EXISTENCE OF mASD CONNECTIONS ON 4-MANIFOLDS WITH CYLINDRICAL ENDS

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ABSTRACT. Taubes' gluing theorems establish the existence of ASD connections on closed, oriented 4-manifolds. We extend these gluing results to the mASD connections of Morgan–Mrowka–Ruberman on oriented 4-manifolds with cylindrical ends. As a corollary, we obtain an ASD-existence result in the presence of degenerate asymptotic flat connections.

1. INTRODUCTION

The results of Taubes [19, 20] on "gluing" establish the existence of non-trivial anti-self dual (ASD) connections on closed, oriented 4-manifolds, provided one works with an $\mathrm{SU}(2)$-bundle with sufficiently high second Chern class. This was extended by Donaldson [3] to a general gluing theorem for connected sums; see also [2, 9]. These gluing results have direct extensions to cylindrical end 4-manifolds, provided one works with ASD connections having a *non-degenerate* flat connection as an asymptotic limit [4]. However, in the absence of such non-degeneracy assumptions, the space of ASD connections on a cylindrical end 4-manifold is generally not well-controlled (e.g., the ASD operator is not Fredholm) and this now-standard gluing formalism breaks down. Nevertheless, the question of existence for ASD connections in this degenerate cylindrical end setting remains well-posed. One of our main results, Theorem A below, establishes one such ASD-existence result in the degenerate setting.

To state this, suppose X is a connected, oriented 4-manifold with cylindrical ends. Thus, we can write $X = X_0 \cup \mathrm{End}\ X$, where X_0 is a compact 4-manifold with boundary N, and $\mathrm{End}\ X \cong [0, \infty) \times N$ is diffeomorphic to a cylinder. We refer to X_0 as the *compact part* and to $\mathrm{End}\ X$ as the *cylindrical ends*. Unless otherwise stated, we allow the case where N has multiple components, or is empty. Fix a metric g on X that is asymptotically cylindrical in the sense described in Section **2A**.

Theorem A. *Assume $b^+(X) \leq 1$. Assume further that the 3-manifold N is connected and satisfies one of the following:*

(i) N is a circle bundle over a surface with positive Euler class: $e(N) > 0$; or
(ii) N has first Betti number at most one: $b_1(N) \leq 1$.

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Then, for any integer $\ell \geq b^+(X) + 1$, the manifold X admits an irreducible ASD*-connection A on a principal $SU(2)$-bundle over X, and A satisfies*

$$\int_X |F_A|^2 \, d\mathrm{vol} = 8\pi^2 \ell.$$

We prove this in Section **6D**. As a concrete example, the hypotheses of Theorem A hold when X_0 is diffeomorphic to the total space of a positive Euler class disk bundle over a surface. To the authors' knowledge, Theorem A (and its extension, Theorem 6.9) is the first general ASD existence result for cylindrical end manifolds that allows for a degenerate flat limit down the end.

Our approach to Theorem A is to (locally) embed the space of ASD connections into the larger space of *modified ASD (*mASD*)* connections of Morgan, Mrowka, and Ruberman [17]. This larger space is obtained by modifying the ASD operator in such a way that one obtains a Fredholm operator whose zero set contains an open set in the space of finite-energy ASD connections; it may also contain some new solutions. It is shown in [17] that, by allowing the auxiliary choices in this construction to vary, every finite-energy ASD connection belongs to some mASD space of connections defined in this way. The other main results of the present paper, stated below, show that the gluing results of Taubes and Donaldson for connected sums have extensions to this mASD setting. We then arrive at Theorem A as a consequence of these mASD-gluing results; the topological hypotheses on N imply that the mASD connections thus obtained are in fact ASD.

Before stating these mASD-gluing results, we give several remarks to help provide further context for this mASD setting.

Remark 1.1. (a) Our primary motivation for developing these gluing results was to use the Morgan–Mrowka–Ruberman "moduli space" of mASD connections to study the action on X of a finite group π. Even in the ASD setting, generic perturbations are usually not π-equivariant, so the standard transversality arguments are not available, and one must appeal to some other approach to handle singularities in the moduli space. As a sequel to this paper, we planned to study the π-equivariant compactification of the "mASD moduli space" as was done in [11], [12], and [13] for the ASD moduli space.

Unfortunately, the mASD operator *fails* to be gauge equivariant in any reasonable sense (see Remark 2.8). This appears to be an oversight in the original text [17] (e.g., see [17, p. 125]), and at present we do not know how to define a suitable gauge quotient of the space of mASD connections that one might call the "mASD moduli space". It is a fundamental and interesting open problem to construct an appropriate mASD-replacement for the ASD moduli space.

(b) The foundational work of Mrowka [18], Morgan–Mrowka–Ruberman [17], and Taubes [21, 22] concerning instantons on cylindrical end 4-manifolds was done shortly before the Seiberg–Witten revolution in gauge theory. One of their striking results in this setting is that a finite-energy ASD connection has a well-defined limiting flat connection on the 3-manifold N "at infinity".

At that time, a central problem was to understand the behaviour of ASD connections under neck-stretching within a closed 4-manifold, as well as the reverse operation in which ASD connections on non-compact 4-manifolds with matching data on their cylindrical ends could be glued together. Indeed, the authors of [17, p. 12] state: "The thickened moduli space seems to provide the correct geometric context for a general gluing theorem for ASD connections, although we do not treat this topic in this book". This point of view was a main ingredient in a paper of Fintushel and Stern [7] (and in as yet unpublished work of Morgan and Mrowka [16]). An account of gluing along cylindrical ends from the perspective of Floer homology was later provided by Donaldson [4], simplified by assuming the presence of a perturbation to avoid degeneracies (see (c), below). We note, however, that the gluing results of the present paper take place on the compact part X_0, and not on the ends.

(c) Researchers have worked around the technical issues involved in gluing in the degenerate setting by various methods. Of these methods, one of the most popular is to perturb the ASD equation on the ends in such a way that all perturbed-ASD connections are asymptotic to non-degenerate perturbed-flat connections [8], [4]. However, this approach has several drawbacks. For one, ASD connections are generally not solutions of perturbed-ASD equations of this type; this can obscure the geometric information one can infer from an abstract existence result for perturbed-ASD connections (e.g., to what extent do these connections depend on the perturbation?). Another drawback is that these perturbation schemes are not well-behaved in the presence of reducible flat connections (e.g., the trivial flat connection), and this limits the applicability of such approaches. For example, a full $\mathrm{SU}(2)$-instanton Floer theory for 3-manifolds N with $b_1(N) \geq 1$ is still lacking, and even the existing instanton Floer theory for integer homology spheres handles the trivial flat connection separately. In summary, a more in-depth understanding of ASD connections with degenerate limits is desired, and we view the results of this paper as being a step in that direction.

To state our gluing results for mASD connections, let G be a compact Lie group and fix a principal G-bundle $E \to X$. We assume that E is translation-invariant on the end; that is, we assume the diffeomorphism $\mathrm{End}\ X \cong [0, \infty) \times N$ is covered by a bundle isomorphism $E|_{\mathrm{End}\ X} \cong [0, \infty) \times Q$ for some principal G-bundle $Q \to N$. We also fix a flat connection Γ on Q.

Given a connection A on E that converges sufficiently fast down the end, one can define a quantity

$$\kappa(E, A|_{\mathrm{End}\ X}) := -\frac{1}{8\pi^2} \int_X \langle F_A \wedge F_A \rangle \in \mathbb{R}$$

that we call the *relative characteristic number of the adapted bundle* $(E, A|_{\mathrm{End}\ X})$; see Section **6A** for more details. If A is ASD, then $\kappa(E, A|_{\mathrm{End}\ X}) = (8\pi^2)^{-1} \int_X |F_A|^2$ equals the usual energy of the connection A. The upshot for us is that the quantity $\kappa(E, A|_{\mathrm{End}\ X})$ is well-defined for a much larger class of connections than those with finite energy. Indeed, this relative characteristic number depends only on the topological type of the adapted bundle $(E, A|_{\mathrm{End}\ X})$, and it is a lift of the Chern–Simons value of the connection on Q to which A is asymptotic. Note that if $\kappa(E, A|_{\mathrm{End}\ X}) \neq 0$, then A is not flat. When E is closed, then this relative characteristic number is actually an integer that depends only on E, and we will simply write it as $\kappa(E)$ (e.g., if $G = \mathrm{SU}(r)$, then $\kappa(E) = c_2(E)\,[X]$ is the second Chern number). We will primarily use $\kappa(E, A|_{\mathrm{End}\ X})$ to keep track of the topological data in our gluing operations, just as the second Chern class keeps track of the underlying bundle type when gluing in the standard SU(2)-setting for ASD connections on closed 4-manifolds.

By making several auxiliary choices, collectively called *thickening data*, one can define the *modified* ASD *(*mASD*) operator*, which is a non-linear Fredholm map s defined on a suitable space of connections on E; see Section 2 for definitions. In particular, we note that this space of connections is defined so that all elements are asymptotic to connections close to Γ. By definition, the mASD connections are those in the zero set of s, and we say that an mASD connection A is *regular* if the linearization of s at A is surjective.

For $k = 1, 2$, suppose X_k is an oriented, cylindrical end 4-manifold equipped with a principal G-bundle $E_k \to X_k$ and thickening data, as above. Let $X = X_1 \# X_2$ be a connected sum of these manifolds, taken at points in the compact parts of the X_k. Then the E_k can be used to form a connected sum bundle $E \to X$, and we equip this with the thickening data induced from that of the E_k; see Section **3A**. Our basic gluing result can be stated as follows.

Theorem B. *For $k = 1, 2$, suppose A_k is a regular* mASD *connection on E_k. Then for any $\epsilon > 0$, the bundle $E = E_1 \# E_2$ admits an* mASD *connection A with the property that the distance between $A|_{X_k \cap X}$ and $A_k|_{X_k \cap X}$ is less than ϵ for $k = 1, 2$. Here the distance is relative to the $L^2_2(N) \times L^p_{1,\delta}(X)$-metric on the space of connections induced from the identification (2.4). Moreover,*

$$\left| \kappa(E, A|_{\mathrm{End}\ X}) - \sum_{k=1}^{2} \kappa(E_k, A_k|_{\mathrm{End}\ X}) \right| < \epsilon. \tag{1.2}$$

This is a special case of Theorem 3.3, which works in the broader setting where the A_k are not necessarily regular. In this broader setting, the connection A need not be mASD, but its failure to be mASD is expressed through an obstruction map. In Theorem 5.1, we extend Theorem B to a gluing result for *families* of regular mASD connections. These results are mASD-extensions of results familiar from the ASD setting; see [5, Section 7.2].

As an application of Theorems B and 5.1, we establish the following existence result, extending that of Taubes [19, 20] to the present cylindrical end mASD situation.

Theorem C. *Assume $G = \mathrm{SU}(2)$ and $b^+(X) \le 1$, and fix an integer $\ell \ge b^+(X) + 1$. Then for every $\epsilon > 0$, there is a principal* SU(2)*-bundle $E \to X$ and an* mASD *connection A on E that is irreducible, and satisfies*

$$|\kappa(E, A|_{\mathrm{End}\,X}) - \ell| < \epsilon. \tag{1.3}$$

If $b^+(X) = 0$, then the connection A is regular. If $b^+(X) = 0$ and X is simply-connected, then gluing produces an open subset of the space of mASD *connections that are in Coulomb gauge relative to some fixed connection.*

The cases $b^+(X) = 0$ and $b^+(X) = 1$ are special cases of Theorem 6.2 and Theorem 6.3, respectively. Structurally, our proof strategies for these are very similar to the analogous statements in the closed case [19, 20] by realizing X as a trivial connected sum $X \cong X \# S^4$. Under the assumption that $b^+(X) = 0$, it follows that the trivial flat connection on X is regular as an mASD connection. It is well-known that the 4-sphere admits irreducible ASD connections of every positive second Chern class, and these are necessarily regular for topological reasons. Then Theorem C for $b^+(X) = 0$ follows from the general gluing result of Theorem B and adjacent results designed to handle gauge transformations (more below). We note also that Theorem 6.2 (the more general version of Theorem C) is proved for an arbitrary compact Lie group G, under mild hypotheses on ℓ and G.

The strategy for our proof of Theorem C when $b^+(X) = 1$ is similar, albeit more involved since the trivial flat connection on X is no longer regular. Thus a careful analysis of the obstruction map of Theorem 3.3 is required. Just as in [20], we glue ASD connections on S^4 at several sites instead of one, and this is sufficient to show that the obstruction vanishes for some choice of gluing parameters. In this analysis, we use the assumption that $G = \mathrm{SU}(2)$. As Taubes mentions [20, p. 518], it is likely that the restriction to $G = SU(2)$ can be removed, but that would call for a different approach. We prove our general existence results only for $b^+ \leq 1$ because (i) these are the cases of interest for our applications, and (ii) extending the discussion to higher values of b^+ would add considerable length to the paper (this can already be seen in [20]).

The appearance of $\epsilon > 0$ in the statements of Theorems B and C is new to this mASD setting. To explain it, we note that in the standard set-up of gluing ASD connections on a closed 4-manifold, the inequality (1.2) would be replaced by the equality $\kappa(E_1 \# E_2) = \kappa(E_1) + \kappa(E_2)$; likewise (1.3) would be replaced by $\kappa(E) = \ell$. The presence of an *inequality* for us reflects a need to freely vary the asymptotic values in order to obtain the mASD connection A. This is at the heart of what makes the mASD set-up a viable candidate for the type of existence statement in Theorem C. For example, when $b^+(X) = 0$, the trivial flat connection is regular only because the mASD operator allows for this variation in the asymptotic values.

In Section 7, we have included a discussion of how Theorem C for $b^+(X) = 0$ provides a "partial compactification" of the space of mASD connections. We also discuss why this compactification is only partial, and what a more complete compactification would require.

As mentioned above, the lack of gauge-equivariance for the mASD operator means that we are *not* free to pass to the quotient modulo gauge. Indeed, to obtain a Fredholm problem for the gluing constructions, we work entirely within a fixed Coulomb slice. Since the natural Coulomb slice varies as the connections vary, this dependence becomes relevant when we glue over families of connections, which is necessary for Theorem C. This is a central obstacle with which we must contend in the present paper: In the usual ASD setting, one could apply suitable gauge transformations that put all nearby ASD connections into the same slice. However, in this mASD setting, the gauge-transformed mASD connections would no longer be mASD. To account for this, we establish a pair of gauge fixing results, Proposition 4.3 and Theorem 4.5, that show that, by a making an additional perturbation, an mASD connection in one Coulomb slice can be perturbed to an mASD connection in a nearby Coulomb slice.

Apart from the failure of gauge equivariance in the mASD setting, the main difference between the mASD and ASD settings is that we now need to handle the additional nonlinearities that arise from the term modifying the ASD operator. The key observation we use for handling this term is that it factors through a finite-dimensional manifold.

Finally, we mention that if Γ is non-degenerate, then every mASD connection with asymptotic value near Γ is in fact ASD. E.g., this non-degeneracy hypothesis is satisfied when N is a rational homology 3-sphere and Γ is the trivial connection. As such, our results recover standard gluing results for ASD connections on cylindrical end manifolds with non-degenerate asymptotic limits; see Sections **2C** and **6D** for more details. More interestingly, there are situations for which Γ is degenerate, but for which every mASD connection with asymptotic value near Γ is ASD. In such cases, our mASD gluing theorem produces an ASD connection. Theorem A is one result of this type.

Contents

2. Background on the thickened moduli space

In this section we give a rapid review of the relevant background material from [17]; we also expand on some of the results of [17], which will assist in our discussion of gluing below. With a few exceptions, we use much of the same notation and set-up established in [17]. To allow for a more streamlined discussion, we assume throughout that the 3-manifold end N is nonempty; however, see Section **2C** for an extension to the case $N = \emptyset$.

We will write $\mathcal{A}(E)$ and $\mathcal{G}(E)$ for, respectively, the spaces of smooth connections and gauge transformations on $E \to X$. When the bundle is clear from context, we will simply write $\mathcal{A}(X)$ and $\mathcal{G}(X)$. Given a connection A, we denote by F_A its curvature, which is a 2-form on X with values in the adjoint bundle $\mathfrak{g}_E$. We will write $\Omega^\ell(X)$, and sometimes Ω^ℓ, for the space of smooth adjoint bundle-valued ℓ-forms on X that are rapidly decaying.

To touch base with constants associated with characteristic classes below, we work relative to an inner product on $\mathfrak{g}$ obtained as follows. Fix a Lie group homomorphism

$$G \longrightarrow \mathrm{SU}(r) \tag{2.1}$$

that is also an immersion. Then the induced map $\mathfrak{g} \hookrightarrow \mathfrak{su}(r)$ is an embedding of Lie algebras. Let $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ denote the inner product on $\mathfrak{g}$ obtained by pulling back the inner product $A \otimes B \mapsto -\mathrm{tr}(AB)$ on $\mathfrak{su}(r)$. This inner product is Ad-invariant, and so determines a metric on the adjoint bundle $\mathfrak{g}_E$.

Notation such as $L^p_k(\Omega^\ell(X), g)$ will denote the L^p_k-Sobolev completion of $\Omega^\ell(X)$, relative to a metric g on X and the above-defined metric on $\mathfrak{g}_E$. When X or g are clear from context, or not relevant, we may drop them from the notation.

2A. Auxiliary choices.

2A.1. *The center manifold.* Fix a metric g_N on N, and a smooth flat connection Γ on the bundle $Q \to N$. Let $U_\Gamma \subseteq L^2_2(\mathcal{A}(N))$ be a coordinate patch centered at Γ, in the sense of [17, Def. 2.3.1]; for our purposes, it suffices to know that U_Γ is a small open neighborhood of Γ in the Coulomb slice $\{\Gamma\} + \ker(d^*_\Gamma)$. As in [17, Lemma 2.5.1], there is a unique $\mathrm{Stab}(\Gamma)$-equivariant map

$$\Theta \colon U_\Gamma \longrightarrow L^2_2(\Omega^0(N))$$

with $\Theta(a) \in (\ker \Delta_\Gamma)^\perp$ and

$$d^*_\Gamma(*F_a - d_a\Theta(a)) = 0.$$

It follows from this last equation, and uniqueness, that if a has higher regularity then so too does $\Theta(a)$.

We will be interested in the densely-defined vector field

$$\nabla f_\Gamma \colon U_\Gamma \longrightarrow TU_\Gamma, \quad a \longmapsto \nabla_a f_\Gamma := - * F_a + d_a\Theta(a).$$

Note that the zeros of ∇f_Γ are precisely the flat connections in U_Γ. (As described in [17, Lemma 2.5.1(1)], this vector field is the (negative) gradient of the restriction to U_Γ

of the Chern–Simons functional, where the gradient is taken relative to a certain inner product that takes into account the possibility of a non-trivial stabilizer of Γ.)

