INTERSECTION FORMS, FUNDAMENTAL GROUPS AND 4-MANIFOLDS

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Abstract. This is a short survey of some connections between the intersection form and the fundamental group for smooth and topological 4-manifolds.

1. Introduction

A classical construction of Kervaire [36] shows that any finitely-presented group can be realized as the fundamental group of a closed, oriented smooth 4-manifold M. However, much less is known about other homotopy invariants of 4-manifolds, such as the second homotopy group $\pi_2(M)$, which inherits a $\mathbf{Z}[\pi_1(M, x_0)]$-module structure via the action of the deck transformations on the universal covering $\widetilde{M}$.

Another basic invariant is the *equivariant intersection form* of a 4-manifold M, defined as the triple $(\pi_1(M, x_0), \pi_2(M), s_M)$, where $x_0 \in M$ is a base-point, and

$$s_M \colon \pi_2(M) \otimes_{\mathbb{Z}} \pi_2(M) \to \mathbb{Z}[\pi_1(M, x_0)]$$

is the form defined by counting intersections of immersed 2-spheres (see [59, Chap. 5]). This pairing is Λ-hermitian, in the sense that for all $\lambda \in \Lambda := \mathbb{Z}[\pi_1(M, x_0)]$ we have

$$s_M(\lambda \cdot x, y) = \lambda \cdot s_M(x, y) \quad \text{ and } \quad s_M(y, x) = \overline{s_M(x, y)}$$

where $\lambda \mapsto \bar{\lambda}$ is the involution on Λ given by $\bar{g} = g^{-1}$ for $g \in \pi_1(M, x_0)$.

The main topics of interest for the present survey are:

(1) To what extent does the fundamental group $\pi_1(M, x_0)$ and the equivariant intersection form s_M determine the topology of a closed, oriented 4-manifold M ?
(2) What special properties hold for the equivariant interesection form if M is a *smooth* 4-manifold ?

The material will be divided into sections according to the complexity of the fundamental group. From now on, all manifolds considered will be closed, connected and oriented.

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2. Simply-connected 4-manifolds

Wall [58], [57] showed in the 1960's that homotopy equivalent, simply-connected smooth 4-manifolds M_1, M_2 are smoothly h-cobordant, and hence are stably diffeomorphic

$$M_1 \# r(S^2 \times S^2) \cong M_2 \# r(S^2 \times S^2)$$

for some integer $r \geq 0$. It is still not known whether the existence of such a stable diffeomorphism actually requires more than one copy of $S^2 \times S^2$ (see [46], [37] for other aspects of smooth h-cobordisms).

Spectacular results concerning 4-manifolds were proved in the 1980's by S. Donaldson and M. Freedman, building on work of Atiyah, Casson, Hitchin, Taubes and Uhlenbeck. If M is simply-connected, then $\pi_2(M) \cong \mathbf{Z}^r$ is a free abelian group and the ordinary intersection form

$$q_M \colon H_2(M; \mathbf{Z}) \times H_2(M; \mathbf{Z}) \to \mathbf{Z}$$

is a symmetric, unimodular bilinear form. The signature of this form, denoted $\operatorname{sign}(M)$, is the difference between the number of positive and negative eigenvalues of a matrix representing q_M.

Freedman [16], [17] proved that any such form is realized by one or two *topological* 4-manifolds. Moreover, M is classified up to homeomorphism by q_M and the Kirby-Siebenmann invariant $KS(M) \in \mathbb{Z}/2$ (see [38] for the definition). Donaldson [6], [7], [8] showed using gauge theory that if q_M is a positive definite form then

$$q_M \cong \langle 1 \rangle \perp \langle 1 \rangle \perp \cdots \perp \langle 1 \rangle$$

is standard, and that the h-cobordant, smooth, simply-connected 4-manifolds are not necessarily diffeomorphic.

These results show a striking difference between smooth and topological 4-manifolds. By combining them, it follows that a smooth, non-spin, simply-connected 4-manifold M is homeomorphic to a connected sum of copies of $\pm\mathbf{CP}^2$. If M is smooth, simply-connected and spin, then M is homeomorphic to a connected sum of copies of $S^2 \times S^2$ and $\pm K3$ surfaces, provided that

$$b_2(M) \geq \frac{11}{8} |\operatorname{sign}(M)|,$$

where $b_2(M) = \operatorname{rank}(H_2(M; \mathbf{Z}))$. The well-known $\frac{11}{8}$-conjecture, still unresolved, states that this inequality always holds for smooth, spin 4-manifolds: the best partial result to date is $b_2(M) \geq \frac{5}{4}|\operatorname{sign}(M)| + 2$, if q_M is indefinite, proved by Furuta [20]. The exciting subsequent developments in the study of smooth, simply-connected 4-manifolds are outside the scope of this survey (there is a large and growing literature: for example, the work of Fintushel-Stern [13], [14], Gompf [21], Friedman-Morgan [19], Kronheimer-Mrowka [41], Ozsváth-Szabó [48], [47], Jongil Park [50], Taubes [53], and Seiberg-Witten [61]).

