le 2$. Take any $\mathbf{w}$ in V_i, there exists unique $k \in \{0, 1, 2\}$ and $\mathbf{z}$ in V_1 with $|z_2|^2 + |z_3|^2 \le \varepsilon$ such that

$$\mathbf{w} = P^{i-1}\varphi(a^k)\mathbf{z}.$$

Hence, $\varphi(a)\mathbf{w} = P^{i-1}\varphi(a^{k+1})\mathbf{z}$ is in V_i and for $\lambda \in S^1$, $\varphi(\lambda)\mathbf{w} = P^{i-1}\varphi(a^k)\varphi(\lambda)\mathbf{z}$ is in V_i as $|\lambda z_2|^2 + |\lambda z_3|^2 \le \varepsilon$. We have

$$\varphi(b)P^{i-1}\varphi(a^k) = P^{i-1}\varphi(a^{-2(i-1)})\varphi(b)\varphi(a^k) = P^{i-1}\varphi(a^{k+i-1})\varphi(b)\varphi(\omega^{-k}). \tag{1}$$

Hence for $i = 1$, $\varphi(b)\mathbf{w} = \varphi(a^k)\varphi(b)\varphi(\omega^{-k})\mathbf{z}$ is in V_i as $|\omega^{-k+1}z_2|^2 + |\omega^{-k+2}z_3|^2 \le \varepsilon$. For $i = 2$, $\varphi(b)\mathbf{w} = P\varphi(a^{k+1})\varphi(b)\varphi(\omega^{-k})\mathbf{z}$ is in V_i as above. Hence the lemma is proved for $i = 1$ and $i = 2$. For $i = 0$, it follows from the definition of V_0.

REMARK 1.5 Observe that each of the subspaces V_1 or V_2 is diffeomorphic to the disjoint union of three copies of $S^1 \times D^4$, since the subset $\{\mathbf{z} \in S^5 : |z_2|^2 + |z_3|^2 \le \varepsilon\} = S^1 \times D^4$.

Now define a subspace $U_i \subset X_i$ for $i = 0, 1$ or 2 by the following Γ-equivariant pullback diagram:

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & X_i \arrow[d, "p_i"] \\ V_i \arrow[r, "t_i"] & Y \end{tikzcd}$$

LEMMA 1.6 *The Γ-action on U_i is free for $i \in \{0, 1, 2\}$.*

Proof. Take two subsets of Γ as follows:

$$A_1 = \left\{b^k z \mid 1 \le k \le 2, z \in S^1\right\},$$
$$A_2 = \left\{a^k b^{-k} z \mid 1 \le k \le 2, z \in S^1\right\}.$$

All elements of Γ except $A_1 \cup A_2$ act freely on X_0. But all the fixed point sets of elements of A_i are in $p_0^{-1}(V_i - \partial V_i)$ for $i \in \{1, 2\}$. Hence Γ acts freely on U_0. Now for any $i \in \{1, 2\}$, all elements of Γ except A_i act freely on V_i, but all the elements of A_i act freely on X_i. Hence Γ acts freely on U_i.

REMARK 1.7 Since U_i is an S^5-bundle over V_i, the subspace $U = U_1 \cup U_2$ is diffeomorphic to a disjoint union of six copies of $S^1 \times D^4 \times S^5$.

LEMMA 1.8 *There is a Γ-equivariant isomorphism $\alpha : \partial U_0 \to \partial U_1 \cup \partial U_2$ as Γ-equivariant 5-dimensional sphere bundles over $\partial V_0 = \partial V_1 \cup \partial V_2$ with structure group $U(3)$.*

Proof. For $m = 1$ and 2, we have

$$\partial V_m = \{P^{m-1}\varphi(a^k)\mathbf{z} \in Y \mid 0 \leq k \leq 2, |z_2|^2 + |z_3|^2 = \varepsilon\},$$

and $\partial V_0 = \partial V_1 \cup \partial V_2$. This means that there is a unique way to write every element of ∂U_0 in the following standard form