For $m \geq 2$, let $\mathcal{H} = \mathcal{H}_\Gamma \subseteq U_\Gamma$ be a Stab(Γ)-invariant $\mathcal{C}^m$-center manifold for the vector field ∇f_Γ, as in [17, Cor. 5.1.4]. In particular, this means that

- $\mathcal{H}_\Gamma$ is a finite-dimensional $\mathcal{C}^m$-manifold containing Γ,
- the tangent space to $\mathcal{H}_\Gamma$ at Γ is the Γ-harmonic space
$$H^1_\Gamma := \ker(d_\Gamma \oplus d^*_\Gamma) \subseteq \Omega^1(N),$$
- ∇f_Γ is tangent to $\mathcal{H}_\Gamma$, and
- every zero of ∇f_Γ sufficiently close to Γ is contained in $\mathcal{H}_\Gamma$.

We denote by $\Xi = \Xi_\Gamma$ the restriction of ∇f_Γ to $\mathcal{H}_\Gamma$.

Fix a compactly supported cutoff function $\beta\colon \mathcal{H} \to [0,1]$ that is identically 1 near Γ. The *trimmed vector field* is given by

$$\Xi^{tr}(h) := \beta(h)\Xi(h).$$

Set $\mathcal{H}_{in} = \beta^{-1}(1)$ and $\mathcal{H}_{out} = \beta^{-1}((0,1])$. We will write $\mathcal{H}_{\Gamma,in}, \mathcal{H}_{\Gamma,out}$ etc. in place of $\mathcal{H}_{in}, \mathcal{H}_{out}$, etc. when the connection Γ is relevant.

Fix a real number $T \geq 1$. The trimmed vector field is complete and so, for each $h \in \mathcal{H}$, there is a unique solution $h_T\colon \mathbb{R} \to \mathcal{H}$ to the flow

$$\frac{d}{dt} h_T(t) = \Xi^{tr}(h_T(t)), \quad h_T(T) = h.$$

We then set

$$\alpha(h) := h_T(t) + \Theta(h_T(t))dt.$$

Depending on context, we may view this as a connection on the submanifold End $X \cong [0,\infty) \times N$, or on the cylinder $\mathbb{R} \times N$. Note that if $h_T(t) \in \mathrm{int}(\mathcal{H}_{in})$ lies in the interior, for some t, then $\alpha(h)$ is ASD in a neighborhood of $\{t\} \times N$.

Lemma 2.2. *For all $h \in \mathcal{H}$, the connection $\alpha(h)$ is in $L^2_{2,loc}(\mathbb{R} \times N) \cap \mathcal{C}^0(\mathbb{R} \times N)$, and hence in $L^p_{1,loc}(\mathbb{R} \times N) \cap \mathcal{C}^0(\mathbb{R} \times N)$ for any $1 \leq p < 4$. Moreover, the map $h \mapsto \alpha(h)$ is $\mathcal{C}^m$ relative to the $L^2_2(N)$-topology on the domain and the $\mathcal{C}^0(\mathbb{R} \times N)$-topology on the codomain.*

Proof. The initial condition h is in $L^2_2(N) \subset \mathcal{C}^0(N)$, by assumption. It then follows from standard regularity arguments for flows that the path h_T is in $L^2_{2,loc} \cap \mathcal{C}^0$ on $\mathbb{R} \times N$. Hence $\alpha(h)$ is in the same space as well, since the regularity of $\Theta(h_T)$ is controlled by that of h_T. That α is $\mathcal{C}^m$ relative to these topologies follows from a similar argument applied to its kth derivative for $1 \leq k \leq m$. The assertion about L^p_1 follows from the embedding $L^2_{2,loc} \hookrightarrow L^p_{1,loc}$, which holds provided $1 \leq p < 4$. □

2A.2. *The choice of metric.* We will use $t\colon$ End $X \to [0,\infty)$ to denote the projection relative to the identification End $X \cong [0,\infty) \times N$. With the use of a cutoff function, we can view t as a smooth real-valued function defined on all of X, which we will denote by the same symbol.

Fix a smooth cylindrical end metric g_0 on X; this means that the restriction

$$g_0|_{\mathrm{End}\ X} = dt^2 + g_N$$

is a product metric, where g_N is the fixed metric on N. Let B be a $\mathcal{C}^3$-neighborhood of g_0 in the space of $\mathcal{C}^{\max(m,3)}$-metrics on X so that the conclusions of [17, Theorem 2.6.3] hold (the proof of Theorem 2.6.3 shows that such a set exists; the details of the theorem will not play an active role in the discussion that follows). Let $\mu_\Gamma^\pm$ be the smallest positive eigenvalue of $\mp * d_\Gamma : \Omega^1(N) \to \Omega^1(N)$. Then we will say that a metric g on X is *asymptotically cylindrical* if $g \in B$ and

$$\| g - g_0 \|_{\mathcal{C}^1(\{t\}\times N)} \leq e^{-\max(\mu_\Gamma^-, \mu_\Gamma^+)t}$$

for all $t \geq 0$. (This is effectively Condition A3 of [17, p. 116].) Throughout, we will always assume our metrics are asymptotically cylindrical in this sense. Note that every cylindrical end metric is automatically asymptotically cylindrical.

2A.3. *Thickening data.* Fix data as in [17, Section 7.2]; we will refer to this as the *thickening data* and denote it by $\mathcal{T}_\Gamma$. In particular, this includes the choice of positive numbers ϵ_0 and δ, that we will describe momentarily. The details of the remaining data in $\mathcal{T}_\Gamma$ will not play an active role in our discussion. For convenience, we also assume that $\mathcal{T}_\Gamma$ includes the choice of the fixed $T \geq 1$ from above.

The key feature for us regarding ϵ_0 is that $|\mathcal{CS}(h_2) - \mathcal{CS}(h_1)| < \epsilon_0/2$ for all $h_1, h_2 \in \mathrm{supp}(\beta)$, where $\mathcal{CS}$ is the Chern–Simons function. For any $\epsilon_0 > 0$, this inequality can be arranged by shrinking the support of β, if necessary. The remaining requirements for ϵ_0 will not be directly relevant to us, but see [17, Definition 4.3.2] for more details. As for δ, we assume $\delta > \mu_\Gamma^-$ and that $\delta/2$ is not an eigenvalue of $*d_\Gamma$. By shrinking the size of the coordinate patch U_Γ, if necessary, we may assume further that $\delta/2$ is not an eigenvalue of $*d_{\Gamma'}$ for any $\Gamma' \in U_\Gamma$. At various times, we may place additional restrictions on δ.

2A.4. *Weighted spaces.* We define the space $L^p_{k,\delta}(X)$ to be the completion of the set of compactly supported smooth forms f on X, relative to the weighted Sobolev norm

$$\|f\|_{L^p_{k,\delta}} := \|e^{\delta t/2} f\|_{L^p_k}.$$

When $p = 2$, this definition recovers the set of norms used in [17]. The subspace of ℓ-forms will be denoted by $L^p_{k,\delta}(\Omega^\ell)$ or $L^p_{k,\delta}(\Omega^\ell(X))$. Following standard conventions, when $k = 0$, we will write L^p_δ in place of $L^p_{0,\delta}$. We note that the norm $\|f\|_{L^p_{k,\delta}}$ is equivalent to the norm given by

$$\sum_{0 \leq j \leq k} \|e^{\delta t/2} \nabla^j f\|_{L^p}.$$

In particular, we can use this equivalence to transfer Sobolev embedding results for L^p_k to the weighted setting; e.g., see the proof of Lemma 3.9.

2B. Gauge theory.

2B.1. *The space of connections.* For $1 \leq p < 4$, define $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ to consist of the connections A on E satisfying the following:

- A has regularity $L^p_{1,loc}$,
- there is some $h \in \mathcal{H}_{out}$ so that $A - \alpha(h) \in L^p_{1,\delta}(\Omega^1(\mathrm{End}\ X))$,
- for each $t \geq T$, the connection $A|_{\{t\}\times N}$ is gauge equivalent to a connection in U_Γ.

This space of connections is generally *not* an affine subspace of $L^p_{1,loc}(\mathcal{A}(E))$; this reflects the nonlinearities in the definition of the map $h \mapsto \alpha(h)$. We give $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ the structure of a $\mathcal{C}^m$-Banach manifold, as in [17, Section 7.2.2]. (Equivalently, this $\mathcal{C}^m$-Banach manifold structure is precisely the one for which the map ι, defined in (2.4), is a $\mathcal{C}^m$-diffeomorphism.) By [17, Prop. 7.2.3] (see also [17, p. 120]), given $A \in \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$, the element $h \in \mathcal{H}_{out}$ from the second bullet point is uniquely determined; this uses the assumption that $\delta > \mu^-_\Gamma$. As such, there is a well-defined map

$$p_T \colon \mathcal{A}^{1,p}(\mathcal{T}_\Gamma) \longrightarrow \mathcal{H}_{out},$$

and this is $\mathcal{C}^m$-smooth.

Remark 2.3. Note that our space $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ consists of connections with weaker regularity than the one in [17, Ch. 7], which is modeled on L^2_2 instead of L^p_1. This changes little as far as the exposition of [17] is concerned; the only significant exception to this is the gauge group, which we will discuss in the next section.

Fix a smooth cutoff function β'' on X supported on $[T-1/2,\infty)\times N$ and identically 1 on $[T,\infty)\times N$. Fix also a smooth reference connection A_{ref} on E; we assume this belongs to the space $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$. Using these objects, we can form the map

$$i \colon \mathcal{H}_{out} \longrightarrow \mathcal{A}^{1,p}(\mathcal{T}_\Gamma), \quad h \longmapsto A_{ref} + \beta''(\alpha(h) - A_{ref}),$$

where $\alpha(h) = h_T(t) + \Theta(h_T(t))dt$ is as above. This map i is $\mathcal{C}^m$-smooth. As in [17, Lemma 10.1.1], it is convenient to introduce the map

$$\iota \colon \mathcal{H}_{out} \times L^p_{1,\delta}(\Omega^1(X)) \longrightarrow \mathcal{A}^{1,p}(\mathcal{T}_\Gamma), \quad (h,V) \longmapsto \iota(h,V) := i(h) + V. \tag{2.4}$$

This map ι is a $\mathcal{C}^m$-diffeomorphism with inverse given by $A \mapsto (p_T(A), A - i(p_T(A)))$. It follows immediately from the definitions that

$$p_T(i(h)) = p_T(\iota(h,V)) = h$$

for all $h \in \mathcal{H}_{out}$ and $V \in L^p_{1,\delta}(\Omega^1)$. We view ι as providing something of a coordinate system on the space of connections.

The tangent space to $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ at A is the space of all 1-forms $W \in L^p_{1,loc}(\Omega^1(X))$ so that there is some $\eta \in T_{p(A)}\mathcal{H}$ with

$$W - (Di)_{p_T(A)}\eta \in L^p_{1,\delta}(X),$$

where $(Di)_h$ is the linearization at $h \in \mathcal{H}_{out}$ of the map $i\colon \mathcal{H}_{out} \to \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$. Linearizing ι at (h, V), we obtain a Banach space isomorphism

$$(D\iota)_{(h,V)}\colon T_h\mathcal{H}_{out} \times L^p_{1,\delta}(\Omega^1(X)) \longrightarrow T_A\mathcal{A}^{1,p}(\mathcal{T}_\Gamma) \qquad (2.5)$$
$$(\eta, V') \longmapsto (Di)_h\eta + V'.$$

The 1-form $(Di)_h\eta$ vanishes on X_0, so the operator norm of $(D\iota)_{(h,V)}$ is independent of the metric on X_0.

2B.2. *The gauge group.* When $2 < p < 4$, we will write $\mathcal{G}^{2,p}_\delta(\Gamma)$ for the set of bundle automorphisms u of E with the property that $u^*A \in \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ for all $A \in \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$. The condition on p ensures that $\mathcal{G}^{2,p}_\delta(\Gamma)$ is a well-defined Banach Lie group that acts $\mathcal{C}^m$-smoothly on $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$; we will only consider $\mathcal{G}^{2,p}_\delta(\Gamma)$ for p satisfying $2 < p < 4$. The proof of [17, Lemma 7.2.7] carries over to this setting to imply that the group $\mathcal{G}^{2,p}_\delta(\Gamma)$ is equal to the space of $L^p_{2,loc}$-gauge transformations with the property that there is some $\tau_u \in \mathrm{Stab}(\Gamma)$, viewed as a t-invariant gauge transformation on End X, so that $u|_{\mathrm{End}\ X} \circ \tau_u^{-1}$ is in $L^p_{2,\delta}(\mathrm{End}\ X)$. The gauge transformation τ_u is uniquely determined by u, and we denote by $\mathcal{G}^{2,p}_\delta \subseteq \mathcal{G}^{2,p}_\delta(\Gamma)$ the (normal) subgroup of all gauge transformations u with $\tau_u = \mathrm{Id}$ equal to the identity.

We will write $\mathrm{Stab}(A)$ for the stabilizer of A under the action of $\mathcal{G}^{2,p}_\delta(\Gamma)$. The center $Z(G)$ of G embeds into $\mathcal{G}^{2,p}_\delta(\Gamma)$ as the set of constant maps $X \to Z(G)$, and we will identify $Z(G)$ with its image in the gauge group. Note that $Z(G)$ is also the center of $\mathcal{G}^{2,p}_\delta(\Gamma)$ and $Z(G) \subseteq \mathrm{Stab}(A)$. We will say that A is *(projectively) irreducible* if $Z(G)$ and $\mathrm{Stab}(A)$ have the same dimension (equivalently, if they have isomorphic Lie algebras). Note that the term "irreducible" is only defined when $2 < p < 4$.

Lemma 2.6. *Fix $2 < p < 4$, and assume $A \in \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ is irreducible. Then there is a neighborhood $U \subseteq \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ of A so that A' is irreducible for all $A' \in U$.*

Proof. We begin with a few preliminaries. Set $\mathcal{A} := \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ and $\mathcal{G} := \mathcal{G}^{2,p}_\delta(\Gamma)$. Linearizing the gauge group action at $A \in \mathcal{A}$, we obtain a map

$$d_A : \mathrm{Lie}(\mathcal{G}) \longrightarrow T_A\mathcal{A}, \quad \phi \longmapsto d_A\phi.$$

Then a connection $A \in \mathcal{A}$ is irreducible if and only if the kernel of d_A equals the Lie algebra $\mathfrak{z} := \mathrm{Lie}(Z(G))$ of the center of $\mathcal{G}$. It follows form the definition of the topologies on $\mathcal{A}$ and $\mathcal{G}$, as well as from standard elliptic estimates for δ-decaying spaces, that the operator d_A is bounded. Moreover, this operator has a range that is closed and admits a complement in $T_A\mathcal{A}$.

Let $H^0_\Gamma := \ker(d_\Gamma)$ be the Lie algebra of $\mathrm{Stab}(\Gamma)$. The center $\mathfrak{z}$ is naturally a subalgebra of H^0_Γ, so we can write

$$H^0_\Gamma = \mathfrak{z} \oplus \mathfrak{z}^\perp$$

where $\mathfrak{z}^\perp$ is the $L^2(N)$-orthogonal complement of $\mathfrak{z}$. Then for $\tau \in H^0_\Gamma$, we will write $\tau^\perp \in \mathfrak{z}^\perp$ for its projection.

Just as the map ι provides "coordinates" for $\mathcal{A}$, there is an analogous Banach space isomorphism

$$\iota_{\Omega^0} : H^0_\Gamma \times L^p_{2,\delta}(\Omega^0) \longrightarrow \mathrm{Lie}(\mathcal{G}), \quad (\tau, \xi) \longmapsto (\tau - \tau^\perp) + \beta'' \tau^\perp + \xi;$$

here we are viewing $\tau - \tau^\perp \in \mathfrak{z}$ as a 0-form on X. This map ι_{Ω^0} takes $\mathfrak{z} \times \{0\}$ isomorphically to $\mathfrak{z} \subseteq \mathrm{Lie}(\mathcal{G})$. Let $\mathrm{Y} \subseteq \mathrm{Lie}(\mathcal{G})$ be the image under ι_{Ω^0} of the complement $\mathfrak{z}^\perp \times L^p_{2,\delta}(\Omega^0)$ to $\mathfrak{z} \times \{0\}$. Then we have a direct sum decomposition

$$\mathrm{Lie}(\mathcal{G}) = \mathfrak{z} \oplus \mathrm{Y}.$$

The key point is that A is irreducible if and only if the restriction

$$d_A| : \mathrm{Y} \longrightarrow T_A \mathcal{A}$$

is injective. We will want to view this operator as a function of A, and for this it would be convenient if $d_A|$ were to have a codomain that is independent of A. Though this is not the case presently, we can arrange for A-independence of the codomain as follows: Let $(D\iota)_A : T_h \mathcal{H} \times L^p_{1,\delta}(\Omega^1) \to T_A \mathcal{A}$ be the linearization of the coordinate map ι. Relative to the L^2_2-inner product on N, we can define the parallel transport map $\mathrm{PT}_h : T_h \mathcal{H} \to T_\Gamma \mathcal{H}$. Letting Id denote the identity on $L^p_{1,\delta}(\Omega^1)$, we will be interested in the operator

$$D_A := (\mathrm{PT}_h \times \mathrm{Id}) \circ (D\iota)_A^{-1} \circ d_A| : \mathrm{Y} \longrightarrow T_\Gamma \mathcal{H} \times L^p_{1,\delta}(\Omega^1).$$

This is a bounded linear map, and expansions of the form $d_A = d_{A'} + [A - A', \cdot]$ show it depends continuously on $A \in \mathcal{A}$ in the operator norm topology on the space $\mathcal{B}(\mathrm{Y}, T_\Gamma \mathcal{H} \times L^p_{1,\delta}(\Omega^1))$ of bounded linear maps from Y to $T_\Gamma \mathcal{H} \times L^p_{1,\delta}(\Omega^1)$. Since Y has finite codomension, and d_A has closed range, the operator D_A has closed range as well.