3. Infinite cyclic fundamental groups

If $\pi_1(M) = \mathbf{Z}$ and $\Lambda = \mathbf{Z}[\mathbf{Z}]$, then $\pi_2(M)$ is a finitely-generated, free Λ-module of rank $b_2(M)$ and s_M is a non-singular hermitian form. The classification theorem for these manifolds uses the full equivariant intersection form.

Theorem 3.1 (Freedman-Quinn [17])**.** *A closed, oriented topological 4-manifold M with $\pi_1(M) = \mathbf{Z}$ is classified up to homeomorphism by s_M and $KS(M)$. Any non-singular hermitian form on a finitely-generated free Λ-module can be realized by one or two manifolds.*

More precisely, such forms are *even* and realized by a unique spin manifold, or *odd* and realized by two non-spin manifolds with different Kirby-Siebenmann invariants. The equivariant intersection form of a connected sum $M = (S^1 \times S^3) \# N$, with a 1-connected manifold N, is said to be *extended from the integers*. In other words, $s_M = q_N \otimes_{\mathbf{Z}} \Lambda$. Conversely, by the classification theorem, any manifold whose equivariant intersection form is extended from the integers must be homeomorphic to a connected sum with $S^1 \times S^3$.

Fintushel and Stern [12] constructed a smooth 4-manifold M, which was homeomorphic but not diffeomorphic to a connected sum with $S^1 \times S^3$. The existence of indecomposable topological 4-manifolds with $\pi_1 = \mathbf{Z}$ and $\chi(M) > 0$ was settled later.

Theorem 3.2 ([23])**.** *There exists a closed, oriented topological 4-manifold M with $\pi_1(M) = \mathbf{Z}$ and $\chi(M) = 4$, and M is not homotopy equivalent to a connected sum $(S^1 \times S^3) \# N$ for any 1-connected N.*

The main step in the proof was the construction of a non-extended hermitian form L on a free Λ-module (using a certain odd, definite, rank 4 form over $\mathbf{Z}[t]$ found by Quebbemann [51, §6]). We also showed that any 4-manifold M with $\pi_1(M) = \mathbf{Z}$ and $b_2(M) - |\operatorname{sign}(M)| \geq 6$ splits off $S^1 \times S^3$ and is determined up to homeomorphism by the explicit invariants b_2, sign, w_2 and KS.

Question. If M is a smooth, closed, oriented 4-manifold with $\pi_1(M) = \mathbf{Z}$, then is s_M extended from the integers ?

This is a natural question after comparing the example $M = M_L$ in Theorem 3.2 with the Fintushel-Stern example.

Theorem 3.3 ([18])**.** *The manifold M_L is not smoothable.*

The idea of the proof is to consider the n-fold cyclic coverings $M_n \to M_L$. Since q_{M_L} is standard of rank 4, and both Euler characteristic and signature multiply by the index of a finite covering, the forms q_{M_n} of rank $= 4n$ are all definite, odd, unimodular forms over $\mathbf{Z}$. This seems to be an interesting series of definite forms: we showed that for $n \geq 3$ they were all non-standard, and for $n = 3, 4$ they were the unique indecomposable odd lattices in dimension 12 and 16 respectively. In any case, by Donaldson's theorem M_n is non-smoothable for $n \geq 3$ and hence M_L is non-smoothable. In [18] we found many more examples of non-extended forms, and manifolds realizing these forms with a wide variety of other infinite fundamental groups.

4. The quadratic 2-type and surgery

In the non simply-connected case, the obvious homotopy invariants are the equivariant intersection form s_M and the first k-invariant

$$k_M \in H^3(\pi;\pi_2(M)),$$

which together with $\pi := \pi_1(M, x_0)$ and π_2 specifies the algebraic 2-type $B = B(M)$ as introduced by MacLane and Whitehead [44]. The space B is a fibration over $K(\pi, 1)$, classified by k_M, with fibre $K(\pi_2(M), 2)$ and there is a 3-connected reference map $\tilde{c}\colon M \to B(M)$ lifting the classifying map $c\colon M \to K(\pi, 1)$ for the universal covering $\widetilde{M} \to M$. In [22] we introduced the *quadratic* 2-*type* of M as the quadruple

$$[\pi_1(M, x_0), \pi_2(M), k_M, s_M]\ .$$

An *isometry* of two such quadruples is an isomorphism on π_1, π_2 inducing an isometry of the equivariant intersection forms, and respecting the k-invariants.

In general, not much is known about these homotopy invariants, but they are related by an exact sequence

$$0 \to H^2(\pi;\Lambda) \to H^2(M;\Lambda) \to \operatorname{Hom}_\Lambda(H_2(M;\Lambda),\Lambda) \to H^3(\pi;\Lambda) \to 0 \tag{1}$$

arising from the universal coefficient spectral sequence. In this sequence, $H^2(M;\Lambda) \cong H_2(M;\Lambda) \cong \pi_2(M)$ by Poincaré duality, and the middle map

$$H^2(M;\Lambda) \to \operatorname{Hom}_\Lambda(H_2(M;\Lambda),\Lambda)$$

is the adjoint of s_M. The radical $R(s_M)$ of the intersection form s_M is isomorphic to the π-module $R(\pi) := H^2(\pi;\Lambda)$, and $\pi_2(M)$ is a finitely-generated Λ-module.