$$(P^{m-1}\varphi(a^k)\mathbf{z}, \mathbf{w}),$$

where $m \in \{1, 2\}$, $k \in \{0, 1, 2\}$ and $|z_2|^2 + |z_3|^2 = \varepsilon$. In addition, $\partial U_n = \partial V_n \times S^5$, for $n = 0, 1$ and 2, with Γ-action given by $g \cdot (\mathbf{z}, \mathbf{w}) = (\varphi(g)\mathbf{z}, \psi_i(g)\mathbf{w})$. We define an isomorphism

$$\alpha : \partial U_0 \to \partial U_1 \cup \partial U_2$$

given by

$$\alpha(P^{m-1}\varphi(a^k)\mathbf{z}, \mathbf{w}) = \left(P^{m-1}\varphi(a^k)\mathbf{z}, \Theta_m(\mathbf{z})\mathbf{w}\right),$$

where

$$\Theta_1(\mathbf{z}) = \frac{1}{\sqrt{\varepsilon(1-\varepsilon)}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{z}_1 z_2 & -\bar{z}_1 z_3 \\ 0 & z_1 \bar{z}_3 & z_1 \bar{z}_2 \end{bmatrix} \in \mathrm{SU}(3),$$

$$\Theta_2(\mathbf{z}) = \frac{1}{\sqrt{\varepsilon(1-\varepsilon)}} \begin{bmatrix} \bar{z}_1 z_2 & -z_1 \bar{z}_3 & 0 \\ \bar{z}_1 z_3 & z_1 \bar{z}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SU}(3).$$

Now it is clear that α is an isomorphism. We just have to check that it is Γ-equivariant.

First, check that α is equivariant under a:

$$\begin{aligned}
\alpha\left(a \cdot \left(P^{m-1}\varphi(a^k)\mathbf{z}, \mathbf{w}\right)\right) &= \alpha\left(\varphi(a)P^{m-1}\varphi(a^k)\mathbf{z}, \psi_0(a)\mathbf{w}\right) \\
&= \alpha\left(P^{m-1}\varphi(a^{k+1})\mathbf{z}, \psi_0(a)\mathbf{w}\right) = \left(P^{m-1}\varphi(a^{k+1})\mathbf{z}, \Theta_m(\mathbf{z})\psi_0(a)\mathbf{w}\right) \\
&= \left(\varphi(a)P^{m-1}\varphi(a^k)\mathbf{z}, \psi_m(a)\Theta_m(\mathbf{z})\mathbf{w}\right) = a \cdot \alpha\left(P^{m-1}\varphi(a^k)\mathbf{z}, \mathbf{w}\right).
\end{aligned}$$

Secondly, check that α is equivariant under b:

$$\begin{aligned}
\alpha\left(b \cdot \left(P^{m-1}\varphi(a^k)\mathbf{z}, \mathbf{w}\right)\right) &= \alpha\left(\varphi(b)P^{m-1}\varphi(a^k)\mathbf{z}, \psi_0(b)\mathbf{w}\right) \\
&= \alpha\left(P^{m-1}\varphi(a^{k+m-1})\varphi(b)\varphi(\omega^{-k})\mathbf{z}, \psi_0(b)\mathbf{w}\right), \text{ by formula (1)}, \\
&= \alpha\left(P^{m-1}\varphi(a^{k+m-1})\begin{bmatrix} \omega^{-k}z_1 \\ \omega^{-k+1}z_2 \\ \omega^{-k+2}z_3 \end{bmatrix}, \psi_0(b)\mathbf{w}\right) = (\star).
\end{aligned}$$

For $m = 1$, we have

$$\begin{aligned}
(\star) &= \left(\varphi(a^k) \begin{bmatrix} \omega^{-k} z_1 \\ \omega^{-k+1} z_2 \\ \omega^{-k+2} z_3 \end{bmatrix}, \frac{1}{\sqrt{\varepsilon(1-\varepsilon)}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{z}_1 \omega z_2 & -\bar{z}_1 \omega^2 z_3 \\ 0 & z_1 \omega \bar{z}_3 & z_1 \omega^2 \bar{z}_2 \end{bmatrix} \psi_0(b) \mathbf{w} \right) \\
&= \left(\varphi(b)\varphi(a^k)\mathbf{z}, \Theta_1(\mathbf{z}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \psi_0(b)\mathbf{w} \right) = \left(\varphi(b)\varphi(a^k)\mathbf{z}, \Theta_1(\mathbf{z})\psi_1(b)\mathbf{w} \right) \\
&= \left(\varphi(b)\varphi(a^k)\mathbf{z}, \psi_1(b)\Theta_1(\mathbf{z})\mathbf{w} \right) = b \cdot \alpha \left(\varphi(a^k)\mathbf{z}, \mathbf{w} \right).
\end{aligned}$$