It follows from the construction that D_A is injective if and only if A is irreducible. Assume that A is irreducible. Then the fact that $\mathrm{im}(d_A)$ has a complement in $T_A \mathcal{A}$ implies that D_A admits a bounded left inverse, which we denote by L_A. In summary, the map

$$\mathcal{A} \longrightarrow \mathcal{B}(\mathrm{Y}, \mathrm{Y}), \quad A' \longmapsto L_A D_{A'}$$

is a continuous map into the space of bounded linear operators on the Banach space Y. It is clearly invertible at $A' = A$. Since the set of invertible bounded linear maps on a Banach space is open, there is some neighborhood $U \subseteq \mathcal{A}$ of A so that $L_A D_{A'}$ is invertible for all $A' \in U$. Thus if $A' \in U$, then A' is irreducible. $\square$

Remark 2.7. Completing $L^p_{2,\delta}$ to $L^p_{1,\delta}$, the map D_A extends to a bounded linear operator of the form

$$D_A : L^p_{1,\delta}(\mathrm{Y}) \longrightarrow T_\Gamma \mathcal{H} \times L^p_\delta(\Omega^1(X)).$$

Let $p^* = 4p/(4-p)$ be the Sobolev conjugate of $p \in (2,4)$. Then one can show that the map $A \mapsto D_A \in \mathcal{B}(L^p_{1,\delta}(\mathrm{Y}), T_\Gamma \mathcal{H} \times L^p_\delta(\Omega^1(X)))$ is continuous in $A = i(h) + V$ relative to the topology $(h, V) \in \mathcal{C}^0(N) \times L^{p^*}_\delta(X)$. The proof we gave for Lemma 2.6 carries over to show that $A' = \iota(h', V')$ is irreducible whenever $A = \iota(h, V)$ is irreducible, and $\|h - h'\|_{\mathcal{C}^0} + \|V - V'\|_{L^{p^*}_\delta}$ is sufficiently small.

2B.3. *The* mASD *equation.* Fix a cut off function β' on X that is identically 1 on the cylinder $[T + 1/2, \infty) \times N$ and supported on the slightly larger set $[T, \infty) \times N$. Consider the map

$$s\colon \mathcal{A}^{1,p}(\mathcal{T}_\Gamma) \longrightarrow L^p_\delta(\Omega^+(X)), \quad A \longmapsto F^+_A - \beta' F^+_{i(p(A))}.$$

We will call s the *modified* ASD *(*mASD*) operator*. The equation $s(A) = 0$ is the *modified* ASD *(*mASD*) equation*, and any A satisfying $s(A) = 0$ will be called *modified* ASD *(*mASD*)*. The map s is $\mathcal{C}^m$ in the specified topologies; see [17, Lemma 7.1.1] and use the fact that the composition of $\mathcal{C}^m$ functions is again $\mathcal{C}^m$.

Remark 2.8. Unfortunately, when G is not abelian, the map s is not generally well-behaved under any suitable gauge group; e.g., it is not equivariant relative to the action of the gauge group of Section **2B**. The issue is that the term $F_{i(p(A))}$ is gauge equivariant relative to the *trivial* G-action on $\mathfrak{g}$, while the term F_A is gauge equivariant relative to the *adjoint* G-action on $\mathfrak{g}$. Consequently, any non-trivial linear combination of these (e.g., as in the above formula for s) is not equivariant relative to either G-action. This issue is apparent even in the smooth compactly supported setting, and hence persists regardless of which Sobolev completion we choose.

The following will help us understand the linearization of s.

Lemma 2.9. *If $A = \iota(h, V)$ for $(h, V) \in \mathcal{H}_{out} \times L^p_{1,\delta}(\Omega^1(X))$, then*

$$s(A) = s(\iota(h, V)) = (1 - \beta') F^+_{i(h)} + d^+_{i(h)} V + \frac{1}{2} [V \wedge V]^+ . \tag{2.10}$$

Proof. This follows from the identity $F^+_{i(h)+V} = F^+_{i(h)} + d^+_{i(h)} V + \frac{1}{2} [V \wedge V]^+$ and the fact that $p_T(\iota(h, V)) = h$. □

2B.4. *A Coulomb slice.* To obtain a Fredholm operator, we will restrict the operator s to a Coulomb (gauge) slice

$$\mathcal{SL} = \mathcal{SL}(A') = \left\{ A' + V \;\middle|\; V \in \ker(d^{*,\delta}_{A'}) \subseteq L^p_{1,\delta}(\Omega^1) \right\}$$

for some fixed connection A'. Here $d^{*,\delta}_A = e^{-t\delta} d^*_A e^{t\delta}$ is the adjoint relative to the L^2_δ-inner product; see [17, Prop. 10.3.1]. We set

$$\widehat{\mathcal{M}} = \widehat{\mathcal{M}}(\mathcal{T}_\Gamma, A') := s^{-1}(0) \cap \mathcal{SL}(A'),$$

which we refer to as the *space of* mASD*-connections*. For us, this will play the role that the ASD moduli space usually plays in the closed setting (though, as discussed in the introduction, this is less than satisfying for global considerations due to its dependence on A'). Elliptic regularity implies that any element of $\widehat{\mathcal{M}}$ has regularity $\mathcal{C}^m$.

The map s is equivariant relative to the action of the finite-dimensional group $\mathrm{Stab}(\Gamma)$. However, the slice $\mathcal{SL}(A')$ is only preserved by the group $\mathrm{Stab}(A')$, which embeds into

$\mathrm{Stab}(\Gamma)$ via the map $u \mapsto \tau_u$ from Section **2B.2**. As such, the space $\widehat{\mathcal{M}}(\mathcal{T}_\Gamma, A')$ only admits an action of the subgroup $\mathrm{Stab}(A') \subseteq \mathrm{Stab}(\Gamma)$.

Consider the restriction of the mASD operator s to this slice $\mathcal{SL}(A')$. Then the linearization at $A \in \mathcal{SL}(A')$ of this restriction is a bounded linear map

$$(Ds)_A \colon T_A\mathcal{SL}(A') \longrightarrow L^p_\delta(\Omega^+).$$

Standard elliptic theory implies that this has closed range, and we denote the cokernel by

$$H^+_{A,\delta} := \mathrm{coker}\ (Ds)_A. \tag{2.11}$$

We will say that an mASD connection A is *regular* if $H^+_{A,\delta} = 0$; note that this condition depends on A' and δ, not just A. We will write

$$\widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A') \subseteq \widehat{\mathcal{M}}(\mathcal{T}_\Gamma, A')$$

for the subset of regular mASD connections. It follows from the implicit function theorem that $\widehat{\mathcal{M}}_{reg}$ is a $\mathcal{C}^m$-smooth manifold.

Despite Remark 2.8, essentially all of the linear analysis of [17, Ch. 8] remains valid, with the index of $(Ds)_A$ used in place of the index of the complex $E_\delta(w)$ defined in [17, Section 8.5]. In particular, the operator $(Ds)_A$ is Fredholm precisely because we have assumed $\delta/2$ is not in the spectrum of $*d_\Gamma|_{\mathrm{Im}(d^*_\Gamma)}$.

Remark 2.12. When $0 < \delta/2 < \mu^-_\Gamma$ and $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$, then the index of $(Ds)_A$ is given by the formal dimension formula appearing in [17, Prop. 8.5.1]. The index for more general compact G can be computed using the strategy outlined in [5, Section 7.1] (e.g., when G is simple and simply-connected, use the data from [1, Table 8.1] to pin down the constants specific to G).

2C. Special cases.

2C.1. *Flat connections.* In this section, we study the operator $(Ds)_A$ and its cokernel in the special case when A is flat. To simplify the discussion, we assume A is in temporal gauge on the end (see [4, p. 15]). We continue to work in the general setting where the metric is asymptotically cylindrical. It follows that, for each $t \geq T$, the restriction $A|_{\{t\}\times N} = \Gamma$ is constantly equal to Γ on the end. Then $A \in \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ and $p(A) = \Gamma$. The associated flow $\alpha(\Gamma) = A$ recovers the flat connection A on End X. This implies A is mASD.

The operator s is defined in terms of the map ι, and we recall that the definition of ι required the choice of a reference connection A_{ref} on X. It is convenient to take $A_{ref} := A$; the reader can check that any other choice does not affect the outcome of the discussion that follows, but only makes the notation at points more complicated. In particular, this gives

$$A = i(\Gamma) = \iota(\Gamma, 0).$$

Let $b^+(X, A)$ be the dimension of a maximal positive definite subspace for the pairing map $q_A \colon \hat{H}^2(\overline{X}, \mathrm{ad}(A)) \otimes \hat{H}^2(\overline{X}, \mathrm{ad}(A)) \to \mathbb{R}$, as in [17, Section 8.7], where $\overline{X}$ is the natural compactification of X obtained by adding a copy of N at infinity. For example, when $A = A_{\text{triv}}$ is the trivial connection on the trivial G-bundle, then $b^+(X, A_{\text{triv}}) = \dim(G) b^+(X)$ is a multiple of the usual self-dual Betti number of X. We will need the following result.

Proposition 2.13. *Assume* $0 < \delta/2 < \mu_\Gamma^-$*, where* μ_Γ^- *is as in Section* **2A***. Then the cokernel* $H^+_{A,\delta}$ *has dimension* $b^+(X, A)$*.*

This is proved in [17, Prop. 8.7.1(4)], however the discussion there does not deal with the linearized operator $(Ds)_A$ directly. In preparation for our gluing arguments below, we will summarize the argument given in [17, Prop. 8.7.1(4)], but from the present perspective. Our proof is sketched below, after we give some preliminary computations that will be useful in their own right.

The restriction $\iota|\colon \mathcal{H} \times \ker(d_A^{*,\delta}) \to \mathcal{SL}(A)$ is a diffeomorphism, essentially by definition. To understand the cokernel $H^+_{A,\delta}$, it suffices to understand the cokernel of the linearization of $s \circ \iota|$. Towards this end, differentiating (2.10) at $A = \iota(\Gamma, 0)$ in the direction of $(\eta, V) \in T_\Gamma \mathcal{H}_{out} \times \ker(d_A^{*,\delta})$ gives

$$(Ds)_A \circ (D\iota)_{(\Gamma,0)}(\eta, V) = (1 - \beta') d_A^+ (Di)_\Gamma \eta + d_A^+ V. \tag{2.14}$$

It follows from the definition of i, and the fact that Γ is flat, that $(Di)_\Gamma \eta = \beta'' \eta$. Hence

$$d_A^+ (Di)_\Gamma \eta = (\partial_t \beta'')(dt \wedge \eta)^+, \tag{2.15}$$

which is zero everywhere except on $(T - 1/2, T) \times N$, where it vanishes if and only if $\eta = 0$. Combining this with (2.14), we therefore have the formula

$$(Ds)_A \circ (D\iota)_{(\Gamma,0)}(\eta, V) = (1 - \beta')(\partial_t \beta'')(dt \wedge \eta)^+ + d_A^+ V. \tag{2.16}$$

This shows that, relative to the coordinates afforded by ι, the leading order term of $(Ds)_A$ is $d_A^+|_{\ker(d_A^{*,\delta})}$, together with a compactly-supported term.

Proof of Proposition 2.13 (sketch). A maximal positive definite subspace for q_A can be realized as the space $H^+(X, \mathrm{ad}(A))$ of self-dual 2-forms $W \in L^2(\Omega^+(X, \mathrm{ad}(A)))$ satisfying $d_A W = 0$ and so that the restriction to any slice $\{t\} \times N$ has trivial Γ-harmonic part; note that this definition does not depend on δ, but see also Lemma 2.17. Similarly, we can represent the cokernel $H^+_{A,\delta}$ of $(Ds)_A$ as the L^2_δ-orthogonal complement $(\mathrm{im}\,(Ds)_A)^{\perp,\delta}$ to the image of $(Ds)_A$. Then Proposition 2.13 follows by showing that the map

$$j\colon (\mathrm{im}\,(Ds)_A)^{\perp,\delta} \longrightarrow H^+(X, \mathrm{ad}(A)), \quad W \longmapsto e^{\delta t} W$$

is well-defined and bijective. That the map j is well-defined follows from the formula in (2.16). Indeed, if $W \in (\mathrm{im}\,(Ds)_A)^\perp$, then we have

$$0 = (W, (Ds)_A(0, V))_\delta = (W, d_A^+ V)_\delta = (d_A^{*,\delta} W, V)_\delta,$$

where $(\cdot,\cdot)_\delta$ is the L^2_δ-inner product. This holds for all $V \in L^2_{1,\delta}(\Omega^1)$, so it follows that $d_A j(W) = -e^{t\delta} * d_A^{*,\delta} W = 0$. Similarly, we have

$$0 = (W, (Ds)_A(\eta, 0))_\delta = (W, (1-\beta')(\partial_t \beta'')(dt \wedge \eta)^+)_\delta.$$

Since $(1-\beta')\partial_t\beta''$ is non-zero on the cylinder $(T-1/2, T) \times N$, and since $\eta \in T_\Gamma \mathcal{H} = H^1_\Gamma$ is allowed to roam freely over the Γ-harmonic space, it follows that the harmonic part of $W|_{\{t\}\times N}$ must vanish for any $t \in (T-1/2, T)$. Bijectivity of j follows from the index calculation of [17, Prop. 8.7.1(4)], which uses the assumption $0 < \delta/2 < \mu_\Gamma^-$. $\square$

We end with the following exponential decay estimate that we will use in Section **6C**.

Lemma 2.17. *For each $W \in H^+(X, \mathrm{ad}(A))$, there is some C so that the restriction $W|_{\{t\}\times N}$ satisfies*

$$\|W|_{\{t\}\times N}\|_{C^0(N)} \leq C e^{-\mu_\Gamma^- t}$$

for all $t \geq 0$. In particular, $H^+(X, \mathrm{ad}(A)) \subseteq L^2_\delta(X)$ for any $\delta < 2\mu_\Gamma^-$.

Proof. It suffices to establish the estimate of the lemma under the assumption that the metric g is cylindrical. Since A is in temporal gauge on the end, its covariant derivative decomposes as $d_A = dt \wedge \partial_t + d_\Gamma$. Standard elliptic estimates for the operator $\Delta_\Gamma = d_\Gamma^* d_\Gamma + d_\Gamma d_\Gamma^*$ on N provide a uniform constant C so that

$$\|v\|_{C^0(N)} \leq C(\|v\|_{L^2(N)} + \|\Delta_\Gamma v\|_{L^2(N)})$$

for all $v \in \Omega^1(N)$.

Fix $W \in H^+(X, \mathrm{ad}(A))$. On End X, we can write $W = dt \wedge v + *_N v$ for some path $v = v(t) \in \Omega^1(N)$ of 1-forms. The condition $d_A W = 0$ implies $d_\Gamma *_N v = 0$ and $*_N d_\Gamma v = \partial_t v$. In particular, the above elliptic estimate implies

$$\|v(t)\|_{C^0(N)} \leq C(\|v(t)\|_{L^2(N)} + \|\partial_t^2 v(t)\|_{L^2(N)}).$$

It suffices to show that $f(t) := \|v(t)\|^2_{L^2(N)} + \|\partial_t^2 v(t)\|^2_{L^2(N)}$ decays exponentially in t at a rate of $2\mu_\Gamma^-$. To see this, differentiate twice to get

$$\begin{aligned} f''(t) &= 2\|\partial_t v(t)\|^2_{L^2(N)} + 2\|\partial_t^3 v(t)\|^2_{L^2(N)} + 2(\partial_t^2 v(t), v(t)) + 2(\partial_t^4 v(t), \partial_t^2 v(t)) \\ &= 4\|d_\Gamma v(t)\|^2_{L^2(N)} + 4\|d_\Gamma \partial_t^2 v(t)\|^2_{L^2(N)}, \end{aligned}$$

where we used $\partial_t v = *_N d_\Gamma v$ and integration by parts. By definition of $H^+(X, \mathrm{ad}(A))$, $v(t)$ is orthogonal to the Γ-harmonic space and $d_\Gamma *_N v(t) = 0$. It follows that $v(t)$ lies in the image of $*_N d_\Gamma$. Moreover, the 2-form W is in L^2, by definition. This combines with the equation $*_N d_\Gamma v = \partial_t v$ to imply that $v(t)$ lies in the span of the negative eigenspaces of d_Γ; the same is true of $\partial_t^2 v(t)$. The definition of μ_Γ^- then implies that

$$f''(t) = 4\|d_\Gamma v(t)\|^2_{L^2(N)} + 4\|d_\Gamma \partial_t^2 v(t)\|^2_{L^2(N)} \geq 4(\mu_\Gamma^-)^2 f(t).$$

Since $f(t)$ is non-negative and converges to 0 as t approaches ∞, it follows from this estimate that $f(t) \leq C e^{-2\mu_\Gamma^- t}$; e.g., see [6, p. 623]. $\square$

2C.2. *Non-degenerate* Γ. Here we assume that Γ is non-degenerate in the sense that the harmonic space $H^1_\Gamma = \{0\}$ is trivial. Then any center manifold necessarily consists of a single point and so, e.g., there is a unique choice of cutoff function β. Then the mASD operator is the ASD operator. Moreover, non-degeneracy implies that any finite-energy ASD connection asymptotic to Γ decays exponentially on the end at a rate of $e^{-\delta t}$ for any $\delta/2 < \mu^-_\Gamma$, where μ^-_Γ is as in Section **2A**; the proof of this assertion is similar to the proof of Lemma 2.17. In particular, in the discussion of Section **2B**, we can work with any such $\mu^-_\Gamma < \delta < 2\mu^-_\Gamma$, and the resulting mASD/ASD space would be independent of this choice of δ. In summary, when Γ is non-degenerate, there is essentially a canonical choice of thickening data $\mathcal{T}_{\Gamma,\mathrm{can}}$. Moreover, if A' is any connection defining a slice, then the associated space of mASD connections

$$\widehat{\mathcal{M}}(\mathcal{T}_{\Gamma,\mathrm{can}}, A') = \left\{ A \in \mathcal{A}(X) \;\middle|\; F^+_A = 0, d^*_{A'}(A - A') = 0, \lim_{t\to\infty} A|_{\{t\}\times N} = \Gamma \right\}$$

is the set of ASD connections in the A'-Coulomb slice that are asymptotic to Γ.

2C.3. *Closed* X. Here we assume that X is closed. We can handle this situation within the framework of mASD connections in two, essentially equivalent, ways. The first is by viewing X as a cylindrical end 4-manifold with an empty end. Then one can check that it makes sense to choose the empty set $\mathcal{T}_\emptyset = \emptyset$ of thickening data, and that, e.g., the mASD space $\widehat{\mathcal{M}}(\mathcal{T}_\emptyset, A')$ is exactly the set of ASD connections on E in the A'-Coulomb slice.

Alternatively, we can remove a point in X, thereby creating a cylindrical end manifold X' with 3-manifold end S^3. By conformal scaling, the metric on X induces a cylindrical end metric on X'. Fixing a trivialization of the bundle at the deleted point, we can take $\Gamma = \Gamma_{\mathrm{triv}}$ to be the trivial connection. This is non-degenerate, and so the discussion from the previous section applies. By the conformal invariance of the ASD equation, and Uhlenbeck's removable singularities theorem [23], the associated mASD space

$$\widehat{\mathcal{M}}(\mathcal{T}_{\Gamma_{\mathrm{triv}},\mathrm{can}}, A'|_{X'}) \cong \widehat{\mathcal{M}}(\mathcal{T}_\emptyset, A')$$

can be canonically identified with the same ASD space from the previous paragraph.