If $\pi := \pi_1(M, x_0)$ is a non-trivial finite group, then $\pi_2(M)$ is a finitely-generated free abelian group with a $\Lambda := \mathbb{Z}\pi$-module structure, as studied in integral representation theory. In general, there are infinitely many non-isomorphic indecomposable integral representations (e.g. for $\pi = \mathbb{Z}/p \times \mathbb{Z}/p$), and there is no known classification. If $\pi_1(M)$ is infinite, the precise structure of $\pi_2(M)$ is unknown except in very special cases, such as $\pi_1(M) = \mathbf{Z}$ mentioned in Section 3.

The study of these modules can be simplified somewhat by considering stable equivalence classes: two modules L_1, L_2 are *stably isomorphic*, denoted $L_1 \simeq_s L_2$, if there exists a free module Λ^r such that $L_1 \oplus \Lambda^r \cong L_2 \oplus \Lambda^r$. For example, the kernel

$$0 \to \Omega^{n+1}\mathbf{Z} \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathbf{Z} \to 0$$

after n-steps in a free resolution $\{F_*\}$ of the trivial module $\mathbf{Z}$ is stably unique by Schanuel's Lemma. For $n = 3$, such modules arise as $\pi_2(K) = H_2(K;\Lambda)$, where K is a finite 2-complex with $\pi_1(K, x_0) = \pi$, and the resolution is obtained from the chain complex $C_*(\widetilde{K})$ of the universal covering. Finite 2-complexes K provide examples of smooth 4-manifolds by taking the boundary of a thickening (i.e. a regular neigbourhood) of K in $\mathbb{R}^5$.

The stabilization operation in algebra has analogues in topology. For 2-complexes, $K \mapsto K \vee S^2$ gives the stabilization $\pi_2(K) \mapsto \pi_2(K) \oplus \Lambda$. Whitehead [60, Theorem

19] showed that any two finite 2-complexes K, K' with isomorphic fundamental groups are stably simple-homotopy equivalent, but the problem of finding the minimal Euler characteristic realized by a 2-complex with given π_1 is still unsolved. This is a cancellation problem. Note that Whitehead's Theorem implies that the stable isomorphism type of the Λ-modules $H_2(K;\Lambda)$ and $H^2(K;\Lambda)$ depend only on the fundamental group, and not on the choice of finite 2-complex K.

It turns out that the stable structure of $\pi_2(M)$ for a 4-manifold is very special.

Theorem 4.2. *Let M be a closed, oriented 4-manifold with fundamental group π. Then $\pi_2(M)$ is stably isomorphic as a Λ-module to a certain extension*

$$\mathcal{E}_M : 0 \to H_2(K;\Lambda) \to E \to H^2(K;\Lambda) \to 0$$

where K is any finite 2-complex with $\pi_1(K, x_0) = \pi$.

Remark 4.3. The boundaries of thickenings of 2-complexes yield trivial extensions. The finite fundamental group case was done in [22, 2.4], and in that case the extension class of $\mathcal{E}_M$ corresponds to the image of the fundamental class $c_*[M] \in H_4(\pi;\mathbf{Z})$ under a natural isomorphism $\theta\colon H_4(\pi;\mathbf{Z}) \cong \operatorname{Ext}^1_\Lambda(H^2(K;\Lambda), H_2(K;\Lambda))$.

Proof. We will use a chain complex argument. By stabilizing $K \mapsto K \vee rS^2$ and $M \mapsto M \# t(S^2 \times S^2)$ if necessary, we may assume that K is the sub-complex of 2-cells of M. Consider the cellular chain complex $C_* = C_*(\widetilde{M})$ of finitely-generated free Λ-modules. We have the exact sequences

$$0 \to \mathcal{Z}_2 \to C_2 \to C_1 \to C_0 \to \mathbf{Z} \to 0$$

and

$$0 \to \mathfrak{B}_3^* \to C_3^* \to C_4^* \to \mathbf{Z} \to 0$$

showing that $\mathfrak{B}_3^* = \operatorname{Hom}_\Lambda(\mathfrak{B}_3, \Lambda)$ is stably isomorphic to the 2-boundaries $\mathfrak{B}_2$. The details here depend on whether π is finite or infinite: in the latter case note that $\operatorname{Ext}^1_\Lambda(\mathfrak{B}_3, \Lambda) \cong H^4(M;\Lambda) = \mathbf{Z}$. We now form the pull-back diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & H^2(M;\Lambda) \arrow[r] & H^2(K;\Lambda) \arrow[r] \arrow[u] & \mathfrak{B}_2 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & H^2(M;\Lambda) \arrow[r] \arrow[u, equal] & E \arrow[r] \arrow[u] & C_2 \arrow[r] \arrow[u] & 0 \\
 & & H_2(K;\Lambda) \arrow[r, equal] \arrow[u] & \mathcal{Z}_2 \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{4}$$

where $\mathcal{Z}_2 = H_2(K;\Lambda)$ since K is the 2-skeleton of M. The middle horizontal sequence splits since C_2 is a free Λ-module. The middle vertical sequence is $\mathcal{E}_M$, and $E \cong \pi_2(M) \oplus C_2$ is a stabilization of $\pi_2(M)$. □