For $m = 2$, we have

$$\begin{aligned}
(\star) &= \left(P\varphi(a^{k+1}) \begin{bmatrix} \omega^{-k} z_1 \\ \omega^{-k+1} z_2 \\ \omega^{-k+2} z_3 \end{bmatrix}, \frac{1}{\sqrt{\varepsilon(1-\varepsilon)}} \begin{bmatrix} \bar{z}_1 \omega z_2 & -z_1 \omega \bar{z}_3 & 0 \\ \bar{z}_1 \omega^2 z_3 & z_1 \omega^2 \bar{z}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \psi_0(b) \mathbf{w} \right) \\
&= \left(\varphi(b) P\varphi(a^k)\mathbf{z}, \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Theta_2(\mathbf{z})\psi_0(b)\mathbf{w} \right) \\
&= \left(\varphi(b) P\varphi(a^k)\mathbf{z}, \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \psi_0(b)\Theta_2(\mathbf{z})\mathbf{w} \right) \\
&= \left(\varphi(b) P\varphi(a^k)\mathbf{z}, \psi_2(b)\Theta_2(\mathbf{z})\mathbf{w} \right) = b \cdot \alpha \left(P\varphi(a^k)\mathbf{z}, \mathbf{w} \right).
\end{aligned}$$

Thirdly, check that α is equivariant under $\lambda \in S^1$:

$$\begin{aligned}
\alpha \left(\lambda \cdot \left(P^{m-1}\varphi(a^k)\mathbf{z}, \mathbf{w} \right) \right) &= \alpha \left(\varphi(\lambda) P^{m-1}\varphi(a^k)\mathbf{z}, \psi_0(\lambda)\mathbf{w} \right) \\
&= \alpha \left(P^{m-1}\varphi(a^k)\lambda\mathbf{z}, \mathbf{w} \right) = \left(P^{m-1}\varphi(a^k)\lambda\mathbf{z}, \Theta_m(\mathbf{z})\mathbf{w} \right) \\
&= \left(\varphi(\lambda) P^{m-1}\varphi(a^k)\mathbf{z}, \psi_m(\lambda)\Theta_m(\mathbf{z})\mathbf{w} \right) = \lambda \cdot \alpha \left(P^{m-1}\varphi(a^k)\mathbf{z}, \mathbf{w} \right).
\end{aligned}$$

Proof of Theorem A. Define a new space X by the following pushout diagram:

$$\begin{array}{ccc}
\partial U_0 \cong \partial U_1 \cup \partial U_2 & \longrightarrow & U_1 \cup U_2 \\
\big\downarrow & & \big\downarrow \\
U_0 & \longrightarrow & X
\end{array}$$

where the isomorphism α from Lemma 1.8 is used to make the identification $\partial U_0 \cong \partial U_1 \cup \partial U_2$. The above pushout diagram can be considered in the category of Γ-equivariant 5-dimensional sphere bundles with the structure group $U(3)$. Hence we see that Γ acts freely on X because the action of

Γ on $U_1 \cup U_2$ and on U_0 are both free. In addition, the base spaces of these bundles are given by the following pushout diagram:

$$\begin{tikzcd} \partial V_0 = \partial V_1 \cup \partial V_2 \arrow[r] \arrow[d] & V_1 \cup V_2 \arrow[d] \\ V_0 \arrow[r] & Y \end{tikzcd}$$

Hence X is a 5-dimensional sphere bundle over $Y = S^5$ with structure group $U(3)$. But $\pi_4(U(3)) = 0$. Hence $X = S^5 \times S^5$.