3. Gluing two mASD connections

Here we state and prove our first gluing result, which discusses gluing together mASD connections over the compact parts of two cylindrical end 4-manifolds. When the connections are not regular, the resulting connection may not be mASD, and its failure to be mASD is captured by a suitable obstruction map. Our set-up is very similar to that of ASD gluing outlined in Donaldson–Kronheimer [5, Section 7.2], to which we refer the reader for more details at various points. When introducing new terms for the analysis, we have tried to keep our notation as consistent with that of [5] as possible. Our emphasis below will be on the new features that arise in the mASD setting. In the present section, the only serious new features arise from the fact that the mASD operator s has a nonlinear term not present in the usual ASD setting; these features manifest themselves in the proofs of the claims appearing in the proof of Theorem 3.3.

3A. Set-up for gluing. Let X_1 and X_2 be oriented cylindrical end 4-manifolds equipped with asymptotically cylindrical metrics as in Section **2A.2**. We will write X_{k0} for compact part of X_k and we set $N_k := \partial X_{k0}$. We will need parameters $\lambda > 0$ and $L > 1$ so that $b := 4L\lambda^{1/2} \ll 1$. The constant L will be fixed later, but we will ultimately be interested in allowing λ to be arbitrarily small. For each k, fix a point

$$x_k \in B_b(x_k) \subset \mathrm{int}(X_{k0})$$

in the interior of the compact part. To simplify the discussion, we assume that the metric on X_k is flat over $B_b(x_k)$; see [3, Section IV(vi)] for how to extend the discussion to handle more general metrics.

Following the approach in [5, Section 7.2.1], we glue along the annuli

$$\Omega_k := B_{L\lambda^{1/2}}(x_k) \backslash B_{L^{-1}\lambda^{1/2}}(x_k),$$

using an "inversion" map $f_\lambda \colon \Omega_1 \to \Omega_2$, to produce a connected sum

$$X = X(L, \lambda) := (X_1 \backslash B_{L^{-1}\lambda^{1/2}}(x_1)) \cup_{f_{L,\lambda}} (X_2 \backslash B_{L^{-1}\lambda^{1/2}}(x_2)).$$

Then X is an oriented cylindrical end 4-manifold with asymptotic 3-manifold $N = N_1 \sqcup N_2$. We will write X_0 for the compact part of X; this is formed by analogously gluing the compact parts X_{k0} of the X_k. The metrics on the X_k can be glued to form a metric on X, and we assume this is done as outlined at the end of p. 293 in [5]. We denote this metric by $g_{L,\lambda}$. Since we are interested in the limiting behavior of this for small λ, we will include the metric in the notation for our various norms and spaces of connections, forms, etc. whenever it is relevant.

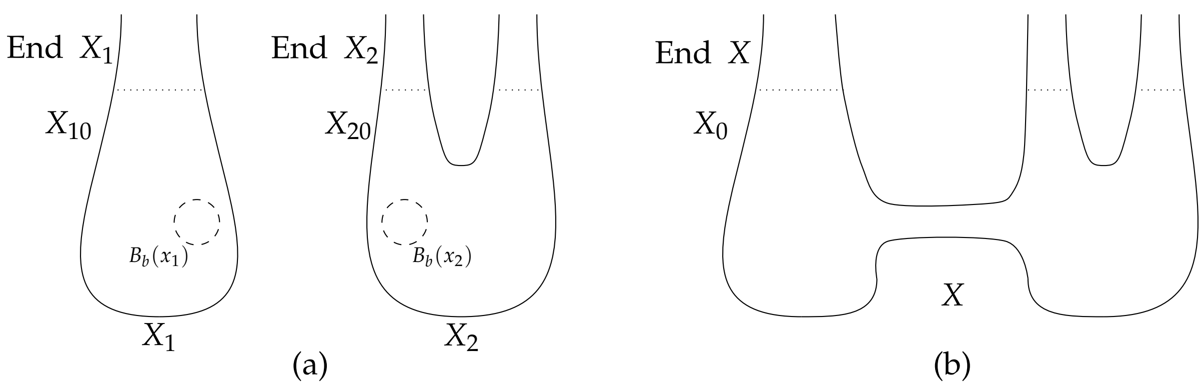

FIGURE 1. Illustrated above are the manifolds X_1, X_2 in (a), and their connected sum X in (b). The 3-manifolds N_1, N_2, and N are unlabeled, but are illustrated as dotted lines in the figure above.

Fix principal G-bundles $E_k \to X_k$ and flat connections $\Gamma_k \in \mathcal{A}(N_k)$ for $k = 1, 2$. These induce a bundle over N as well as a flat connection Γ on N. Fix $\delta > 0$ as in Section **2A.3** associated to this flat connection Γ. It follows that, for $k = 1, 2$, the quantity $\delta/2$ is not in the spectrum of $- * d_{\Gamma_k}|_{\mathrm{Im}(d^*_{\Gamma_k})}$. Let $\mathcal{T}_{k,\Gamma_k}$ be thickening data for E_k with this δ.

Fix an isomorphism $\rho\colon (E_1)_{x_1} \to (E_2)_{x_2}$ of G-spaces, as well as flat connections $A_{\flat,k}$ for E_k over $B_b(x_k)$. Using these flat connections, we can extend ρ to a bundle isomorphism $E_1|_{\Omega_1} \cong E_2|_{\Omega_2}$ covering f_λ. It is with this bundle isomorphism that we glue the E_k over the Ω_k to obtain a bundle

$$E = E(\rho, L, \lambda) \longrightarrow X(L, \lambda).$$

Since the gluing takes place away from the cylindrical end, the thickening data $\mathcal{T}_{k,\Gamma_k}$ for the E_k induce thickening data $\mathcal{T}_\Gamma$ for E.

Fix $1 \leq p < 4$ and suppose that, for each k, we have a smooth mASD connection

$$A_k \in \mathcal{A}^{1,p}(\mathcal{T}_{k,\Gamma_k})$$

on X_k. By performing the cutting off procedure described in Sections 7.2.1 and 4.4.5 of [5], we can form a connection A'_k on E_k that is equal to A_k outside of the ball $B_b(x_k)$ and equal to the flat connection $A_{\flat,k}$ inside of the ball $B_{b/2}(x_k)$. Then the A'_k patch together to determine a smooth connection $A' = A'(A_1, A_2)$ on E; this depends on ρ, L, λ and the A_k. It follows that A' is equal to A_k in $X_k \backslash B_b(x_k) \subseteq X$ and that A' is approximately mASD:

$$\|s(A')\|_{L^p_\delta(X, g_{L,\lambda})} \leq C b^{4/p}, \tag{3.1}$$

where C is a constant independent of L, λ; see (7.2.36) in [5]. We will refer to A' as the *preglued connection*. We define the maps i and ι of Section **2B.1** in terms of A'.

Remark 3.2. Assume $2 < p < 4$ and set $p^* = 4p/(4-p)$. By [5, Eq. (7.2.37)], as $b \to 0$, the connections A'_k converge in $L^{p^*}_\delta$ to A_k. In particular, by Remark 2.7, if A_k is irreducible, then so too is A'_k, provided b is sufficiently small. The stabilizer group of A' is contained in that of A'_k, and so it follows that A' is irreducible when either of A_1 or A_2 is irreducible and b is sufficiently small.

Set $H^+_k := H^+_{A_k,\delta}$; see (2.11). As described in [5, p. 290], we can choose lifts

$$\sigma_k\colon H^+_k \longrightarrow L^p_\delta(\Omega^+(X_k))$$

so that the operator $(Ds)_{A_k} \oplus \sigma_k$ is surjective. Moreover, we can do this in such a way that, for every $v \in H^+_k$, the form $\sigma_k(v)$ is supported in the complement of the ball $B_{2b}(x_k)$. Set

$$H^+ := H^+_1 \oplus H^+_2$$

and consider the linear map

$$\sigma := \sigma_1 \oplus \sigma_2\colon H^+ \longrightarrow L^p_\delta(\Omega^+(X), g_{L,\lambda}).$$

Relative to the $L^p_\delta(X, g_{L,\lambda})$-norm on H^+, this map σ is bounded with a bound independent of L, λ.

3B. **Gluing two connections.** The main result of this section is as follows.

Theorem 3.3. *Assume* $2 \leq p < 4$. *Fix* ρ, δ, *thickening data, and* mASD *connections* A_1, A_2 *as in Section* **3A**. *Then there are constants* $C, C', L, \lambda_0 > 0$ *so that the following holds for each* $0 < \lambda < \lambda_0$.

Let $A' = A'(A_1, A_2)$ *be the preglued connection constructed from* ρ, L, λ, *and the* A_k.

(a) There is a $\mathcal{C}^m$*-map* $J_{A_1,A_2} \colon L^p_\delta(\Omega^+(X), g_{L,\lambda}) \to \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ *that satisfies* $J_{A_1,A_2}(0) = A'$. *The first* m *derivatives of* $\xi \mapsto J_{A_1,A_2}(\xi)$ *are bounded in operator norm by a bound that is independent of* λ.

(b) There is a linear map $\pi \colon L^p_\delta(\Omega^+(X), g_{L,\lambda}) \to H^+$ *satisfying* $\sigma \circ \pi \circ \sigma = \sigma$ *and*

$$\|\pi\xi\|_{H^+} \leq C\|\xi\|_{L^p_\delta(X,g_{L,\lambda})} \qquad \forall \xi \in \Omega^+(X).$$

(c) There is a unique 2-form $\xi(A_1, A_2) \in L^p_\delta(\Omega^+(X))$ *so that*

$$\|\xi(A_1, A_2)\|_{L^p_\delta(X,g_{L,\lambda})} \leq Cb^{4/p} \tag{3.4}$$

and so that the connection $\mathcal{J}(A_1, A_2) := J_{A_1,A_2}(\xi(A_1, A_2))$ *satisfies*

$$s(\mathcal{J}(A_1, A_2)) = -\sigma\pi\xi(A_1, A_2). \tag{3.5}$$

In particular, for $k = 1, 2$ *the connection* $\mathcal{J}(A_1, A_2)$ *is close to* A_k *on* $X_k \backslash B_{L\lambda^{1/2}}(x_k) \subseteq X$ *in the sense that*

$$\|\iota^{-1}(\mathcal{J}(A_1, A_2)) - \iota^{-1}(A_k)\|_{L^2_2(N_k)\times L^p_{1,\delta}(X_k\backslash B_{L\lambda^{1/2}}(x_k))} \leq C'b^{4/p}. \tag{3.6}$$

If A_1 *and* A_2 *are regular, then so is* $\mathcal{J}(A_1, A_2)$. *In this case,* $\mathcal{J}(A_1, A_2))$ *is* mASD *and the maps* $(A_1, A_2, \xi) \mapsto J_{A_1,A_2}(\xi)$ *and* $(A_1, A_2) \mapsto \xi(A_1, A_2)$ *are both* $\mathcal{C}^m$*-smooth, relative to the specified topologies.*

If $p > 2$ *and either* A_1 *or* A_2 *is irreducible, then so is* $\mathcal{J}(A_1, A_2)$.

Before getting to the proof, we briefly discuss the maps appearing in this theorem, and their analogues in the standard ASD theory; precise definitions of these maps are given in the proof, below. The map π is a measure of the failure of A' to be regular. It serves the same role here and enjoys the same properties as the map of the same name [5, pp. 290—291] in the ASD setting. As for J, this map is formed from a near-right inverse P of the linearization $Ds_{A'} : T_{A'}\mathcal{A} \to \Omega^+$, composed with an exponential map $T_{A'}\mathcal{A} \to \mathcal{A}$. As an example, in the special case where Γ is non-degenerate (so mASD = ASD), the relevant space of connections $\mathcal{A}$ is an affine space, and this exponential map is simply given by the affine action. In this case, the map J_{A_1,A_2} simplifies to $J_{A_1,A_2}(\xi) = A'(A_1, A_2) + P\xi$, just as in the usual ASD setting [5, p. 289].

The object $\mathcal{J}(A_1, A_2)$ is the glued connection we are after. Equation (3.5) expresses the degree to which this connection is mASD. In particular, the obstruction map mentioned in the introduction can be taken to be the map $(A_1, A_2) \mapsto \pi \circ \xi(A_1, A_2)$. Finally we mention that the map diffeomorphism ι^{-1} appears in (3.6) only to make explicit the sense in which $\mathcal{J}(A_1, A_2)$ approximates the A_k away from the gluing points.

The proof of Theorem 3.3 that we adopt relies on the following two lemmas.

Lemma 3.7. *Let $S\colon B \to B$ be a $\mathcal{C}^m$-map on a Banach space B with $S(0) = 0$ and*

$$\|S(\xi_1) - S(\xi_2)\| \leq \kappa(\|\xi_1\| + \|\xi_2\|)\|\xi_1 - \xi_2\|, \tag{3.8}$$

for some $\kappa > 0$ and all $\xi_1, \xi_2 \in B_1(0) \subset B$ in the unit ball. Then for each $\eta \in B$ with $\|\eta\| < 1/(10\kappa)$, there is a unique $\xi \in B$ with $\|\xi\| \leq 1/(5\kappa)$ such that

$$\xi + S(\xi) = \eta.$$

Moreover, if $\eta = \eta(a)$ depends $\mathcal{C}^m$-smoothly on a parameter a, then the solution $\xi = \xi(a)$ depends $\mathcal{C}^m$-smoothly on this parameter as well.

Proof. The existence and uniqueness follows from the contraction mapping principle and is carried out in [5, Lemma 7.2.23]. The $\mathcal{C}^m$-smooth dependence of ξ on a follows from, e.g., the discussion in [14, Section I.5]. □

The remaining lemma will be used to establish the nonlinear estimate (3.8) in our mASD setting. In its statement, we write $p^* = 4p/(4-p)$ for the Sobolev conjugate of $p < 4$.

Lemma 3.9. *Assume $2 \leq p < 4$. There is a constant C so that if $L, \lambda > 0$ are any constants for which the connected sum X is defined, then*

$$\|fg\|_{L^p_\delta(X, g_{L,\lambda})} \leq C\|f\|_{L^{p^*}_\delta(X, g_{L,\lambda})}\|g\|_{L^{p^*}_\delta(X, g_{L,\lambda})}$$

for all real-valued functions $f, g \in L^{p^*}_\delta(X)$.

Proof. Writing $X = X_0 \cup \text{End } X$, it suffices to show that there is a uniform constant C so that

$$\begin{aligned} \|fg\|_{L^p(X_0, g_{L,\lambda})} &\leq C\|f\|_{L^{p^*}(X_0, g_{L,\lambda})}\|g\|_{L^{p^*}(X_0, g_{L,\lambda})}, \\ \|fg\|_{L^p_\delta(\text{End } X, g_{L,\lambda})} &\leq C\|f\|_{L^{p^*}_\delta(\text{End } X, g_{L,\lambda})}\|g\|_{L^{p^*}_\delta(\text{End } X, g_{L,\lambda})}. \end{aligned}$$

We begin with the estimate over End X. Note that the metric $g_{L,\lambda}$ is independent of λ over this region, so we do not need to worry about showing that any such constant C is independent of λ. To obtain the estimate, use the definition of the δ-dependent norms, together with Hölder's inequality to get

$$\begin{aligned} \|fg\|_{L^p_\delta(\text{End } X)} &= \|e^{t\delta/2}fg\|_{L^p(\text{End } X)} \\ &= \|(e^{-t\delta/2}e^{t\delta/2}f)(e^{t\delta/2}g)\|_{L^p(\text{End } X)} \\ &\leq \|e^{-t\delta/2}(e^{t\delta/2}f)\|_{L^4(\text{End } X)}\|e^{t\delta/2}g\|_{L^{p^*}(\text{End } X)} \end{aligned}$$

Since $2 \leq p < 4$, we have $4 \leq p^* < \infty$. Hence there is some $4 < r \leq \infty$ with $4^{-1} = r^{-1} + (p^*)^{-1}$. Then we can use Hölder's inequality again to continue the above:

$$\|fg\|_{L^p_\delta(\text{End } X)} \leq \|e^{-t\delta/2}\|_{L^r(\text{End } X)}\|f\|_{L^{p^*}_\delta(\text{End } X)}\|g\|_{L^{p^*}_\delta(\text{End } X)}.$$

Then the requisite estimate holds with $C = \|e^{-t\delta/2}\|_{L^r(\text{End } X)}$, which is plainly finite.

As for the estimate over X_0, the same type of argument gives

$$\|fg\|_{L^p(X_0, g_{L,\lambda})} \leq \text{vol}(X_0, g_{L,\lambda})^{1/r}\|f\|_{L^{p^*}(X_0, g_{L,\lambda})}\|g\|_{L^{p^*}(X_0, g_{L,\lambda})}.$$

As discussed on p. 293 of [5], the condition $p \geq 2$ implies that $\mathrm{vol}(X_0, g_{L,\lambda})$ can be taken to be independent of L, λ, provided $L\lambda^{1/2}$ is uniformly bounded from above (which is necessarily the case whenever the connected sum is defined). □

Proof of Theorem 3.3. Our intention is to apply Lemma 3.7. To do this, we need to recast solving $s(A) = 0$ for A into solving an equation for a self-mapping S of a Banach space. Ultimately, the Banach space will be the codomain of the mASD operator s, and S will be the quadratic part of s, precomposed with the linearization of s at A'.

We begin this process by passing to a local chart on $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ (recall from Section **2B.1** that this space of connections is generally *not* an affine space). For this, write

$$A' = \iota(h', V') = i(h') + V'$$

for $(h', V') \in \mathcal{H}_{\Gamma,out} \times L^p_{1,\delta}(\Omega^1(X))$. Let $\exp_{h'} \colon B_\epsilon(0) \subset T_{h'}\mathcal{H} \to \mathcal{H}$ be the exponential map associated to the $L^2_2(N)$-metric on $\mathcal{H} := \mathcal{H}_\Gamma$; here $\epsilon > 0$ is small enough so that the exponential is well-defined. This is all taking place on the 3-manifold of N, and so this exponential and this ϵ are manifestly independent of L, λ. Coupling this exponential on $\mathcal{H}$ with the exponential on Ω^1 given by the affine action, we obtain a map

$$\begin{array}{rcl} \exp_{(h',V')} \colon B_\epsilon(0) \times L^p_{1,\delta}(\Omega^1(X)) & \longrightarrow & \mathcal{H} \times L^p_{1,\delta}(\Omega^1(X)) \\ (\eta, V) & \longmapsto & (\exp_{h'}(\eta), V' + V). \end{array}$$

The chart for $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ that we will use is $\iota \circ \exp_{(h',V')}$.