Surgery theory as developed by Browder, Novikov, Sullivan and Wall [59] provides a powerful framework for classifying manifolds of dimension ≥ 5 within a fixed homotopy type. However, in dimension 4 there are serious obstacles arising from the failure of the Whitney trick. One approach, developed by Cappell and Shaneson [3], is based on Wall's idea of studying smooth 4-manifolds after stabilization with copies of $S^2 \times S^2$. The drawback is that information about the original (unstabilized) homotopy type is lost in the process.

Freedman's work [16] fully established 4-dimensional surgery theory for topological manifolds whose fundamental groups do not "grow" too quickly. This class includes the poly-cyclic by finite groups, but it is not known at present if 4-dimensional topological surgery theory works for (non-cyclic) free fundamental groups. Note that Donaldson's results show that smooth surgery theory definitely does not work in dimension 4, and there are s-cobordant smooth 4-manifolds which are not diffeomorphic.

The modified surgery theory of M. Kreck [39] requires less initial information about the homotopy type: for example, one can try to classify smooth 4-manifolds which have the same algebraic 2-type (up to smooth s-cobordism). In this theory, the key step is to compute certain bordism groups $\Omega_4(B,\xi)$, where ξ is a bundle over B whose pullback $c^*(\xi) \cong \nu_M$ is the stable normal bundle of M. For such computations there are a variety of methods available, including the Atiyah-Hirzebruch and Adams spectral sequences. If two manifolds $[M_1 . \tilde{c}_1]$, $[M_2, \tilde{c}_2]$ are bordant over the type (B,ξ), then the triviality of an algebraically defined invariant implies that M_1 and M_2 are smoothly s-cobordant (see [39, Theorem B]).

One possible way of analysing the final step is to note that the relation $[M_1, \tilde{c}_1] = [M_2, \tilde{c}_2] \in \Omega_4(B,\xi)$ implies that

$$M_1 \# r_1(S^2 \times S^2) \cong M_2 \# r_2(S^2 \times S^2)$$

are stably diffeomorphic [39, Cor. 3], with control on the reference maps to B (see [40], [5] for further applications of stabilization). The *cancellation* problem is to find techniques for removing $S^2 \times S^2$ factors from both sides. In algebra, cancellation theorems for modules and quadratic forms over noetherian rings were proved by Bak [1], Bass [2], Stafford [52] and Vaserstein [56]. In [23], we realized that these algebraic results could be combined with the constructions by [3, 1.5] of self-diffeomorphisms of 4-manifolds to prove cancellation theorems for certain 4-manifolds. For example, the integral group rings $\mathbf{Z}[\pi]$ of poly-cyclic by finite groups are noetherian rings, but the group ring of a free group on 2 generators is not noetherian. This theme has recently been taken up again in [4].

5. Finite fundamental group

In early joint work with M. Kreck [23] we studied the topology of 4-manifolds with finite fundamental groups, and obtained a good description of the homotopy types within a prescribed algebraic 2-type. We also showed that there are only finitely many homeomorphism types of closed, oriented 4-manifolds with given finite π_1, and given Euler characteristic (see [22, p. 87]). To obtain precise classification results up to homeomorphism we needed to restrict to special fundamental groups. We say that M has w_2-type (I) if $w_2(\widetilde{M}) \neq 0$, w_2-type (II) if $w_2(M) = 0$, and w_2-type (III) if $w_2(M) \neq 0$ but $w_2(\widetilde{M}) = 0$.

Theorem 5.1 ([22], [24]). *Closed, oriented topological 4-manifolds with finite cyclic fundamental groups are classified up to homeomorphism by $\pi_1(M)$, q_M, the w_2-type, and $KS(M)$.*

This is a generalization of Freedman's Theorem in the simply-connected case (where the w_2-type is determined by the intersection form). One interesting consequence is that certain automorphisms of the cohomology ring of a smooth 4-manifold are induced by self-homeomorphisms but not by a self-diffeomorphism (Donaldson's work is used to rule out a self-diffeomorphism, see the example [22, p.87]).

However, for more complicated fundamental groups we can not expect a classification in terms of the ordinary intersection form q_M on $H_2(M; \mathbf{Z})$ (see [54], [55]). Here is a sample result involving the quadratic 2-type.

Theorem 5.2. *Closed, oriented, topological 4-manifolds with $w_2(M) = 0$ and odd order finite fundamental groups are classified up to homeomorphism by the simple isometry class of the quadratic 2-type $[\pi_1(M), \pi_2(M), k_M, s_M]$.*

Proof. This is essentially an exercise in the methods of [39] and [25], and the information needed to use the odd order assumption is provided by [26, Section 4]. The definition of *simple isometry class* will be explained below.