2. Proof of Theorem B

Let E denote any finite odd-order subgroup of the exceptional Lie group G_2. To construct a free E-action on $S^{11} \times S^{11}$, we start with the free E-action on G_2 given by left multiplication. Now consider the principal fibre bundle

$$S^3 = \mathrm{SU}(2) \to G_2 \to G_2/\mathrm{SU}(2) = V_2(\mathbb{R}^7)$$

with structure group SU(2) over the Stiefel manifold $V_2(\mathbb{R}^7)$. This fibre bundle can be identified with the sphere bundle of an associated 2-dimensional complex vector bundle ξ. By construction, the space

$$Z(\xi) = G_2 \times_{\mathrm{SU}(2)} \mathbb{C}^2$$

is the total space of the vector bundle ξ, where SU(2) acts on $\mathbb{C}^2$ via the standard representation, and freely off the zero section. It follows that the group G_2 acts on $Z(\xi)$ through left multiplication, and freely off the zero section. We therefore obtain a free smooth G_2-action on the total space Y of the sphere bundle

$$S^{11} \to Y \to V_2(\mathbb{R}^7)$$

of the complex vector bundle $\xi \oplus \xi \oplus \xi$. This action can be restricted to any finite subgroup of G_2, but the equivariant surgery construction given below to obtain a free action on $S^{11} \times S^{11}$ is valid only for the odd order subgroups E of G_2.

LEMMA 2.1 *Y is a smooth, closed, parallelizable manifold diffeomorphic to $S^{11} \times V_2(\mathbb{R}^7)$.*

Proof. Consider the fibre bundle

$$\mathrm{SU}(3)/\mathrm{SU}(2) \to G_2/\mathrm{SU}(2) \to G_2/\mathrm{SU}(3)$$

which is equivalent to

$$S^5 \to V_2(\mathbb{R}^7) \to S^6.$$

By [8, Prop. 7.5], the tangent bundle along the fibers of the total space $V_2(\mathbb{R}^7)$ is equivalent to ξ after adding a trivial line bundle. It is known that the total space $V_2(\mathbb{R}^7)$ is parallelizable [9, Corollary], and the tangent bundle of the base S^6 is stably trivial. Therefore, ξ is stably trivial over $V_2(\mathbb{R}^7)$, which means that the 12-plane bundle $\xi \oplus \xi \oplus \xi$ is trivial over $V_2(\mathbb{R}^7)$ as the dimension of $V_2(\mathbb{R}^7)$ is 11. This proves Y is diffeomorphic to $S^{11} \times V_2(\mathbb{R}^7)$. We also know that the tangent bundle of S^{11} is stably trivial, hence Y parallelizable.

LEMMA 2.2 *Y is 4-connected and has the integral homology of $S^{11} \times S^{11}$, except for the groups $H_5(Y; \mathbf{Z}) = H_{16}(Y; \mathbf{Z}) = \mathbf{Z}/2$.*

Proof. The proof is easy using Lemma 2.1 and the fact that $V_2(\mathbb{R}^7)$ is 4-connected, with integral homology given as follows

$$H_q(V_2(\mathbb{R}^7)) = \begin{cases} \mathbf{Z} & \text{if } q = 0 \text{ or } q = 11 \\ \mathbf{Z}/2 & \text{if } q = 5 \\ 0 & \text{otherwise} \end{cases} .$$

We will now show how to perform E-equivariant framed surgery on Y to obtain a free E-action on $S^{11} \times S^{11}$. In the successive steps, we remove the interior of an equivariant framed embedding of $E \times S^k \times D^{22-k}$ and attach $E \times D^{k+1} \times S^{22-k-1}$ along their common boundaries.

This is an equivariant version of the original spherical modification construction of Milnor [16, 19] which formed the starting point for surgery theory, as developed by Browder, Novikov, Sullivan and Wall (see [27] or the short overview in [14, §7]). We remark that non-simply connected surgery is carried out equivariantly in the universal covering of a manifold, where the equivariance is with respect to the action of the fundamental group as deck transformations.

In order to carry out E-equivariant framed surgery on Y, we will need a partial equivariant trivialization of the normal bundle of Y to produce the framings. Let $X = Y/E$ and ν_X be the classifying map of the stable normal bundle of X. Since Y is 4-connected by Lemma 2.2, we can construct the classifying space BE by adding k-cells to X for $k > 5$. Let $B = BE^{(12)} \cup X$, where $BE^{(12)}$ denotes the 12-skeleton of BE. We have a pullback diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & \widetilde{B} \arrow[d] \\ X \arrow[r, hook] & B \end{tikzcd}$$

of universal coverings. The assumption that E has odd order will now be used for the first time.