Remark 3.10. Throughout the proof that follows, we will work with the $L^2_2(N)$-norm on $T_{h'}\mathcal{H}$; we will often not keep track of this in the notation. Note that this choice of norm is effectively immaterial since $\mathcal{H}$ is finite-dimensional and so any two norms are equivalent, provided they are well-defined.

Consider the map

$$\begin{array}{rcl} \widetilde{s} \colon T_{h'}\mathcal{H} \times L^p_{1,\delta}(\Omega^1(X), g_{L,\lambda}) & \longrightarrow & L^p_\delta(\Omega^+(X), g_{L,\lambda}) \\ (\eta, V) & \longmapsto & s(\iota(\exp_{(h',V')}(\eta, V))), \end{array}$$

which is the map s relative to the chart just described. This satisfies $\widetilde{s}(0,0) = s(A')$, and so (3.1) gives

$$\|\widetilde{s}(0,0)\|_{L^p(X,g_{L,\lambda})} \leq C b^{4/p}. \tag{3.11}$$

Write $(D\widetilde{s})_{(\eta,V)}$ for the linearization of $\widetilde{s}$ at (η, V). The definition of σ implies that the operator $(D\widetilde{s})_{(0,0)} \oplus \sigma$ is surjective, and the following provides appropriate uniform estimates for a right inverse of this operator.

Claim 1: For $2 \leq p < 4$, there are constants $C_1, \lambda_0 > 0$, and $L > 1$, as well as linear maps

$$\begin{array}{rcl} P \colon L^p_\delta(\Omega^+(X), g_{L,\lambda}) & \longrightarrow & T_{h'}\mathcal{H} \times L^p_{1,\delta}(\Omega^1(X), g_{L,\lambda}) \\ \pi \colon L^p_\delta(\Omega^+(X), g_{L,\lambda}) & \longrightarrow & H^+ \end{array}$$

so that $P \oplus \pi$ is a right inverse to $(D\widetilde{s})_{(0,0)} \oplus \sigma$ that, for all $0 < \lambda < \lambda_0$, satisfies

$$\|(P \oplus \pi)\xi\|_{(L^2_2(N)\times L^{p^*}_\delta(X,g_{L,\lambda}))\oplus L^p_\delta(X,g_{L,\lambda})} \le C_1\|\xi\|_{L^p_\delta(X,g_{L,\lambda})}, \quad \forall \xi \in \Omega^+(X). \tag{3.12}$$

This claim also has an extension to some $p < 2$; see Corollary 3.23 for details. We will prove Claim 1 shortly. At the moment, we will show how we use it to finish the proof of the theorem. To prove Theorem 3.3 (a), set

$$J(\xi) = J_{A_1,A_2}(\xi) := \iota(\exp_{(h',V')}(P\xi)).$$

Clearly $J(0) = A'$ and J is an immersion near 0. That the derivatives of J are bounded uniformly (λ-independent) follows from the fact that P is uniformly bounded (by the claim) and the fact that the maps ι and $\exp_{(h',V')}$ are uniformly bounded. The map in Theorem 3.3 (b) is the map π from Claim 1.

We now prove Theorem 3.3 (c). Define $S\colon L^p_\delta(\Omega^+(X), g_{L,\lambda}) \to L^p_\delta(\Omega^+(X), g_{L,\lambda})$ by

$$S(\xi) := \widetilde{s}(P\xi) - (D\widetilde{s})_{(0,0)}P\xi - \widetilde{s}(0,0).$$

This is $\mathcal{C}^m$ and is the nonlinear part of the map $\widetilde{s} \circ P$. The following claim says that this map satisfies the requisite nonlinear estimates.

Claim 2: The map S satisfies the hypotheses of Lemma 3.7 with a constant κ that is independent of $0 < \lambda < \lambda_0$.

Once again, we defer the proof of this claim until after we have finished the argument for Theorem 3.3. It follows from Claim 2, Lemma 3.7, and the estimate (3.11) that, provided λ is sufficiently small, there is a unique $\xi = \xi(A_1, A_2) \in L^p_\delta(\Omega^+)$ satisfying

$$\xi + S(\xi) = -\widetilde{s}(0,0), \quad \|\xi\|_{L^p_\delta(X)} \le 1/(5\kappa). \tag{3.13}$$

Setting $\|\cdot\|_{L^p_\delta} := \|\cdot\|_{L^p_\delta(X,g_{L,\lambda})}$, and using (3.8), we get

$$\|\xi\|_{L^p_\delta} \le \|\widetilde{s}(0,0)\|_{L^p_\delta} + \|S(\xi)\|_{L^p_\delta} \le Cb^{4/p} + \frac{1}{5}\|\xi\|_{L^p_\delta}.$$

This implies the requisite estimate on ξ. Unraveling the definitions, we also have

$$s(\mathcal{J}(A_1,A_2)) = \widetilde{s}(P\xi) = \widetilde{s}(0,0) + (D\widetilde{s})_{(0,0)}P\xi + S(\xi) = -\xi + (D\widetilde{s})_{(0,0)}P\xi = -\sigma\pi\xi,$$

where $\mathcal{J}(A_1,A_2) := J(\xi)$, by definition. This finishes the proof of (c).

To prove the estimate (3.6), use the fact that A_k agrees with $A' = A'(A_1,A_2)$ on $X_k \backslash B_{L\lambda^{1/2}}(x_k)$, and then use the estimates on J, ξ to get

$$\|\iota^{-1}(\mathcal{J}(A_1,A_2)) - \iota^{-1}(A')\|_{L^2_2(N)\times L^p_{1,\delta}(X)} \le C''\|\xi\|_{L^p_\delta(X)} \le C'b^{4/p}. \tag{3.14}$$

When the A_k are regular, then the map π is the zero map so $\mathcal{J}(A_1,A_2)$ is automatically mASD by (3.5). In this case, the operator $(D\iota)_{(h',V')} \circ P$ is a right inverse to $(Ds)_{A'}$, essentially by definition. Then it follows from (3.14) that $(D\iota)_{(h',V')} \circ P$ is an approximate right inverse to $(Ds)_{\mathcal{J}(A_1,A_2)}$ and so $\mathcal{J}(A_1,A_2)$ is regular. The $\mathcal{C}^m$-smooth dependence of J on the A_k follows from Remark 3.17 (a) below, and the $\mathcal{C}^m$-smoothness of ξ follows from the $\mathcal{C}^m$-smoothness assertion of Lemma 3.7.

Finally, assume A_1 or A_2 is irreducible (assume $p > 2$ so this term is defined). It follows from Remark 3.2 that A' is irreducible as well. The irreducibility of $\mathcal{J}(A_1, A_2)$ then follows from (3.14) and Lemma 2.6.

To finish the proof of Theorem 3.3, it therefore suffices to verify the claims; we begin with Claim 1. Let

$$h_k = p_T(A_k) \in \mathcal{H}_k := \mathcal{H}_{\Gamma_k, out}$$

where p_T is the map from Section **2B.1**. Similar to what we did above over X, for each k, we can form a map

$$\widetilde{s}_k \colon T_{h_k}\mathcal{H}_k \times L^p_{1,\delta}(\ker(d^{*,\delta}_{A_k})) \longrightarrow L^p_\delta(\Omega^+(X_k))$$

by precomposing s with ι and the exponential $\exp_{h_k}$ for $\mathcal{H}_k$ based at h_k. (Note we have restricted to a Coulomb slice here; this will be used in Remark 3.17.) Linearizing at $(0,0)$, and coupling with σ_k, we obtain a bounded linear map

$$D_k := (D\widetilde{s}_k)_{(0,0)} \oplus \sigma_k \colon \Big(T_{h_k}\mathcal{H}_k \times L^p_{1,\delta}(\ker(d^{*,\delta}_{A_k}))\Big) \oplus H^+_k \longrightarrow L^p_\delta(\Omega^+(X_k)).$$

Standard elliptic theory for δ-decaying spaces, and the finite-dimensionality of $\mathcal{H}_k$, imply that D_k restricts to a bounded map of the form

$$D_k : \Big(T_{h_k}\mathcal{H}_k \times L^p_{\ell+1,\delta}(\ker(d^{*,\delta}_{A_k}))\Big) \oplus H^+_k \to L^p_{\ell,\delta}(\Omega^+)$$

for each $\ell \geq 0$; see [15]. Moreover, the definition of σ_k implies that this restriction is surjective, and so the "Laplacian"

$$D_k D^{*,\delta}_k : L^p_{2,\delta}(\Omega^+) \longrightarrow L^p_\delta(\Omega^+)$$

is a Banach space isomorphism, where $D^{*,\delta}_k$ is the adjoint of D_k relative to the δ-decaying L^2-inner products on the domain and codomain. It follows from these observations that the formula

$$P_k := D^{*,\delta}_k\big(D_k D^{*,\delta}_k\big)^{-1} \colon L^p_\delta(\Omega^+) \longrightarrow \Big(T_{h_k}\mathcal{H}_k \times L^p_{1,\delta}(\ker(d^{*,\delta}_{A_k}))\Big) \oplus H^+_k$$

defines a bounded right inverse for D_k. Coupling this with the embedding $L^p_{1,\delta} \hookrightarrow L^{p^*}_\delta$, it follows that there is a constant c_k with

$$\|P_k\xi\|_{(L^2_2(N_k)\times L^{p^*}_\delta(X_k))\oplus L^p_\delta(X_k)} \leq c_k\|\xi\|_{L^p_\delta(X_k)}, \quad \forall \xi \in \Omega^+(X_k). \tag{3.15}$$

The argument at this stage is almost identical to that given in [5, Section 7.2.3] (see also [5, Prop. 7.2.18]); however, we supply some of the details since we will refer to them again in Section **3C**. The operators P_1, P_2 can be glued together (using a carefully constructed cut-off function supported on the gluing region; see [5, p. 294]) to produce an operator

$$Q \colon L^p_\delta(\Omega^+(X), g_{L,\lambda}) \longrightarrow \Big(T_{h'}\mathcal{H} \times L^p_{1,\delta}(\Omega^1(X), g_{L,\lambda})\Big) \oplus H^+$$

that satisfies

$$\|Q\xi\|_{(L^2_2(N)\times L^{p^*}_\delta(X,g_{L,\lambda}))\oplus L^p_\delta(X,g_{L,\lambda})} \leq (c_1+c_2)\|\xi\|_{L^p_\delta(X,g_{L,\lambda})}, \quad \forall \xi \in \Omega^+(X).$$

Moreover, Q is an approximate right inverse to $(D\widetilde{s})_{(0,0)} \oplus \sigma$ in the sense that

$$((D\widetilde{s})_{(0,0)} \oplus \sigma) \circ Q = I + R$$

for some R satisfying

$$\|R(\xi)\|_{L^p_\delta} \leq \epsilon(L,b,p)\|\xi\|_{L^p_\delta} \tag{3.16}$$

where $\epsilon(L,b,p) \to 0$ as $L \to \infty$ and $b \to 0$ (the assumption $p \geq 2$ is used here to establish this decay property for $\epsilon(L,b,p)$, see [5, pp. 293,294]). Choose $L > 1, \lambda_0 > 0$ so that $\epsilon(L, 4L\lambda_0^{1/2}, p) < 1/3$. Then $Q(I+R)^{-1}$ is a right inverse to $(D\widetilde{s})_{(0,0)} \oplus \sigma$ and has operator norm at most $C_1 = 3(c_1+c_2)$. Then we can write this right inverse as $Q(I+R)^{-1} = P \oplus \pi$, where the splitting corresponds to the direct sum decomposition of the codomain of Q. The estimate (3.12) immediately follows, so this finishes the proof of Claim 1.

Remark 3.17. (a) It is not hard to show from the construction outlined in [5] that, when the A_k are regular, then the right inverse P depends $\mathcal{C}^m$-smoothly on the A_k. The key observation here is that, though many choices have been made in this construction (e.g., cutoff functions), the only ones that depend on the A_k are the choices of σ_k, but these can be taken to be zero when the A_k are regular.

(b) Note that the $\Omega^1(X_k)$-component of the operator P_k takes values in the kernel $\ker(d^{*,\delta}_{A_k})$. It follows from this (see the gluing construction outlined in [5]) that, in the complement of the gluing region, the $\Omega^1(X)$-component of $P\xi$ lies in the kernel $\ker(d^{*,\delta}_{A'})$ in the following sense: Let $U_k \subset X_k$ be an open set in the complement of the gluing region, so we can view $U_k \subset X$ as a subset of the glued 4-manifold. Fix $\xi \in \Omega^+(X)$ and write $(\eta, V) = P\xi \in T_{h'}\mathcal{H} \times L^p_{1,\delta}(\Omega^1(X))$ for the components of $P\xi$. Then

$$\left(d^{*,\delta}_{A'} V\right)\Big|_{U_k} = d^{*,\delta}_{A_k}(V|_{U_k}) = 0.$$

On the gluing region, the quantity $d^{*,\delta}_{A'}V$ need not be zero, but it depends only on the values of $P\xi$, and not its derivatives. In particular, since P is bounded, there is a uniform bound of the form

$$\|d^{*,\delta}_{A'}V\|_{L^p_\delta} \leq C\|\xi\|_{L^p_\delta}.$$

(c) As mentioned in the previous remark, it is generally not the case that V lies in the kernel of $d^{*,\delta}_{A'}$ on all of X, due to the cutoff functions. Thus the map J_{A_1,A_2} generally does not take values in the slice $\mathcal{SL}(A'(A_1,A_2))$, nor any fixed slice depending only on the A_k. In particular, the only slice to which the connection $\mathcal{J}(A_1,A_2)$ of Theorem 3.3 clearly belongs is the slice $\mathcal{SL}(\mathcal{J}(A_1,A_2))$ centered at itself. We will revisit this in Section 5.

Now we move on to prove Claim 2. Fix L, λ_0 as in Claim 1 and we assume $\lambda \in (0, \lambda_0)$. We clearly have $S(0) = 0$, so it suffices to show that S satisfies the quadratic estimate (3.8) for a uniform constant κ. For this, note that by Lemma 2.2 and Taylor's Theorem, we can write

$$i(\exp_{h'}(\eta)) = i(h') + (Di)_{h'}\eta + q_{h'}(\eta)$$

where $q_{h'} \colon T_{h'}\mathcal{H} \to L^p_{1,loc}(X) \cap \mathcal{C}^0(X)$ vanishes to first order. Since $\mathcal{H}_{out}$ is finite-dimensional, we can quantify this relative to any metric with respect to which the terms are well-defined. In particular, there is a constant C_2 so that

$$\|q_{h'}(\eta_1) - q_{h'}(\eta_2)\|_{\mathcal{C}^0(X)} \leq C_2(\|\eta_1\|_{L^2_2(N)} + \|\eta_2\|_{L^2_2(N)})\|\eta_1 - \eta_2\|_{L^2_2(N)} \tag{3.18}$$

for all $\eta_1, \eta_2 \in T_{h'}\mathcal{H}$. Note that $q_{h'}(\eta)$ need not decay to zero down the ends of X, since $i(h')$ and $i(\exp_{h'}(\eta))$ generally do not converge to the same connection at infinity. However, on the compact part we have

$$q_{h'}(\eta)|_{X_0} = 0. \tag{3.19}$$

Indeed, on X_0 the connection $i(h)$ equals the reference connection for all $h \in \mathcal{H}_{out}$, and i vanishes to all but the zeroth order on X_0.

To verify (3.8), fix $\xi_1, \xi_2 \in L^p_\delta(\Omega^+(X), g_\lambda)$ with $\|\xi_j\|_{L^p_\delta} \leq 1$ and set

$$(\eta_j, V_j) := P\xi_j \in T_{h'}\mathcal{H} \times L^p_{1,\delta}(\Omega^1(X)).$$

Then using the definition of S and the formula (2.10), we can write

$$\begin{array}{rcl} S(\xi_j) & = & \frac{1}{2}\left[V_j \wedge V_j\right]^+ + \frac{1-\beta'}{2}\left[(Di)_{h'}\eta_j + q_{h'}(\eta_j) \wedge (Di)_{h'}\eta_j + q_{h'}(\eta_j)\right]^+ \\ & & +(1-\beta')d^+_{i(h')}q_{h'}(\eta_j) + \left[V' \wedge q_{h'}(\eta_j)\right]^+ \\ & & + \left[V_j \wedge q_{h'}(\eta_j)\right]^+ + \left[V_j \wedge (Di)_{h'}\eta_j\right]^+. \end{array} \tag{3.20}$$

(These are the higher order terms in the mASD operator s, expressed in terms of V_j and η_j.) It suffices to show that each term on the right satisfies an estimate of the form (3.8). Below we set $\|\cdot\|_{L^p_\delta} := \|\cdot\|_{L^p_\delta(X, g_\lambda)}$.

We begin with the first term on the right of (3.20). This shows up in the ASD setting as well (see [5, p. 289]), but our argument is a bit more involved due to the non-compactness of X. For this, we use Lemma 3.9 to get

$$\begin{array}{rcl} \frac{1}{2}\|\,[V_1 \wedge V_1]^+ - [V_2 \wedge V_2]^+\,\|_{L^p_\delta} & = & \frac{1}{2}\|\,[V_1 + V_2 \wedge V_1 - V_2]^+\,\|_{L^p_\delta} \\ & \leq & \mathfrak{c}_\mathfrak{g}\|\,|V_1 + V_2||V_1 - V_2|\,\|_{L^p_\delta} \\ & \leq & \mathfrak{c}_\mathfrak{g} C_{3.9}\Big(\|V_1\|_{L^{p^*}_\delta} + \|V_2\|_{L^{p^*}_\delta}\Big)\|V_1 - V_2\|_{L^{p^*}_\delta} \end{array}$$

where $C_{3.9}$ is the constant from Lemma 3.9 and $\mathfrak{c}_\mathfrak{g}$ is defined by

$$\mathfrak{c}_\mathfrak{g} := \sup_{(\nu_1, \nu_2, \nu_3)} |\langle \nu_1, [\nu_2, \nu_3]\rangle| \tag{3.21}$$

with the supremum running over all $v_j \in \mathfrak{g}$ with $|v_j| = 1$. Since V_j is a component of $P\xi_j$, we can then use the estimate of Claim 1 to continue the above and get

$$\tfrac{1}{2}\| \,[V_1 \wedge V_1]^+ - [V_2 \wedge V_2]^+ \,\|_{L^p_\delta} \quad \leq \quad \mathfrak{c}_{\mathfrak{g}} C_{3.9} C_1^2 \Big(\|\xi_1\|_{L^p_\delta} + \|\xi_2\|_{L^p_\delta} \Big) \|\xi_1 - \xi_2\|_{L^p_\delta},$$

which is the desired estimate.