Here are some details of the steps in the argument. We first notice that the normal 2-type for such a spin manifold M is $B \times BTOPSPIN$, where $B = B(M)$ is the algebraic 2-type. Since $\pi := \pi_1(M)$ has odd order,

$$\Omega_4^{TOPSPIN}(K(\pi, 1)) = \mathbf{Z} \oplus H_4(\pi; \mathbf{Z}),$$

and the stable homeomorphism class of M is determined by its signature and the image of the fundamental class $c_*[M] \in H_4(\pi; \mathbf{Z})$. If M and M' have isometric quadratic 2-types, then there exists an isometry $\alpha\colon s_M \cong s_{M'}$ respecting the k-invariants. We use α to identify the 2-types $B(M) \cong B(M')$, and conclude that the images of their fundamental classes agree by [26, p. 168]. Hence M and M' are spin bordant over B, and there exists a stable homeomorphism

$$h\colon M \# r(S^2 \times S^2) \cong M' \# r(S^2 \times S^2)$$

respecting the reference maps to $K(\pi, 1)$. If α is a simple isometry of the quadratic 2-types, then we claim that this data will allow us to construct another stable homeomorphism h' between these manifolds, with the additional property that the induced isometry h'_* of equivariant intersection forms induces the identity on the hyperbolic summands (in fact, we will obtain $h'_* = (\alpha \oplus 1)$). With that additional property, one can attach handles to both domain and range to obtain an s-cobordism between M and M' (see [39, §4]).

To modify the homeomorphism h we proceed as follows. Let $M_r := M \# r(S^2 \times S^2)$ denote the r-fold stabilization of M. By composition, we obtain an element

$$\beta := (\alpha \oplus 1)^{-1} \circ h_* \in \mathrm{Isom}[\pi_1(M_r), \pi_2(M_r), k_{M_r}, s_{M_r}].$$

The braid diagram of [26, p. 168], combined with [26, Theorem B], now shows that $\beta = \phi_*$ for some $\phi \in \mathrm{Aut}_\bullet(M_r)$, such that ϕ is induced by an inertial h-cobordism $(W; M_r, -M_r)$. We have further assumed that α is a *simple* isometry of the quadratic 2-types. By definition, this means that the Whitehead torsion $\tau(\phi) = 0 \in \mathrm{Wh}(\mathbb{Z}\pi)$, and hence $\tau(W, M_r) = u \in \mathrm{Wh}(\mathbb{Z}\pi)$ is self-dual ($\bar{u} = u$). Note that this definition of a simple isometry is independent of the choice of h-cobordism inducing ϕ. From the exact sequences in the proof of [26, 4.1], and the fact that the discriminant map $L_6^h(\mathbb{Z}\pi) \to \widehat{H}^0(\mathbb{Z}/2; \mathrm{Wh}(\mathbb{Z}\pi))$ is surjective (since π has odd order), we can realize the self-equivalence ϕ by an s-cobordism W': if necessary, we modify our first choice by the action of $L_6^h(\mathbb{Z}\pi)$ on $\mathcal{H}(M)$. It follows that the homotopy self-equivalence induced by W' is realized by a self-homeomorphism $f\colon M_r \to M_r$. We now define $h' := h \circ f^{-1}$ and notice that $h'_* = \alpha \oplus 1$, as required. □

Stabilization and cancellation techniques can also be used effectively for manifolds with arbitrary finite fundamental groups (see [25]). For 4-manifolds, the connected sum operation gives the stabilization

$$\pi_2(M \# (S^2 \times S^2)) = \pi_2(M) \oplus \Lambda \oplus \Lambda$$

where the equivariant intersection form is stabilized by adding a hyperbolic plane

$$H(\Lambda) = (\Lambda \oplus \Lambda, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$$

The cancellation problem for 4-manifolds with finite fundamental group has the following optimal solution:

Theorem 5.3 ([25]). *Let M, M' be closed, oriented topological 4-manifolds with finite fundamental group. If $M = M_0 \# (S^2 \times S^2)$, and*

$$M \# r(S^2 \times S^2) \cong M' \# r(S^2 \times S^2)$$

then $M \cong M'$.

Note that even in the simply-connected case, non-isomorphic forms can become isomorphic after adding a hyperbolic plane, so the statement is best possible.

6. Fundamental groups of aspherical 2-complexes

A finitely-presented group π is *geometrically 2-dimensional* ($\text{g-dim}\,\pi \le 2$) if there exists a finite aspherical 2-complex with fundamental group π. Examples of geometrically 2-dimensional groups include free groups, 1-relator groups (e.g. surface groups) and small cancellation groups [43], provided they are torsion-free, as well as many word-hyperbolic groups, see also [31, 2.3], [34, §10].