LEMMA 2.3 *Since E has odd order, the normal bundle $\nu_X : X \to BSO$ is the restriction of a bundle $\nu : B \to BSO$.*

Proof. The successive obstructions to extending the classifying map $\nu_X : X \to BSO$ of the stable normal bundle of X to a map from B to BSO lie in the groups

$$H^k(B, X; \pi_{k-1}(BSO))$$

for $k \geq 6$. We claim that these obstructions vanish since E has odd order. For $6 \leq k \leq 7$, we have $\pi_{k-1}(BSO) = 0$. For $8 \leq k \leq 11$, by considering Lemma 2.2 and the cohomology long exact sequence of the pair (B, X) with coefficients in any abelian group A, we get $H^k(B, X; A) = 0$. Finally for $k = 12$, we have $\pi_{11}(BSO) = 0$, so we may extend ν_X over B.

Let $B' = BE^{(11)} \cup X \subseteq B$, and still denote the restriction of ν to B' by ν.

LEMMA 2.4 *The pullback $\tilde{\nu}$ of ν by the map $\widetilde{B}' \to B'$ is trivial.*

Proof. The normal bundle ν_Y of Y is trivial, hence it is enough to extend a null homotopy of the map ν_Y to a null homotopy of $\tilde{\nu}$. The successive obstructions for this extension problem lie in the groups

$$H^k(\widetilde{B}', Y; \pi_k(BSO))$$

for $k \geq 6$. We claim that these obstructions also vanish. For $6 \leq k \leq 7$, we have $\pi_k(BSO)) = 0$. For $8 \leq k \leq 10$, by considering Lemma 2.2 and the cohomology long exact sequence of the pair $(\widetilde{B}', Y)$ with coefficients in any abelian group A, we get $H^k(\widetilde{B}', Y; A) = 0$. Since $\pi_{11}(BSO) = 0$, we are done.

Let $\mathbf{H}(L)$ denote the standard skew-hermitian hyperbolic form on the module $L \oplus L^*$. The following uses surgery below the middle dimension, a standard procedure in surgery theory [**16**, §5; **27**, Chap. 1].

LEMMA 2.5 *After preliminary surgeries on X, we can obtain a smooth manifold M with the following properties*:

(1) *$\widetilde{M}$ is stably parallelizable.*
(2) *The classifying map $c : M \to BE$ induces an isomorphism $\pi_1(M) \cong E$.*
(3) *$\pi_i(M) = 0$ for $1 < i < 11$.*
(4) *The intersection form*

$$(\pi_{11}(M), s_M) \cong \mathbf{H}(\mathbf{Z}) \perp (F, \lambda)$$

for some non-singular skew-hermitian form λ on a finitely generated free $\mathbf{Z}E$-module F.

Proof. Lemma 2.3 gives a bundle $\nu : B' \to BSO$. We will perform a sequence of surgeries over (B', ν), so that in particular the bundle ν pulls back to the stable normal bundle of the trace of the surgeries. By Lemma 2.4, the resulting manifold M at any stage of these surgeries has universal covering $\widetilde{M}$ stably parallelizable.

The first step is surgery to kill a generator of $\pi_5(X) = \mathbf{Z}/2$. We use the short exact sequence

$$0 \to \langle 2, I \rangle \to \mathbf{Z}E \to \mathbf{Z}/2 \to 0$$

of $\mathbf{Z}E$-modules, where I denotes the augmentation ideal of $\mathbf{Z}E$, to keep track of the effect of the first step of the E-equivariant framed surgery on Y. The result of the first step is a manifold M such that