Now we move on to the second term in (3.20). Set $r(\eta) := (Di)_{h'}\eta + q_{h'}(\eta)$, so we want to bound the L^p_δ-norm of

$$\tfrac{1-\beta'}{2}\Big([r(\eta_1) \wedge r(\eta_1)]^+ - [r(\eta_2) \wedge r(\eta_2)]^+ \Big) = \tfrac{1-\beta'}{2} [r(\eta_1) + r(\eta_2) \wedge r(\eta_1) - r(\eta_2)]^+$$

in terms of the right-hand side of (3.8). Note that this is supported on the compact cylinder $\mathrm{Cyl}_0 := [T - 1/2, T + 1/2] \times N$, and so its L^p_δ-norm is bounded by a constant times

$$\begin{aligned} &\Big(\textstyle\sum_{j=1}^2 \|r(\eta_j)\|_{L^{p^*}_\delta(\mathrm{Cyl}_0)} \Big) \|r(\eta_1) - r(\eta_2)\|_{L^{p^*}_\delta(\mathrm{Cyl}_0)} \\ &\qquad \leq \|e^{t\delta/2}\|^2_{L^{p^*}(\mathrm{Cyl}_0)} \Big(\textstyle\sum_{j=1}^2 \|r(\eta_j)\|_{\mathcal{C}^0(\mathrm{Cyl}_0)} \Big) \|r(\eta_1) - r(\eta_2)\|_{\mathcal{C}^0(\mathrm{Cyl}_0)} \end{aligned}$$

By Lemma 2.2, this is bounded by a constant times

$$\Big(\|\eta_1\|_{L^2_2(N)} + \|\eta_2\|_{L^2_2(N)} \Big) \|\eta_1 - \eta_2\|_{L^2_2(N)} \leq C_1^2 \Big(\|\xi_1\|_{L^p_\delta} + \|\xi_2\|_{L^p_\delta} \Big) \|\xi_1 - \xi_2\|_{L^p_\delta},$$

as desired.

The estimate for the third term, $(1 - \beta')d^+_{i(h')} q_{h'}(\eta_j)$, is similar and we leave it to the reader. Moving on to the fourth term in (3.20), recall from (3.19) that $q_{h'}(\eta_j)$ vanishes on X_0. This observation combines with (3.18) and then Claim 1 to give

$$\begin{aligned} &\| \,[V' \wedge q_{h'}(\eta_1)]^+ - [V' \wedge q_{h'}(\eta_2)]^+ \,\|_{L^p_\delta} \\ &\qquad = \| \,[V' \wedge q_{h'}(\eta_1) - q_{h'}(\eta_2)]^+ \,\|_{L^p_\delta(\mathrm{End}\ X)} \\ &\qquad \leq \mathfrak{c}_{\mathfrak{g}} \|V'\|_{L^p_\delta(\mathrm{End}\ X)} \|q_{h'}(\eta_1) - q_{h'}(\eta_2)\|_{\mathcal{C}^0(\mathrm{End}\ X)} \\ &\qquad \leq \mathfrak{c}_{\mathfrak{g}} C_2 \|V'\|_{L^p_\delta(\mathrm{End}\ X)} \Big(\|\eta_1\|_{L^2_2(N)} + \|\eta_2\|_{L^2_2(N)} \Big) \|\eta_1 - \eta_2\|_{L^2_2(N)} \\ &\qquad \leq \mathfrak{c}_{\mathfrak{g}} C_2 C_1^2 \|V'\|_{L^p_\delta(\mathrm{End}\ X)} \Big(\|\xi_1\|_{L^p_\delta} + \|\xi_2\|_{L^p_\delta} \Big) \|\xi_1 - \xi_2\|_{L^p_\delta} \end{aligned}$$

This is the desired estimate for this term because $\|V'\|_{L^p_\delta(\mathrm{End}\ X)}$ is plainly independent of λ and the ξ_j.

The remaining two terms are the most difficult to bound. This is because (i) these terms involve both the infinite-dimensional terms V_i as well as the finite-dimensional terms $q_{h'}(\eta_j)$ and $(Di)_{h'}\eta_j$, and (ii) neither of these finite-dimensional terms generally decays to zero at infinity (nor do the differences $q_{h'}(\eta_1) - q_{h'}(\eta_2)$ and $(Di)_{h'}\eta_1 - (Di)_{h'}\eta_j$). The main estimate we need is the following, which we will see is equivalent to the fact that the operator $d^+_{A'}$ is Fredholm (on the appropriate spaces) with our choice of δ.

Claim 3: There is some $T_1 \gg T + 3/2$ and a constant C_3 so that

$$\|V\|_{L^p_\delta([T_1,\infty)\times N)} \leq C_3 \|d^+_{A'} V\|_{L^p_\delta([T_1-1,\infty)\times N)}$$

for all $V \in L^p_{1,\delta}(\Omega^1(X))$ with $d^{*,\delta}_{A'} V|_{[T_1-1,\infty)\times N} = 0$.

We prove Claim 3 after we finish our estimates for the last two terms in (3.20). The argument we give applies to both of these last two terms, so we focus on establishing the estimate for the second-to-last term:

$$\begin{aligned} &[V_1 \wedge q_{h'}(\eta_1)]^+ - [V_2 \wedge q_{h'}(\eta_2)]^+ \\ &\quad = \tfrac{1}{2}\Big([V_1 - V_2 \wedge q_{h'}(\eta_1) + q_{h'}(\eta_2)]^+ + [V_1 + V_2 \wedge q_{h'}(\eta_1) - q_{h'}(\eta_2)]^+ \Big). \end{aligned}$$

It suffices to bound the L^p_δ-norm of each term on the right by the right-hand side of (3.8); we will carry this out for $[V_1 + V_2 \wedge q_{h'}(\eta_1) - q_{h'}(\eta_2)]^+$, the other term is similar. Since the $q_{h'}(\eta_j)$ are supported on End X, we do not need to worry whether our constants are λ-dependent. With T_1 as in Claim 3, write

$$\begin{aligned} &\| [V_1 + V_2 \wedge q_{h'}(\eta_1) - q_{h'}(\eta_2)]^+ \|_{L^p_\delta} \\ &\quad \leq \Big(\| [V_1 + V_2 \wedge q_{h'}(\eta_1) - q_{h'}(\eta_2)]^+ \|_{L^p_\delta([0,T_1]\times N)} \\ &\qquad\qquad + \| [V_1 + V_2 \wedge q_{h'}(\eta_1) - q_{h'}(\eta_2)]^+ \|_{L^p_\delta([T_1,\infty)\times N)} \Big). \end{aligned} \tag{3.22}$$

Set $\mathrm{Cyl}_1 := [0, T_1] \times N$ and estimate the first term on the right as follows:

$$\begin{aligned} &\| [V_1 + V_2 \wedge q_{h'}(\eta_1) - q_{h'}(\eta_2)]^+ \|_{L^p_\delta(\mathrm{Cyl}_1)} \\ &\quad \leq \mathfrak{c}_\mathfrak{g} (\|V_1\|_{L^{p^*}_\delta(\mathrm{Cyl}_1)} + \|V_2\|_{L^{p^*}_\delta(\mathrm{Cyl}_1)}) \|q_{h'}(\eta_1) - q_{h'}(\eta_2)\|_{L^4_\delta(\mathrm{Cyl}_1)} \\ &\quad \leq \mathfrak{c}_\mathfrak{g} \|e^{t\delta/2}\|_{L^4(\mathrm{Cyl}_1)} (\|V_1\|_{L^{p^*}_\delta} + \|V_2\|_{L^{p^*}_\delta}) \|q_{h'}(\eta_1) - q_{h'}(\eta_2)\|_{C^0(\mathrm{Cyl}_1)} \\ &\quad \leq \mathfrak{c}_\mathfrak{g} C_2 C_1^3 \|e^{t\delta/2}\|_{L^4(\mathrm{Cyl}_1)} (\|\xi_1\|_{L^p_\delta} + \|\xi_2\|_{L^p_\delta})^2 \|\xi_1 - \xi_2\|_{L^p_\delta}, \end{aligned}$$

which is the desired estimate for this terms since we have assumed $\|\xi_j\|_{L^p_\delta} \leq 1$, and so

$$(\|\xi_1\|_{L^p_\delta} + \|\xi_2\|_{L^p_\delta})^2 \leq 2(\|\xi_1\|_{L^p_\delta} + \|\xi_2\|_{L^p_\delta}).$$

As for the remaining term on the right of (3.22), note that it is bounded by a constant times

$$\begin{aligned} &\Big(\|V_1\|_{L^p_\delta([T_1,\infty)\times N)} + \|V_2\|_{L^p_\delta([T_1,\infty)\times N)} \Big) \|q_{h'}(\eta_1) - q_{h'}(\eta_2)\|_{C^0([T_1,\infty)\times N)} \\ &\quad \leq C_2 C_1^2 \Big(\|V_1\|_{L^p_\delta([T_1,\infty)\times N)} + \|V_2\|_{L^p_\delta([T_1,\infty)\times N)} \Big) (\|\xi_1\|_{L^p_\delta} + \|\xi_2\|_{L^p_\delta}) \|\xi_1 - \xi_2\|_{L^p_\delta} \end{aligned}$$

We will therefore be done with the proof of Claim 2 if we can show that the terms $\|V_j\|_{L^p_\delta([T_1,\infty)\times N)}$ are bounded. For this, note that the linearization of $\widetilde{s}$ can be written as

$$\begin{aligned} (D\widetilde{s})_{(0,0)}(\eta, V) &= d^+_{A'}((Di)_{h'}\eta + V) - \beta' d^+_{i(h')}(Di)_{h'}\eta \\ &= d^+_{A'} V + (1 - \beta') d^+_{i(h')}(Di)_{h'}\eta + [V' \wedge (Di)_{h'}\eta]^+ . \end{aligned}$$

Note also that, by Remark 3.17 (b), the 1-form V_j lies in the kernel of $d^{*,\delta}_{A'}$ on End X. We can therefore use Claim 3 and the above formula for $(D\widetilde{s})_{(0,0)}$ to write

$$\begin{aligned}\|V_j\|_{L^p_\delta([T_1,\infty)\times N)} &\leq C_3\|d^+_{A'}V_j\|_{L^p_\delta([T_1-1,\infty)\times N)} \\ &\leq C_3\Big(\|(D\widetilde{s})_{(0,0)}(\eta_j,V_j)\|_{L^p_\delta} + \|\left[V'\wedge(Di)_{h'}\eta_j\right]^+\|_{L^p_\delta}\Big),\end{aligned}$$

where we also used the fact that $\beta' = 1$ on $[T_1-1,\infty)\times N$. Since $(\eta_j, V_j) = P\xi_j$ and $P\oplus\pi$ is a right-inverse for $(D\widetilde{s})_{(0,0)}\oplus\sigma$, we can continue this as

$$\begin{aligned}&\leq C_3\Big(\|\xi_j\|_{L^p_\delta} + \|\sigma\pi\xi_j\|_{L^p_\delta} + \mathfrak{c}_\mathfrak{g}\|V'\|_{L^p_\delta}\|(Di)_{h'}\eta_j\|_{\mathcal{C}^0}\Big) \\ &\leq C_3\Big(1 + C_{\sigma\pi} + \mathfrak{c}_\mathfrak{g}C_{(Di)_{h'}}\|V'\|_{L^p_\delta}\|\eta_j\|_{L^2_2(N)}\Big) \\ &\leq C_3\Big(1 + C_{\sigma\pi} + \mathfrak{c}_\mathfrak{g}C_{(Di)_{h'}}C_1\|V'\|_{L^p_\delta}\Big),\end{aligned}$$

where $C_{\sigma\pi}$ and $C_{(Di)_{h'}}$ are the operator norms of $\sigma\pi$ and $(Di)_{h'}$, respectively (we are viewing the latter as a map $L^2_2(N)\to\mathcal{C}^0(X)$; see Lemma 2.2 and the definition of i). This is the uniform bound we are after, and thus finishes the proof of Claim 2.

Finally, we prove Claim 3. Let $h'_T\colon [T,\infty)\to\mathcal{H}_{out}$ denote the flow of the trimmed vector field Ξ^{tr} with $h'_T(T) = h'$. Let

$$h'_\infty := \lim_{t\to\infty} h'_T(t) \in \mathcal{H}_{out}$$

be the limiting connection of this flow. This is a connection on N, but we will view it as a connection on End $X = [0,\infty)\times N$ that is constant in the t-direction. Let $\mathcal{X}$ be the $L^p_{1,\delta}$-completion of the space of 1-forms on X supported on End X, and let $\mathcal{Y}$ be the L^p_δ-completion of the elements of $\Omega^+\oplus\Omega^0$ supported on End X (so the elements of $\mathcal{X}$ and $\mathcal{Y}$ vanish on the compact part). Then the map

$$d^+_{h'_\infty}\oplus d^{*,\delta}_{h'_\infty}\colon \mathcal{X}\longrightarrow\mathcal{Y}$$

is bounded and elliptic. We have assumed that $\delta/2$ is not in the spectrum of $*d_h$, so it follows that the above operator has trivial kernel (it also has trivial cokernel, though we do not need this). In particular, there is a constant C_5 so that

$$\|V\|_{L^p_\delta} \leq \|V\|_{L^p_{1,\delta}} \leq C_5\Big(\|d^+_{h'_\infty}V\|_{L^p_\delta} + \|d^{*,\delta}_{h'_\infty}V\|_{L^p_\delta}\Big)$$

for all $V\in\mathcal{X}$. The connection A' is $\mathcal{C}^0$-asymptotic to h'_∞. In particular, we can choose T_1 large enough so that

$$\|A' - h'_\infty\|_{\mathcal{C}^0([T_1-1,\infty)\times N)} < 1/(6\mathfrak{c}_\mathfrak{g}C_5).$$

Then if $V \in \mathcal{X}$ is supported on $[T_1 - 1, \infty) \times N$ and in the kernel of $d^{*,\delta}_{A'}$, we have

$$\begin{aligned}
\|V\|_{L^p_\delta([T_1-1,\infty)\times N)} &\leq C_5\Big(\|d^+_{h'_\infty} V\|_{L^p_\delta([T_1-1,\infty)\times N)} + \|d^{*,\delta}_{h'_\infty} V\|_{L^p_\delta([T_1-1,\infty)\times N)}\Big) \\
&= C_5\Big(\|d^+_{A'} V + [h'_\infty - A' \wedge V]^+\|_{L^p_\delta([T_1-1,\infty)\times N)} \\
&\qquad\qquad + \|\,[h'_\infty - A' \wedge *V]\,\|_{L^p_\delta([T_1-1,\infty)\times N)}\Big) \\
&\leq C_5\|d^+_{A'} V\|_{L^p_\delta([T_1-1,\infty)\times N)} + \tfrac{1}{3}\|V\|_{L^p_\delta([T_1-1,\infty)\times N)}.
\end{aligned}$$

Then Claim 3 follows (with $C_3 := 3C_5/2$) from this estimate and a cutoff function. $\square$

3C. **Extensions to $p < 2$.** In our existence result of Section **6C**, we will need extensions to $p < 2$ of the estimates (3.12) and (3.4); we state and prove the relevant extensions here. In fact, all we will need is an extension to $p = 4/3$; we leave any more general extensions to the interested reader. Throughout this section, we fix data as in the statement of Theorem 3.3.

For the first result, let $L > 1, \lambda_0 > 0$, and $P \oplus \pi$ be as in the statement of Claim 1 appearing in the proof of Theorem 3.3.

Corollary 3.23. *There is a constant C so the following holds for all $0 < \lambda < \lambda_0$ and $\xi \in \Omega^+(X)$:*

$$\|(P \oplus \pi)\xi\|_{(L^2_2(N)\times L^2_\delta(X,g_{L,\lambda}))\oplus L^{4/3}_\delta(X,g_{L,\lambda})} \leq C\|\xi\|_{L^{4/3}_\delta(X,g_{L,\lambda})}.$$

Proof. We refer to the notation established in the proof of Claim 1. Momentarily suppressing Sobolev completions, let

$$(P \oplus \pi)^{*,\delta} : \Big(T_{h'}\mathcal{H} \times \Omega^1(X)\Big) \oplus H^+ \longrightarrow \Omega^+(X)$$

be the formal adjoint of $P \oplus \pi$, relative to the L^2_δ-inner product on X. By the duality isometries $(L^2_\delta(X))^* \cong L^2_\delta(X)$, $(L^4_\delta(X))^* \cong L^{4/3}_\delta(X)$, and $(L^2_2(N))^* \cong L^2_{-2}(N)$, we will be done if we can establish a uniform bound of the form

$$\|(P \oplus \pi)^{*,\delta}(\eta, V, \mu)\|_{L^4_\delta(X)} \leq C\left(\|\eta\|_{L^2_{-2}(N)} + \|V\|_{L^2_\delta(X)} + \|\mu\|_{L^4_\delta(X)}\right).$$

Since $T_{h'}\mathcal{H}$ is finite-dimensional, there is a bound of the form $\|\eta\|_{L^2_2(N)} \leq C'\|\eta\|_{L^2_{-2}(N)}$ for all $\eta \in T_{h'}\mathcal{H}$. It therefore suffices to show

$$\|(P \oplus \pi)^{*,\delta}(\eta, V, \mu)\|_{L^4_\delta(X)} \leq C''\left(\|\eta\|_{L^2_2(N)} + \|V\|_{L^2_\delta(X)} + \|\mu\|_{L^4_\delta(X)}\right) \tag{3.24}$$

for a uniform constant C''. This is precisely the estimate of Claim 1, except with the adjoint operator $(P \oplus \pi)^{*,\delta}$ in place of $P \oplus \pi$. We will show that the proof of Claim 1 can be sufficiently modified to hold for this adjoint.

Towards this end, note that the adjoint of $P_k = D^{*,\delta}_k(D_k D^{*,\delta}_k)^{-1}$ is given by $P^{*,\delta}_k = (D_k D^{*,\delta}_k)^{-1} D_k$ and so satisfies

$$\|P^{*,\delta}_k(\eta_k, V_k, \mu_k)\|_{L^4_\delta(X_k)} \leq c_k\left(\|\eta_k\|_{L^2_2(N_k)} + \|V_k\|_{L^2_\delta(X_k)} + \|\mu_k\|_{L^4_\delta(X_k)}\right).$$

Just as before, these can be glued together to form an operator $Q^{*,\delta}$ that satisfies

$$\|Q^{*,\delta}(\eta, V, \mu)\|_{L^4_\delta(X)} \leq (c_1 + c_2)\left(\|\eta\|_{L^2_2(N)} + \|V\|_{L^2_\delta(X)} + \|\mu\|_{L^4_\delta(X)}\right). \tag{3.25}$$

Moreover, it is not hard to see that this gluing can be done so that $Q^{*,\delta}$ is exactly the formal L^2_δ-adjoint of the operator Q appearing in the proof of Claim 1. Then the defining formula $P \circ \pi = Q(I + R)^{-1}$ implies

$$(P \circ \pi)^* = (I + R^{*,\delta})^{-1} Q^{*,\delta}; \tag{3.26}$$

here $R^{*,\delta}$ is the formal adjoint of R and so satisfies $\|R^{*,\delta}\xi\|_{L^2_\delta(X)} = \|R\xi\|_{L^2_\delta(X)}$. Then the estimate (3.24) follows from (3.26), (3.25), and (3.16). $\square$

For the second and last of the extensions we need, let $\xi(A_1, A_2) \in L^2_\delta(\Omega^+(X))$ be as in the conclusion of Theorem 3.3 (c).