Recall that the radical $R(s_M)$ of the equivariant intersection form s_M is isomorphic to the π-module $R(\pi) := H^2(\pi;\Lambda)$. A closed oriented 4-manifold M will be called *minimal* if the equivariant intersection form on $\pi_2(M)$ vanishes, or equivalently, if $\pi_2(M) = R(s_M) \cong R(\pi)$. It turns out that a thickening of an aspherical 2-complex for π gives a minimal, smooth 4-manifold M_0 with fundamental group π, whenever $\text{g-dim}\,\pi \leq 2$ (see [27, Lemma 3.7]). For example, the manifolds $\#_r(S^1 \times S^3)$ and $S^2 \times \Sigma$ are minimal, where Σ denotes an oriented surface of genus ≥ 1.

In a series of papers [31], [33], [32], [34], [35], J. Hillman investigated the homotopy classification of Poincaré 4-complexes under various fundamental group assumptions. In the case of g-dim ≤ 2, the problem was reduced to the minimal case, where the homotopy classification was completed for free or surface fundamental groups (see also [49] where these cases were studied from a different viewpoint).

In recent joint work with Matthias Kreck and Peter Teichner (described below), we used the modified surgery approach to obtain classification results for topological 4-manifolds with geometrically 2-dimensional fundamental groups, up to homeomorphism (in favourable cases) or s-cobordism.

A particular nice family of examples if provided by the solvable Baumslag-Solitar groups

$$BS(k) := \{a, b \,|\, aba^{-1} = b^k\}, \quad k \in \mathbb{Z}.$$

The groups $BS(k)$ have geometrical dimension ≤ 2 because the 2-complex corresponding to the above presentation is aspherical. The easiest cases are

$$BS(0) = \mathbb{Z}, \quad BS(1) = \mathbb{Z} \times \mathbb{Z}, \quad \text{and } BS(-1) = \mathbb{Z} \rtimes \mathbb{Z},$$

and these are the only Poincaré duality groups in this family. Each $BS(k)$ is solvable, so is a "good" fundamental group for topological 4-manifolds [16]. This implies that Freedman's s-cobordism theorem is available to complete the homeomorphism classification. This had been done previously only for the three special cases above, see [17] for $BS(0)$, and [33] for $BS(\pm 1)$, using a more classical surgery approach.

Theorem 6.1 ([27, Theorem A])**.** *For closed oriented* 4*-manifolds with solvable Baumslag-Solitar fundamental groups, and given w_2-type and Kirby-Siebenmann invariant, any isometry between equivariant intersection forms can be realized by a homeomorphism.*

In particular, we showed that a minimal 4-manifold is unique up to homeomorphism and established some relations between the invariants in general (based in part on [55]). For fundamental groups π with $H_4(\pi;\mathbb{Z}) = 0$ we showed that the signature is determined by s_M via the formula $\text{sign}(M) = \text{sign}(s_M \otimes_\Lambda \mathbb{Z})$. This formula does not hold for arbitrary 4-manifolds, as one can see from examples of surface bundles over surfaces with nontrivial signature (but vanishing π_2).

For $\pi_1(M) = BS(k)$, type (III) can only occur if k is odd. In this case, we gave a generalization of Rochlin's formula (see [27, Corollary 6.10]):

$$KS(M) \equiv \text{sign}(M)/8 + \text{Arf}(M) \pmod 2$$

where $\text{Arf}(M) \in \mathbb{Z}/2$ is a codimension 2 Arf invariant. In contrast, for spin manifolds $KS(M) \equiv \text{sign}(M)/8 \pmod 2$.

We also proved a realization theorem for hermitian forms in this setting. If M has fundamental group $BS(k)$, then the quotient module $\pi_2(M)^\dagger := \pi_2(M)/R(s_M)$ is a finitely-generated, stably-free Λ-module, and the equivariant intersection form s_M is non-singular on this quotient. It turns out that any such hermitian form can be realized by one or two 4-manifolds.

A close inspection of the arguments shows that we used a number of special facts about the Baumslag-Solitar groups. For more general fundamental groups π, we need to assume the corresponding properties for its algebraic K-theory and L-theory.

Definition 6.2. A group π satisfies properties (W-AA) whenever

1. The Whitehead group $\mathrm{Wh}(\pi)$ vanishes,
2. The assembly map $A_5\colon H_5(\pi;\mathbb{L}_0) \to L_5(\mathbb{Z}\pi)$ is surjective.
3. The assembly map $A_4\colon H_4(\pi;\mathbb{L}_0) \to L_4(\mathbb{Z}\pi)$ is injective.

Note that these properties (and more) do hold whenever the group π satisfies the Farrell-Jones isomorphism conjectures [11] (see [42] for a survey of results on these conjectures).

Theorem 6.3 ([27, Theorem C])**.** *Let π be a geometrically 2-dimensional group satisfying properties* (W-AA)*. For closed oriented 4-manifolds with fundamental group π, and given Kirby-Siebenmann invariant, any isometry between equivariant intersection forms inducing an isomorphism of w_2-types can be realized by an s-cobordism.*

The w_2-type mentioned in this statement is actually a refinement of the notion defined in Section 4, in which we now keep track of the class $w \in H^2(\pi;\mathbb{Z}/2)$ determining $w_2(M)$ in type (III).

7. Some questions

Here are a few questions and problems concerning smooth and topological 4-manifolds with non-trivial fundamental group.