$\pi_6(M) = \langle 2, I\rangle$. We have a short exact sequence

$$0 \to \mathbf{Z}E \to \langle 2, N\rangle \to \mathbf{Z}/2 \to 0,$$

where the module $\langle 2, N\rangle$ is projective over $\mathbf{Z}E$ since E has odd order [24, §6]. Now Schanuel's lemma shows that

$$\langle 2, N\rangle \oplus \langle 2, I\rangle = \mathbf{Z}E \oplus \mathbf{Z}E$$

is free over $\mathbf{Z}E$, so $\langle 2, I\rangle$ is a finitely generated projective $\mathbf{Z}E$-module with stable inverse $\langle 2, N\rangle$. The effect of the subsequent surgeries to make $\widetilde{M}$ highly connected is just to replace a projective module $\pi_i(M) = Q$ at each step with its stable inverse $\pi_{i+1}(M') = Q'$, for $i < 10$. At the last of these steps, where we eliminate $\pi_{10}(M)$, the result is an expression

$$(\pi_{11}(M), s_M) \cong \mathbf{H}(\mathbf{Z}) \perp (P, \lambda'),$$

where (P, λ') is a non-singular skew-hermitian form on $P = Q \oplus Q^*$, and $Q \cong \langle 2, N\rangle$. The projective modules $\langle r, N\rangle$, for r prime to $|E|$, generate the Swan subgroup $T(\mathbf{Z}E) \subseteq \widetilde{K}_0(\mathbf{Z}E)$ of the projective class group. Now Swan [24, Lemma 6.1] proved that

$$\mathbf{Z} \oplus \langle r, N\rangle \cong \mathbf{Z} \oplus \mathbf{Z}E$$

for any r prime to $|E|$ and that

$$\langle 2, N\rangle \oplus \langle r, N\rangle \cong \mathbf{Z}E \oplus \mathbf{Z}E$$

if $2r \equiv 1 \pmod{|E|}$. After surgery on a null-homotopic 10-sphere in M, we obtain $M' = M\#(S^{11} \times S^{11})$, whose equivariant intersection form is

$$(\pi_{11}(M'), s_{M'}) \cong \mathbf{H}(\mathbf{Z}) \perp (P, \lambda') \perp \mathbf{H}(\mathbf{Z}E).$$

However, note that

$$\mathbf{H}(\mathbf{Z}) \perp \mathbf{H}(\mathbf{Z}E) = \mathbf{H}(\mathbf{Z} \oplus \mathbf{Z}E) \cong \mathbf{H}(\mathbf{Z} \oplus \langle r, N\rangle) = \mathbf{H}(\mathbf{Z}) \perp \mathbf{H}(\langle r, N\rangle).$$

Now $(F, \lambda) := \mathbf{H}(\langle r, N\rangle) \perp (P, \lambda')$ is a non-singular skew-hermitian form on a finitely generated free $\mathbf{Z}E$-module.

We next observe that the equivariant intersection form $(\pi_{11}(M), s_M)$ has a quadratic refinement $\mu : \pi_{11}(M) \to \mathbf{Z}E/\{\nu + \bar{\nu}\}$, in the sense of [27, Theorem 5.2]. Since E has odd order, this follows because the universal covering $\widetilde{M}$ has stably trivial normal bundle. We therefore obtain an element (F, λ, μ) of the surgery obstruction group (see [27, p. 49] for the essential definitions). In the splitting $(\pi_{11}(M), s_M, \mu) = H(\mathbf{Z}) \perp (F, \lambda, \mu)$, we may assume that the Arf invariant of the summand $H(\mathbf{Z})$ is zero. This follows by construction, since the preliminary surgeries can be done away from an embedded sphere

$$S^{11} \times * \subset S^{11} \times V_2(\mathbb{R}^7) = Y$$

with trivial normal bundle. We need to check the discriminant of the form (F, λ, μ).

LEMMA 2.6 *We obtain an element*

$$(F, \lambda, \mu) \in L'_2(\mathbf{Z}E)$$

of the weakly simple surgery obstruction group.

Proof. A non-singular, skew-hermitian quadratic form (F, λ, μ) represents an element in $L'_2(\mathbf{Z}E)$ provided that its discriminant lies in $\ker(\mathrm{Wh}(\mathbf{Z}E) \to \mathrm{Wh}(\mathbf{Q}E))$. But the equivariant symmetric Poincaré chain complex $(C(M), \varphi_0)$ is chain equivalent, after tensoring with the rationals $\mathbf{Q}$, to its rational homology complex [**21**, §4]. Therefore, the image of the discriminant of $(\pi_{11}(M) \otimes \mathbf{Q}, s_M)$ equals the image of the torsion of φ_0, which vanishes in $\mathrm{Wh}(\mathbf{Q}E)$ because closed manifolds have simple Poincaré duality [**27**, Theorem 2.1].