Corollary 3.27. *There are $C, \lambda_0' > 0$, so that the following holds for all $0 < \lambda < \lambda_0'$:*

$$\|\xi(A_1, A_2)\|_{L^{4/3}_\delta(X, g_{L,\lambda})} \leq C\lambda^{3/2}.$$

Proof. Setting $\xi := \xi(A_1, A_2)$, the identity in (3.13) gives

$$\|\xi\|_{L^{4/3}_\delta(X)} \leq \|\widetilde{s}(0,0)\|_{L^{4/3}_\delta(X)} + \|S(\xi)\|_{L^{4/3}_\delta(X)}.$$

The estimate (3.1) holds with $p = 4/3$, so the same is true of (3.11); that is,

$$\|\widetilde{s}(0,0)\|_{L^{4/3}_\delta(X)} \leq C_1 b^3$$

for a uniform constant C_1, where $b = 4L\lambda^{1/2}$. To estimate $S(\xi)$, note that the formula (3.20) implies that $S(\xi)$ is quadratically bounded in $P\xi$. Then we can argue as we did in the proof of Claim 2, but use Hölder's inequality $\|fg\|_{L^{4/3}_\delta} \leq \|f\|_{L^4_\delta}\|g\|_{L^2}$, to get a uniform estimate of the form

$$\|S(\xi)\|_{L^{4/3}_\delta(X)} \leq C_2\|P\xi\|_{L^4_\delta(X)}\|P\xi\|_{L^2_\delta(X)}.$$

By (3.12) and Corollary 3.23, this implies

$$\|S(\xi)\|_{L^{4/3}_\delta(X)} \leq C_3 C_2 \|\xi\|_{L^2_\delta(X)}\|\xi\|_{L^{4/3}_\delta(X)}.$$

It follows from (3.4) that we can assume $\|\xi\|_{L^2_\delta(X)} < (2C_3C_2)^{-1}$, provided $\lambda > 0$ is sufficiently small. In summary, this implies

$$\|\xi\|_{L^{4/3}_\delta(X)} \leq C_1 b^3 + \frac{1}{2}\|\xi\|_{L^{4/3}_\delta(X)},$$

from which the corollary follows with $C = 128L^3C_1$. $\square$

4. Gauge fixing and the mASD condition

In the next section, we will find ourselves in the situation where we have an mASD connection A and a nearby connection A_{ref}. We will want to find a gauge transformation u so that u^*A is in the Coulomb slice of A_{ref}. The issue is that, due to the failure of the mASD equation to be gauge invariant, the connection u^*A will no longer be mASD. Nevertheless, we will show in this section that, when A is regular, the connection u^*A is close to a unique mASD connection that lies in the A_{ref}-Coulomb slice. This is made precise in Theorem 4.5, which extends the discussion to handle connections A that are not regular by means of an obstruction map. To accomplish this, we first prove a general gauge fixing result that is tailored to our setting; this is stated in Proposition 4.3.

4A. **Gauge fixing.** We begin by refining our choices of δ and the cut-off function β used to create $\mathcal{H}_{out}$. For the former, we assume $\delta/2$ is not in the spectrum of the de Rham operator d on real-valued functions. It then follows from Sobolev embedding that, for each $1 < q < 4$, there is a constant $\mathfrak{c}_q$ so that

$$\|f\|_{L^q_\delta(X)} + \|f\|_{L^{q^*}_\delta(X)} \leq \mathfrak{c}_q \|df\|_{L^q_\delta(X)} \tag{4.1}$$

for all compactly supported real-valued smooth functions f, where $q^* = 4q/(4-q)$ is the Sobolev conjugate.

As for the cutoff function $\beta\colon \mathcal{H} \to [0,1]$, we assume this is chosen so that it has small support in the sense that

$$\sup_{h,h_0 \in \mathrm{supp}(\beta)} \|h - h_0\|_{\mathcal{C}^0(N)} + \|\Theta(h) - \Theta(h_0)\|_{\mathcal{C}^0(N)} < \frac{1}{2\mathfrak{c}_2\mathfrak{c}_\mathfrak{g}}, \tag{4.2}$$

where $\mathfrak{c}_\mathfrak{g}$ is the constant from (3.21) and $\mathfrak{c}_2$ is the constant from (4.1) with $q = 2$.

The main gauge fixing result we will need is as follows.

Proposition 4.3. *Fix $2 < p < 4$, set $p^* = 4p/(4-p)$, and assume δ, β are as above. There are constants $C, \epsilon > 0$ so that if $A = \iota(h, V)$ and $A_{ref} = \iota(h_{ref}, V_{ref})$ are in $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ and satisfy*

$$\|V - V_{ref}\|_{L^{p^*}_\delta(X)} + \|d^{*,\delta}_{A_{ref}}(V - V_{ref})\|_{L^p_\delta(X)} < \epsilon,$$

then there is a unique $\mu = \mu(A, A_{ref}) \in L^p_{2,\delta}(\Omega^0(X))$ so that

$$\exp(\mu)^*A \in \mathcal{SL}(A_{ref}), \quad \text{and} \quad \|d^{*,\delta}_{A_{ref}} d_A \mu\|_{L^p_\delta(X)} \leq C \|d^{*,\delta}_{A_{ref}}(V - V_{ref})\|_{L^p_\delta(X)}.$$

Moreover, this 0-form $\mu(A, A_{ref})$ depends $\mathcal{C}^m$-smoothly on the pair (A, A_{ref}).

Proof. We will show below that

$$\|\mu\|_{\mathcal{X}} := \|d^{*,\delta}_{A_{ref}} d_A \mu\|_{L^p_\delta(X)}$$

defines a norm on the space $\Omega^0(X)$ of smooth rapidly decaying adjoint bundle-valued 0-forms. Assuming this for now, we denote by $\mathcal{X}$ the completion of $\Omega^0(X)$ relative to

$\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{Y}$ be the completion of $\Omega^0(X)$, but relative to the norm $\|\cdot\|_{\mathcal{Y}} := \|\cdot\|_{L^p_\delta(X)}$. Since $p > 2$, the map

$$\begin{array}{rcl} \mathcal{F}\colon \mathcal{A}^{1,p}(\mathcal{T}_\Gamma) \times \mathcal{A}^{1,p}(\mathcal{T}_\Gamma) \times \mathcal{X} & \longrightarrow & \mathcal{Y} \\ (A, A_{ref}, \mu) & \longmapsto & d^{*,\delta}_{A_{ref}}(u^*A - i(p(u^*A)) - V_{ref}) \end{array}$$

is $\mathcal{C}^m$-smooth, where we have set $u = \exp(\mu) \in \mathcal{G}^{2,p}_\delta$. Note that, relative to the product structure given by ι via (2.4), the quantity $u^*A - i(p(u^*A))$ is the $L^p_{1,\delta}(\Omega^1(X))$-component (i.e., non-center manifold-component) of u^*A, and so $u^*A \in \mathcal{SL}(A_{ref})$ for $u = \exp(\mu)$ if and only if $\mathcal{F}(A, A_{ref}, \mu) = 0$. It therefore suffices to solve $\mathcal{F}(A, A_{ref}, \mu) = 0$ for μ. For this, we have that $\mu = 0$ is an approximate solution since

$$\mathcal{F}(A, A_{ref}, 0) = d^{*,\delta}_{A_{ref}}(V - V_{ref}),$$

which we have assumed is bounded by ϵ. The linearization in the third component of $\mathcal{F}$ at $(A, A_{ref}, 0)$ is the operator

$$\mu \longmapsto d^{*,\delta}_{A_{ref}} d_A \mu.$$

This has operator norm 1 relative to the norms on $\mathcal{X}$ and $\mathcal{Y}$. In particular, it is invertible and so the proposition follows from the inverse function theorem (e.g., precompose $\mathcal{F}$ in the third component with the inverse of $d^{*,\delta}_{A_{ref}} d_A$ and then use Lemma 3.7).

All that remains is to show that $\|\cdot\|_{\mathcal{X}}$ defines a norm; it suffices to show that the operator

$$d^{*,\delta}_{A_{ref}} d_A\colon L^p_{2,\delta} \longrightarrow L^p_\delta$$

is injective. For this, suppose μ lies in its kernel and let $(\cdot,\cdot)_\delta$ be the δ-dependent L^2-inner product. Then

$$0 = (d^{*,\delta}_{A_{ref}} d_A \mu, \mu)_\delta = (d_A\mu, d_{A_{ref}}\mu)_\delta = \|d_{A_{ref}}\mu\|^2_{L^2_\delta} + \Big([A - A_{ref}, \mu], d_{A_{ref}}\mu\Big)_\delta.$$

Hence

$$\|d_{A_{ref}}\mu\|_{L^2_\delta} \leq \|[A - A_{ref}, \mu]\|_{L^2_\delta}.$$

The definition of ι gives $A - A_{ref} = \beta''(h - h_{ref} + (\Theta(h) - \Theta(h_{ref}))dt) + V - V_{ref}$. Then Hölder's inequality and (4.2) allow us to continue the above inequality to get

$$\begin{array}{rcl} \|d_{A_{ref}}\mu\|_{L^2_\delta} & \leq & \mathfrak{c}_{\mathfrak{g}}\big(\|V - V_{ref}\|_{L^4}\|\mu\|_{L^4_\delta} + \|h - h_{ref} + (\Theta(h) - \Theta(h_{ref}))dt\|_{C^0}\|\mu\|_{L^2_\delta}\big) \\ & \leq & \mathfrak{c}_{\mathfrak{g}}\|e^{-\delta t/2}\|_{L^r}\|V - V_{ref}\|_{L^{p^*}_\delta}\|\mu\|_{L^4_\delta} + \frac{1}{2\mathfrak{c}_2}\|\mu\|_{L^2_\delta} \end{array}$$

where r is defined by $r^{-1} + (p^*)^{-1} = 4^{-1}$. Using (4.1) with $f = |\mu|$, and then Kato's inequality $|d|\mu|| \leq |d_{A_{ref}}\mu|$, we can use the above to get

$$\begin{array}{rcl} \|\mu\|_{L^2_\delta} + \|\mu\|_{L^4_\delta} & \leq & \mathfrak{c}_2\|d|\mu|\|_{L^2_\delta} \\ & \leq & \mathfrak{c}_2\|d_{A_{ref}}\mu\|_{L^2_\delta} \\ & \leq & \mathfrak{c}_2\mathfrak{c}_{\mathfrak{g}}\|e^{-\delta t/2}\|_{L^r}\|V - V_{ref}\|_{L^{p^*}_\delta}\|\mu\|_{L^4_\delta} + \frac{1}{2}\|\mu\|_{L^2_\delta}. \end{array}$$

When $\epsilon < 1/(2\mathfrak{c}_2\mathfrak{c}_{\mathfrak{g}}\|e^{-\delta t/2}\|_{L^r})$, this implies that $\mu = 0$. □

Remark 4.4. The operator $d^{*,\delta}_{A_{ref}} d_A \colon L^p_{2,\delta} \to L^p_\delta$ is Fredholm, and we have just seen that it has trivial kernel under the hypotheses of the proposition. It then follows from the embedding $L^p_{2,\delta} \subseteq L^p_\delta \cap \mathcal{C}^0$ that there is a constant C so that

$$\|\mu\|_{L^p_\delta(X)} + \|\mu\|_{\mathcal{C}^0(X)} \leq C\|d^{*,\delta}_{A_{ref}} d_A \mu\|_{L^p_\delta(X)}$$

for all $\mu \in L^p_{2,\delta}(\Omega^0(X))$. It follows from arguments similar to those just used that this constant can be chosen to be independent of A, A_{ref}, provided these connections satisfy the hypotheses of Proposition 4.3.

4B. **Recovering the mASD condition within a slice.** Throughout this section, we assume $2 < p < 4$, and δ, β are chosen as in Section **4A**.

As suggested in the introduction to this section, we will use Proposition 4.3 to put mASD connections into a fixed nearby slice, but this process will generally not preserve the mASD condition. The following theorem is our main readjustment tool that will recover the mASD condition, while simultaneously preserving the slice condition. To state it, use the L^2_δ-inner product to identify the cokernel $H^+_{A,\delta} = \text{coker } (Ds)_A$ with the subset of A-harmonic self-dual forms in $L^p_\delta(\Omega^+(X))$. We denote by

$$\sigma_A \colon H^+_{A,\delta} \longrightarrow L^p_\delta(\Omega^+(X)), \qquad \pi_A \colon L^p_\delta(\Omega^+(X)) \longrightarrow H^+_{A,\delta}$$

the inclusion and L^2_δ-orthogonal projection, respectively. (These maps will play a role analogous to the one played by σ and π in Section 3.) It follows that $(Ds)_A \oplus \sigma_A$ maps surjectively onto $L^p_\delta(\Omega^+(X))$.

Theorem 4.5. *Fix $A_{ref} = \iota(h_{ref}, V_{ref}) \in \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$. Then there are constants $C, \epsilon > 0$ so that the following holds for all $A = \iota(h, V)$ satisfying*

$$\|h - h_{ref}\|_{L^2_2(N)} + \|V - V_{ref}\|_{L^{p^*}_\delta(X)} + \|d^{*,\delta}_{A_{ref}}(V - V_{ref})\|_{L^p_\delta(X)} < \epsilon. \tag{4.6}$$

(a) There is a $\mathcal{C}^m$-map $K_A \colon L^p_\delta(\Omega^+(X)) \to \mathcal{SL}(A_{ref})$ that restricts to an embedding on a neighborhood U of 0.

(b) If $s(A) = 0$ then there is a unique 2-form $\zeta(A) \in U \subseteq L^p_\delta(\Omega^+(X))$ so that

$$\|\zeta(A)\|_{L^p_\delta(X)} \leq C\|d^{*,\delta}_{A_{ref}}(V - V_{ref})\|_{L^p_\delta(X)}$$

and so that the connection $\mathcal{K}(A) := K_A(\zeta(A))$ satisfies

$$s(\mathcal{K}(A)) = -\sigma_A \pi_A \zeta(A).$$

In particular, the connection $\mathcal{K}(A)$ is close to A in the sense that there is a constant C' so that

$$\|\iota^{-1}(\mathcal{K}(A)) - \iota^{-1}(A)\|_{L^2_2(N) \times L^p_{1,\delta}(X)} \leq C'\|d^{*,\delta}_{A_{ref}}(V - V_{ref})\|_{L^p_\delta(X)}.$$

If either A or A_{ref} is regular, then they both are regular and so is $\mathcal{K}(A)$. In this case, the connection $\mathcal{K}(A)$ is mASD *and the maps $(A, \zeta) \mapsto K_A(\zeta)$ and $A \mapsto \zeta(A)$ are both $\mathcal{C}^m$-smooth, relative to the specified topologies.*

If A is irreducible, then so is $\mathcal{K}(A)$.

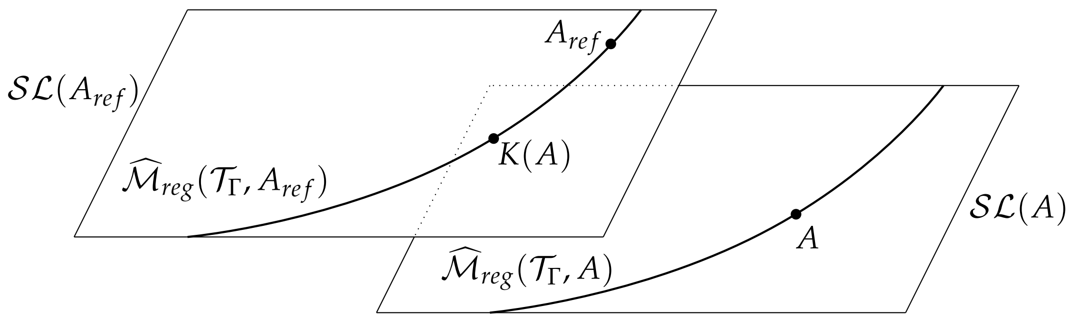

FIGURE 2. Pictured above is the special case of Theorem 4.5 where A is regular. The curved lines represent the spaces of regular mASD connections in the slices $\mathcal{SL}(A_{ref})$ and $\mathcal{SL}(A)$, respectively.

Proof. Take $\epsilon > 0$ to be no larger than the epsilon from the statement of Proposition 4.3. Then it follows from that proposition and Remark 4.4 that, given $A = \iota(h, V)$ with

$$s(A) = 0, \quad \text{and} \quad \|V - V_{ref}\|_{L^{p^*}_\delta(X)} + \|d^{*,\delta}_{A_{ref}}(V - V_{ref})\|_{L^p_\delta(X)} < \epsilon,$$

there is a unique $\mu \in L^p_{2,\delta}(\Omega^0)$ so that $\exp(\mu)^* A \in \mathcal{SL}(A_{ref})$ and

$$\|\mu\|_{L^p_\delta} + \|\mu\|_{\mathcal{C}^0} \leq C_1 \|d^{*,\delta}_{A_{ref}}(V - V_{ref})\|_{L^p_\delta}.$$

Set $u = \exp(\mu)$ and write $u^* A = \iota(h_A, V_A)$ for $h_A \in \mathcal{H}_{out}$ and $V_A \in L^p_{1,\delta}(\Omega^1)$. Let $\exp_{h_A} \colon B_\epsilon(0) \subseteq T_{h_A}\mathcal{H} \to \mathcal{H}$ be the exponential map for the center manifold based at h_A, and extend this to a map

$$\begin{array}{rcl} \exp_{(h_A,V_A)} \colon B_\epsilon(0) \times L^p_{1,\delta}(\ker(d^{*,\delta}_{A_{ref}})) & \longrightarrow & \mathcal{H} \times L^p_{1,\delta}(\ker(d^{*,\delta}_{A_{ref}})) \\ (\eta, V) & \longmapsto & (\exp_{h_A}(\eta), V_A + V), \end{array}$$

which is a $\mathcal{C}^m$-diffeomorphism in a neighborhood of $(0,0)$. Using this, define

$$\widetilde{s} \colon T_{h_A}\mathcal{H} \times L^p_{1,\delta}(\ker(d^{*,\delta}_{A_{ref}})) \longrightarrow L^p_\delta(\Omega^+), \quad (\eta, V) \longmapsto s(\iota(\exp_{(h_A,V_A)}(\eta, V))).$$

By definition of σ_A, the operator $(Ds)_A \oplus \sigma_A$ is surjective. The operators $(Ds)_A$ and $(Ds)_{u^*A}$ are approximately equal when u is $\mathcal{C}^0$-close to the identity (equivalently, when $\|\mu\|_{\mathcal{C}^0}$ is small). It follows that, when ϵ is sufficiently small, the operator $(Ds)_{u^*A} \oplus \sigma_A$ is also surjective, as is $(D\widetilde{s})_{(0,0)} \oplus \sigma_A$. Then we can choose a right inverse to $(D\widetilde{s})_{(0,0)} \oplus \sigma_A$ of the form $P \oplus \pi_A$, where π_A is the projection to $H^+_{A,\delta}$. For $\zeta \in L^p_\delta(\Omega^+(X))$, define

$$K_A(\zeta) := \iota(\exp_{(h_A,V_A)}(P\zeta)).$$

This proves (a) in the statement of the theorem, by taking $U \subseteq L^p_\delta(\Omega^+(X))$ to be small enough so that $P(U) \subseteq B_\epsilon(0) \times \ker(d^{*,\delta}_{A_{ref}})$.