1. For a smooth 4-manifold M with geometrically 2-dimensional fundamental group, is M homeomorphic to $M_0 \# N$, where M_0 is minimal and N is simply-connected ? In other words, is the equivariant intersection form s_M always extended from the integers ?
2. Construct distinct smooth structures on indecomposable, non-simply connected 4-manifolds. Is there a minimal 4-manifold with more than one smooth structure ?
3. For a given group π, there exist 4-manifolds $M(\alpha)$ with $\pi_1(M) = \pi$ and $c_*[M]$ a given element $\alpha \in H_4(\pi;\mathbf{Z})$. How does the minimal possible Euler characteristic and signature of $M(\alpha)$ depend on the class α ?
4. For a given group π, does there exists a stable range constant $c(\pi)$, with the property that a stable homeomorphism or diffeomorphism $M_1 \# r(S^2 \times S^2) \cong M_2 \# r(S^2 \times S^2)$ between manifolds with fundamental group π admits cancellation of at least one copy of $S^2 \times S^2$ (up to s-cobordism) whenever $r > c(\pi)$?
5. Compare the actions of $\mathrm{Diff}(M)$ and $\mathrm{Homeo}(M)$ on the equivariant intersection form of a smooth 4-manifold.

Remark 7.1. There are many interesting problems related to the study the existence and uniqueness of non-free finite group actions on smooth or topological 4-manifolds. One may ask, for example, which equivariant intersection forms are realized by smooth actions of finite cyclic groups on simply-connected 4-manifolds. For topological actions there is a satisfactory picture, particularly for cyclic groups of prime order (see Edmonds [9], [10], Edmonds-Ewing [9], and McCooey [45]). For smooth actions, there are restrictions detected by equivariant gauge theory [28], [29] and the answer is interesting even for the permutation representations which arise for actions on connected sums of $\mathbf{CP}^2$'s (see [30, 1.18]). A striking contrast between smooth and topological actions is shown by the recent paper of Finstushel, Stern and Sunukjian [15], where infinite families of topologically equivalent but smoothly distinct cyclic group actions are constructed on 4-manifolds with non-trivial Seiberg-Witten invariants.

References

[1] A. Bak, *On modules with quadratic forms*, Algebraic K-Theory and its Geometric Applications (Conf., Hull, 1969), Springer, Berlin, 1969, pp. 55–66.

[2] H. Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[3] S. E. Cappell and J. L. Shaneson, *On four dimensional surgery and applications*, Comment. Math. Helv. **46** (1971), 500–528.

[4] D. Crowley and J. Sixt, *Stably diffeomorphic manifolds and* $l_{2q+1}(Z[\pi])$, (arXiv:0808.2008v2 [math.GT]), 2008.

[5] J. F. Davis, *The Borel/Novikov conjectures and stable diffeomorphisms of 4-manifolds*, Geometry and topology of manifolds, Fields Inst. Commun., vol. 47, Amer. Math. Soc., Providence, RI, 2005, pp. 63–76.

[6] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), 279–315.

[7] ——, *Irrationality and the h-cobordism conjecture*, J. Differential Geom. **26** (1987), 141–168.

[8] ——, *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J. Differential Geom. **26** (1987), 397–428.

[9] A. L. Edmonds, *Aspects of group actions on four-manifolds*, Topology Appl. **31** (1989), 109–124.

[10] ——, *Periodic maps of composite order on positive definite 4-manifolds*, Geom. Topol. **9** (2005), 315–339 (electronic).

[11] F. T. Farrell and L. E. Jones, *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc. **6** (1993), 249–297.

[12] R. Fintushel and R. J. Stern, *A fake 4-manifold with* $\pi_1 = \mathbf{Z}$ *and* $b^+ = 4$, Turkish J. Math. **18** (1994), 1–6.

[13] ——, *Knots, links, and 4-manifolds*, Invent. Math. **134** (1998), 363–400.

[14] ——, *Double node neighborhoods and families of simply connected 4-manifolds with* $b^+ = 1$, J. Amer. Math. Soc. **19** (2006), 171–180 (electronic).

[15] R. Fintushel, R. J. Stern, and N. Sunukjian, *Exotic group actions on simply connected smooth 4-manifolds*, (arXiv: 0902.0963 [math.GT]), 2009.

[16] M. H. Freedman, *The disk theorem for four-dimensional manifolds*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983) (Warsaw), PWN, 1984, pp. 647–663.

[17] M. H. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton University Press, Princeton, NJ, 1990.

[18] S. Friedl, I. Hambleton, P. Melvin, and P. Teichner, *Non-smoothable four-manifolds with infinite cyclic fundamental group*, Int. Math. Res. Not. IMRN (2007), 20 pages.