Proof of Theorem B. We now have a smooth closed manifold $[M, c]$ whose equivariant intersection form $(\pi_{11}(M), s_M)$ contains (F, λ, μ), as described above. However, since E has odd order, an element in the surgery obstruction group $L'_2(\mathbf{Z}E)$ is zero provided that its multisignature and ordinary Arf invariant both vanish (this is a result of Bak and Wall, see [**26**, Cor. 2.4.3]). The multisignature invariant is trivial since M is a closed manifold [**27**, 13B]. The ordinary Arf invariant of (F, λ, μ) equals the Arf invariant of $\widetilde{M}$, which vanishes since 22 is not of the form $2^k - 2$ (a famous result of Browder [**10**]). We can now do surgery to obtain a representative $[M, c]$ which has $\widetilde{M} = S^{11} \times S^{11} \# \Sigma$, where Σ is a homotopy 22-sphere. Note that the p-component of π_{22}^S is zero for $p \geq 3$ [**22**, p. 5], so we can get the standard smooth structure on $S^{11} \times S^{11}$.

3. Concluding remarks

In this final section, we will make some additional remarks about the group theory, and explain the significance of constructing actions for our families $\mathcal{P}$ and $\mathcal{E}$ of finite groups, as a step towards answering our original question.

(1) Blackburn has given a classification of p-groups of rank 2. Here we restate his result for 3-groups (see Theorem 4.1 in [**7**] and Theorem 3.1 in [**17**]). If G is a rank 2 3-group of order 3^k, then one of the following holds:
   (a) G is a metacyclic 3-group;
   (b) $G = P(k)$, $k \geq 3$, a group in $\mathcal{P}$;
   (c) $G = B(k, \epsilon)$, $k \geq 4$;
   (d) G is a 3-group of maximal class.
   The 3-groups listed in the first item all act freely and smoothly on a product of two equidimensional spheres [**18**, p. 538]. An explicit construction and the proof of Theorem A show that the groups in the second item on this list act freely on $S^5 \times S^5$. Theorem B shows that the group $B(4, -1)$ in the third item also acts freely on a product of two equidimensional spheres, but of dimension $S^{11} \times S^{11}$.

(2) It was shown by Benson and Carlson [**6**, Theorem 4.4] that free actions of a rank 2 group on a product of two equidimensional spheres could not be ruled out by cohomological methods alone. Hence the arguments given for certain non-existence claims in [**3**, **4**, **25**, **28**] about extraspecial p-groups are not valid. In fact, Theorems A and B applied to the extraspecial 3-group $E(3)$ of order 27 and exponent 3 give specific counterexamples to the results claimed in these papers. The possible sphere dimensions for this group $E(3)$, not previously ruled out

by cohomological methods, are of the form $S^{6r-1} \times S^{6r-1}$, and our examples show existence in the first two cases ($r = 1, 2$).

(3) For any prime number p, the group $E(p)$ is a subgroup of G_2, but $E(2) \cong A_4$. Since A_4 is ruled out by [20], Theorem B shows that the group $E(p)$ can act freely and smoothly on a product of two equidimensional spheres if and only if $p > 2$. More information about the odd-order subgroups of G_2 can be found in [12] (the finite subgroups are not all contained in SU(3), but we do not know if this is true for the odd-order subgroups). The result of Oliver [20] was also proved and extended by Carlsson [11] and Silverman [23].

(4) Let G be a group in $\mathcal{P}$ or $\mathcal{E}$. Let $\mathrm{axe}(G)$ be the minimum number of linear representations of G required for G to act freely on a product of spheres where the action on each sphere is induced from one of these representations. By [5, Proposition 3.3], it easy to see that $\mathrm{axe}(G) \geq 3$. Hence G cannot act freely on a product of two spheres, with a linear action on each sphere. Moreover G is not a subgroup of Sp(2), hence the free actions constructed in [1] will not be on a product of two equidimensional spheres. We also remark that G cannot be written as a product of two groups with periodic cohomology, while all the subgroups of G can. So the families $\mathcal{P}$ and $\mathcal{E}$ are two infinite families of minimal new examples not included in [15].

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