[19] R. Friedman and J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 27, Springer-Verlag, Berlin, 1994.
[20] M. Furuta, *Monopole equation and the $\frac{11}{8}$-conjecture*, Math. Res. Lett. **8** (2001), 279–291.
[21] R. E. Gompf, *Smooth 4-manifolds and symplectic topology*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 548–553.
[22] I. Hambleton and M. Kreck, *On the classification of topological 4-manifolds with finite fundamental group*, Math. Ann. **280** (1988), 85–104.
[23] ______, *Smooth structures on algebraic surfaces with cyclic fundamental group*, Invent. Math. **91** (1988), 53–59.
[24] ______, *Cancellation, elliptic surfaces and the topology of certain four-manifolds*, J. Reine Angew. Math. **444** (1993), 79–100.
[25] ______, *Cancellation of hyperbolic forms and topological four-manifolds*, J. Reine Angew. Math. **443** (1993), 21–47.
[26] ______, *Homotopy self-equivalences of 4-manifolds*, Math. Z. **248** (2004), 147–172.
[27] I. Hambleton, M. Kreck, and P. Teichner, *Topological 4-manifolds with geometrically 2-dimensional fundamental groups*, (arXiv:0802.0995v1 [math.GT]), 2008.
[28] I. Hambleton and R. Lee, *Perturbation of equivariant moduli spaces*, Math. Ann. **293** (1992), 17–37.
[29] ______, *Smooth group actions on definite 4-manifolds and moduli spaces*, Duke Math. J. **78** (1995), 715–732.
[30] I. Hambleton and M. Tanase, *Permutations, isotropy and smooth cyclic group actions on definite 4-manifolds*, Geom. Topol. **8** (2004), 475–509.
[31] J. A. Hillman, *Four-manifolds, geometries and knots*, Geometry & Topology Monographs, vol. 5, Geometry & Topology Publications, Coventry, 2002.
[32] ______, PD_4*-complexes with free fundamental group*, Hiroshima Math. J. **34** (2004), 295–306.
[33] ______, PD_4*-complexes with fundamental group of* PD_2*-group*, Topology Appl. **142** (2004), 49–60.
[34] ______, PD_4*-complexes with strongly minimal models*, Topology Appl. **153** (2006), 2413–2424.
[35] ______, *Strongly minimal* PD_4*-complexes*, Topology Appl. (2009), (to appear: arXiv:math.GT 0712.1069).
[36] M. A. Kervaire, *Smooth homology spheres and their fundamental groups*, Trans. Amer. Math. Soc. **144** (1969), 67–72.
[37] R. Kirby, *Akbulut's corks and h-cobordisms of smooth, simply connected 4-manifolds*, Turkish J. Math. **20** (1996), 85–93.
[38] R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press, Princeton, N.J., 1977, With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.
[39] M. Kreck, *Surgery and duality*, Ann. of Math. (2) **149** (1999), 707–754.
[40] M. Kreck, W. Lück, and P. Teichner, *Stable prime decompositions of four-manifolds*, Prospects in topology (Princeton, NJ, 1994), Ann. of Math. Stud., vol. 138, Princeton Univ. Press, Princeton, NJ, 1995, pp. 251–269.
[41] P. B. Kronheimer and T. S. Mrowka, *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Differential Geom. **41** (1995), 573–734.
[42] W. Lück and H. Reich, *The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 703–842.
[43] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1977 edition.
[44] S. MacLane and J. H. C. Whitehead, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. U. S. A. **36** (1950), 41–48.
[45] M. P. McCooey, *Symmetry groups of four-manifolds*, Topology **41** (2002), 835–851.

[46] J. W. Morgan and Z. Szabó, *Complexity of 4-dimensional h-cobordisms*, Invent. Math. **136** (1999), 273–286.

[47] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) **159** (2004), 1159–1245.

[48] ———, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), 1027–1158.

[49] M. Pamuk, *Homotopy self-equivalences of four-manifolds*, Ph.D. thesis, McMaster University, 2008.

[50] J. Park, *Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$*, Invent. Math. **159** (2005), 657–667.

[51] H.-G. Quebbemann, *Definite lattices over real algebraic function domains*, Math. Ann. **272** (1985), 461–475.

[52] J. T. Stafford, *Absolute stable rank and quadratic forms over noncommutative rings*, K-Theory **4** (1990), 121–130.

[53] C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994), 809–822.

[54] P. Teichner, *Topological four manifolds with finite fundamental group*, Ph.D. thesis, University of Mainz, 1992.

[55] ———, *On the signature of four-manifolds with universal covering spin*, Math. Ann. **295** (1993), 745–759.

[56] L. N. Vaseršteĭn, *Stabilization of unitary and orthogonal groups over a ring with involution*, Mat. Sb. (N.S.) **81 (123)** (1970), 328–351.

[57] C. T. C. Wall, *Diffeomorphisms of 4-manifolds*, J. London Math. Soc. **39** (1964), 131–140.

[58] ———, *On simply-connected 4-manifolds*, J. London Math. Soc. **39** (1964), 141–149.

[59] ———, *Surgery on compact manifolds*, second ed., American Mathematical Society, Providence, RI, 1999, Edited and with a foreword by A. A. Ranicki.

[60] J. H. C. Whitehead, *Simplicial spaces, nuclei and m-groups*, Proc. Lond. Math. Soc. **45** (1939), 243–327.

[61] E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), 769–796.

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