To prove (b), we use the same implicit function theorem argument as in Theorem 3.3. Namely, set
$$S(\zeta) := \widetilde{s}(P\zeta) - (D\widetilde{s})_{(0,0)}P\zeta - \widetilde{s}(0,0).$$
The argument of Claim 2 in the proof of Theorem 3.3 carries over to show that S satisfies the quadratic estimate of Lemma 3.7. We will show in a moment that there is a uniform constant C_2 so that
$$\|\widetilde{s}(0,0)\|_{L^p_\delta} \leq C_2 \|d^{*,\delta}_{A_{ref}}(V - V_{ref})\|_{L^p_\delta}. \tag{4.7}$$
From this and Lemma 3.7 it follows that, by assuming ϵ is sufficiently small, there is a unique $\zeta(A)$ so that
$$\zeta(A) + S(\zeta(A)) = -\widetilde{s}(0,0).$$
As we argued in the proof of Theorem 3.3, this $\zeta(A)$ satisfies the assertions of (b). The regularity and irreducibility assertions also follow as in Theorem 3.3.

It therefore suffices to verify (4.7). Use the assumption that $s(A) = F^+_A - \beta F^+_{i(h)} = 0$ vanishes to write
$$\widetilde{s}(0,0) = s(u^*A) = \beta(\mathrm{Ad}(\exp(u^{-1})) - \mathrm{Id})F^+_{i(h)}.$$
By shrinking ϵ further still, we may suppose $\|\mu\|_{C^0} \leq 1$. Then the Taylor expansion for the exponential $u = \exp(\mu)$ gives
$$\begin{aligned} \|\widetilde{s}(0,0)\|_{L^p_\delta} &\leq C_3 \|F^+_{i(h)}\|_{C^0} \|\mu\|_{L^p_\delta} \\ &\leq C_1 C_3 \|F^+_{i(h)}\|_{C^0} \|d^{*,\delta}_{A_{ref}}(V - V_{ref})\|_{L^p_\delta}. \end{aligned}$$
The quantity $\|F^+_{i(h)}\|_{C^0}$ is bounded independent of h since $\mathcal{H}_{out}$ is compact. □

5. Gluing regular families

Throughout this section, we work with the space $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ for fixed $2 < p < 4$. We assume that δ and the cutoff function β are chosen as in Section **4A**. We also assume $\delta/2 < \mu^-_\Gamma$, so the index formula discussed in Remark 2.12 applies.

We freely refer to the notation of Section 3. For $k = 1, 2$, fix a precompact open set
$$G_k \subseteq \widehat{\mathcal{M}}_{reg}(\mathcal{T}_{k,\Gamma_k}, A_{ref,k})$$
of regular mASD connections on X_k relative to some reference connection $A_{ref,k}$. Since the G_k are precompact, we can fix $L, \lambda > 0$ so that conclusions of Theorem 3.3 hold for all $(A_1, A_2) \in G_1 \times G_2$. (In our applications of the material of this section, the values of L and λ will be fixed, so we do not keep track of them in the notation.) Then Theorem 3.3 produces a regular mASD connection $\mathcal{J}(A_1, A_2) \in \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$.

Ideally, we would want to view the mapping $(A_1, A_2) \mapsto \mathcal{J}(A_1, A_2)$ as a function from $G_1 \times G_2$ into a fixed mASD space. However, since the Coulomb slice to which $\mathcal{J}(A_1, A_2)$ belongs depends on (A_1, A_2) (cf. Remark 3.17 (c)), it is more natural to realize this mapping as a section of a bundle. Towards this end, set
$$\mathcal{E} := \left\{ (A_1, A_2, A) \;\middle|\; A_k \in G_k, \;\; A \in \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, \mathcal{J}(A_1, A_2)) \right\}.$$

Let $\Pi\colon \mathcal{E} \to G_1 \times G_2$ be the projection to the first two factors. Then the map

$$\Psi(A_1, A_2) := (A_1, A_2, \mathcal{J}(A_1, A_2))$$

is clearly a section of the map Π.

Theorem 5.1.

(a) For all sufficiently small $\lambda > 0$, there is a neighborhood $\mathcal{U} \subseteq \mathcal{E}$ of the image of Ψ so that the restriction $\Pi|_{\mathcal{U}}\colon \mathcal{U} \to G_1 \times G_2$ is a locally trivial fiber bundle. The fibers of $\Pi|_{\mathcal{U}}$ can be identified with open subsets of $\widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$ for some A_{ref}.

More specifically, every $(A_{10}, A_{20}) \in G_1 \times G_2$ is contained in an open neighborhood $\mathcal{V} \subseteq G_1 \times G_2$ so that the following holds. Let $A_{ref} = A'(A_{10}, A_{20})$ be the preglued connection, and consider the map $\mathcal{K}$ from Theorem 4.5, defined relative to this reference connection A_{ref}. Then the map

$$\Pi^{-1}(\mathcal{V}) \cap \mathcal{U} \longrightarrow \mathcal{V} \times \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref}), \quad (A_1, A_2, A) \mapsto (A_1, A_2, \mathcal{K}(A)) \tag{5.2}$$

is a well-defined $\mathcal{C}^m$-diffeomorphism onto an open subset of the codomain, and this map produces a local trivialization of $\Pi|_{\mathcal{U}}$ over $\mathcal{V}$.

(b) The map

$$\Phi := \mathcal{K} \circ \mathcal{J} : \mathcal{V} \longmapsto \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$$

is a $\mathcal{C}^m$-diffeomorphism onto an open subset of the codomain. If A_1 or A_2 is irreducible, then the connection $\Phi(A_1, A_2)$ is also irreducible.

Part (b) can be restated by saying that, relative to the local trivilization of (a), the map Ψ is a local $\mathcal{C}^m$-diffeomorphism onto an open subset of the fiber. This is an mASD version of the familiar result for ASD connections that gluing produces an open subset of the ASD moduli space for a connected sum. See Figure 3 for an illustration of the fiber bundle in (a), and Figure 4 for an illustration of the specified trivialization, as well as the map Φ.

Remark 5.3. (a) We will also be interested in the case where G_2 consists of a single point (and so not necessarily an *open* set in the mASD space). In this case, Theorem 5.1 (a) continues to hold verbatim. Theorem 5.1 (b) holds as stated, with the exception that the map Φ is now only a $\mathcal{C}^m$-embedding (it need not have open image). There is no significant change in the proofs to account for this extension.

(b) Recall the fiber isomorphism ρ from the beginning of Section 3. The usual ASD gluing results (e.g., those of [20, 19, 5]) allows ρ to vary as a "gluing parameter". In that setting, this is necessary for obtaining a local submersion into the ASD moduli space. Since we are working in a gauge slice, as opposed to a moduli space quotiented out by a gauge group, this is not appropriate for the current setting. Similarly, there is no need here to give reducibles any special treatment. That said, ρ will play an active role in our existence result of Section **6C**.

(c) We have chosen to phrase Theorem 5.1 in terms of the reference connection $A_{ref} = A'(A_{10}, A_{20})$ given by the preglued connection. This is only in preparation for our applications below, and this specific choice is by no means necessary. Indeed, the proof

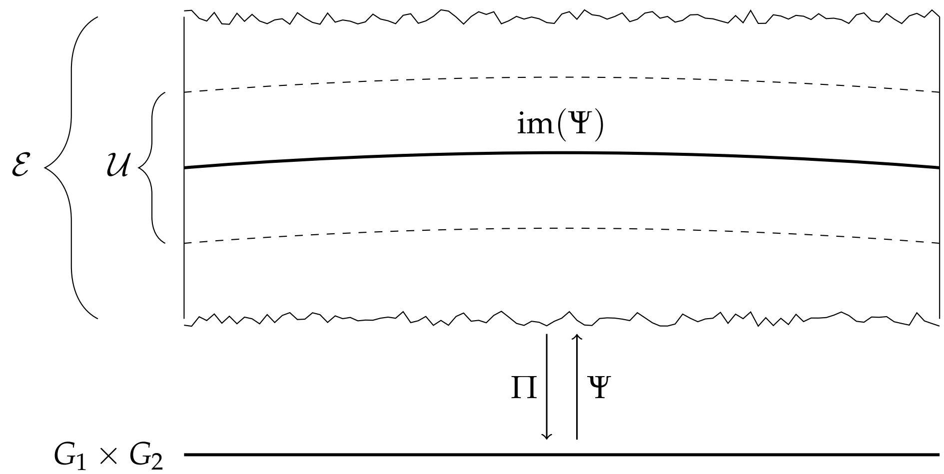

FIGURE 3. The above picture illustrates the fiber bundle $\Pi|_{\mathcal{U}} : \mathcal{U} \to G_1 \times G_2$ obtained by restricting the projection $\Pi : \mathcal{E} \to G_1 \times G_2$ to the open submanifold $\mathcal{U} \subseteq \mathcal{E}$. The fibers are $\mathcal{C}^m$-diffeomorphic to $\widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$.

will show the connection A_{ref} can be replaced by any connection that is sufficiently close to $\mathcal{J}(A_{10}, A_{20})$ in the sense that the coordinates of A_{ref} and $\mathcal{J}(A_{10}, A_{20})$ satisfy the estimate (5.12).

We begin by giving several technical lemmas in Section **5A**, which are used to prove that Φ is an immersion. The proof of Theorem 5.1 is given in Section **5B**.

5A. **Immersion lemmas.** Our ultimate goal is to show that the map Φ is an immersion. Recall this is made up of the maps $\mathcal{J}$ and $\mathcal{K}$, and hence of the maps J, ξ, K, ζ of Theorems 3.3 and 4.5. Each of the four lemmas below establishes an estimate on the derivative of one of these latter four maps. To state the lemmas, we introduce the following seminorms on the tangent space $T_A\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$: Fix an open subset $U \subseteq X$ containing End X and let $W \in T_A\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$. Using the isomorphism (2.5), we can identify W with a pair

$$(\eta, V) \in T_{p(A)}\mathcal{H}_\Gamma \times L^p_{1,\delta}(\Omega^1(X)).$$

Then set

$$\|W\|_{\mathcal{L}(U);A} := \|\eta\|_{L^2_2(N)} + \|V\|_{L^p_\delta(U)} + \|d_A^+ V\|_{L^p_\delta(U)} + \|d_A^{*,\delta} V\|_{L^p_\delta(U)},$$

where the derivatives defining the norm $\|\cdot\|_{L^2_2(N)}$ on $T_{p(A)}\mathcal{H}_\Gamma$ are defined using the connection Γ. Then $\|\cdot\|_{\mathcal{L}(U);A}$ is a continuous seminorm. These seminorms are gauge-invariant in the sense that

$$\|W\|_{\mathcal{L}(U);A} = \|\mathrm{Ad}(u^{-1})W\|_{\mathcal{L}(U);u^*A} \tag{5.4}$$

for all gauge transformations $u \in \mathcal{G}^{2,p}_\delta(\Gamma)$; here, via a slight abuse of notation, we are writing $\mathrm{Ad}(u^{-1})W$ for the linearization in the direction of W of the map $A \mapsto u^*A$. We use similar notation on the X_k. We note that if $U = X$, then $\|\cdot\|_{\mathcal{L}(X);A}$ is a norm that

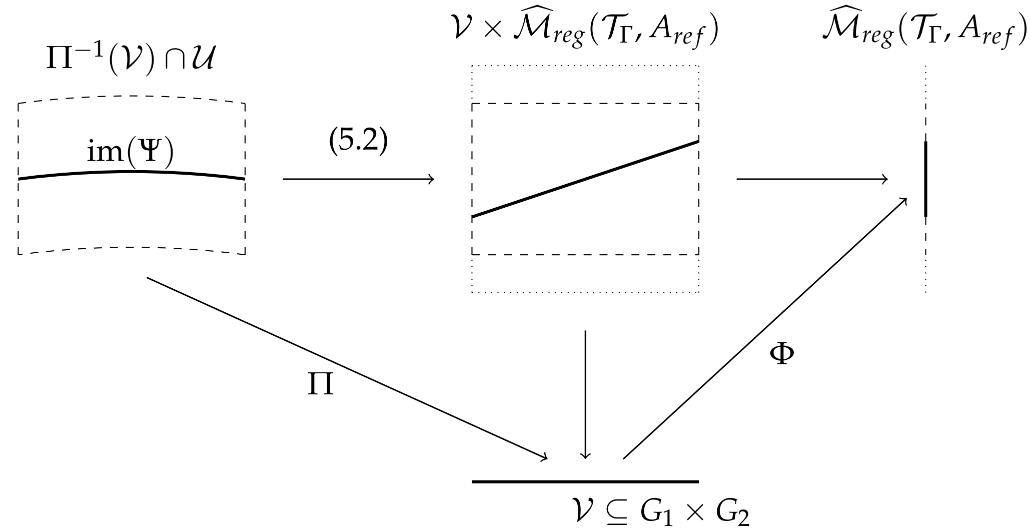

FIGURE 4. The horizontal arrow on the left is the $\mathcal{C}^m$-diffeomorphism (5.2) trivializing $\Pi|_{\mathcal{U}}$ over $\mathcal{V}$. This is a $\mathcal{C}^m$-diffeomorphism onto the region in $\mathcal{V} \times \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$ represented by the dashed lines. It takes $\mathrm{im}(\Psi)$, represented by the solid arc, $\mathcal{C}^m$-diffeomorphically onto the region in $\mathcal{V} \times \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$ represented by the solid diagonal line. The horizontal arrow on the right is the projection of $\mathcal{V} \times \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$ to the second factor. This takes the region represented by the dotted (resp., dashed) lines in the domain onto the region represented by the dotted (resp., dashed) line in the codomain. It restricts to a $\mathcal{C}^m$-diffeomorphism from the region represented by the solid diagonal line onto the region in the codomain represented by the solid line. The vertical arrow is the projection of $\mathcal{V} \times \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$ to the first factor. This restricts to a $\mathcal{C}^m$-diffeomorphism from the region represented by the solid diagonal line onto $\mathcal{V}$. The map Φ is a $\mathcal{C}^m$-diffeomorphism onto the region in its codomain represented by the solid line.

induces the topology on $T_A\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$. When the metric $g_{L,\lambda}$ is relevant, we will include it in the notation by writing $\|\cdot\|_{\mathcal{L}(X,g_{L,\lambda});A}$.

Our first lemma deals with the map $(A_1, A_2, \xi) \mapsto J_{A_1,A_2}(\xi)$. To first order, this map is the sum of the pregluing map $(A_1, A_2) \mapsto A'(A_1, A_2)$ together with a map that is bounded in ξ. We now quantify this to an extent that is sufficient for our purposes.

Lemma 5.5. *Fix connections $A_k \in G_k$ for $k = 1, 2$. There are constants $C, L, \lambda_0, \epsilon > 0$ so that the following holds for all $0 < \lambda < \lambda_0$ and all $\xi \in L^p_\delta(\Omega^+(X), g_{L,\lambda})$ with $\|\xi\|_{L^p_\delta(X,g_{L,\lambda})} < \epsilon$. Let $DJ_{(A_1,A_2,\xi)}(W_1, W_2, x)$ denote the linearization at (A_1, A_2, ξ) in the direction (W_1, W_2, x) of the map*

$$(A_1, A_2, \xi) \longmapsto J_{A_1,A_2}(\xi)$$

from Theorem 3.3 (a). Then

$$\sum_{k=1}^{2} \|W_k\|_{\mathcal{L}(X_k);A_k} \leq C\Big(\|DJ_{(A_1,A_2,\xi)}(W_1,W_2,x)\|_{\mathcal{L}(X,g_{L,\lambda}),A'} + \|x\|_{L^p_\delta(X,g_{L,\lambda})}\Big)$$

for all $W_k \in T_{A_k}G_k$ and all $x \in L^p_\delta(\Omega^+(X), g_{L,\lambda})$, where $A' := A'(A_1, A_2)$ is the preglued connection. The constants $C, L, \lambda_0, \epsilon$ can be chosen to depend continuously on the $A_k \in G_k$.

The next lemma shows that the $(A_1, A_2) \mapsto \xi(A_1, A_2)$ depends minimally on the connections A_1, A_2. In the next section, this will combine with the previous lemma to show that the map $\mathcal{J}(A_1, A_2) = J_{A_1,A_2}(\xi(A_1, A_2))$ is approximately the pregluing map $(A_1, A_2) \mapsto A'(A_1, A_2)$ for λ small; at this point it will follow that $\mathcal{J}$ is an immersion.

Lemma 5.6. *Fix $A_k \in G_k$ for $k = 1, 2$. Then there are constants $C, L, \lambda_0 > 0$ so that the following holds for all $0 < \lambda < \lambda_0$. Let $D\xi_{(A_1,A_2)}(W_1, W_2)$ denote the linearization at (A_1, A_2) in the direction (W_1, W_2) of the map*

$$(A_1, A_2) \longmapsto \xi(A_1, A_2)$$

from Theorem 3.3 (c). Then this satisfies

$$\|D\xi_{(A_1,A_2)}(W_1, W_2)\|_{L^p_\delta(X,g_{L,\lambda})} \leq b^{4/p} C \sum_{k=1}^{2} \|W_k\|_{\mathcal{L}(X_k);A_k}$$

for all $W_k \in T_{A_k}G_k$, where $b = 4L\lambda^{1/2}$. The constants C, L, λ_0 can be chosen to depend continuously on the $A_k \in G_k$.

These next two lemmas are analogues of the previous two, but for the operator $\mathcal{K}(A) = K_A(\zeta(A))$ in place of $\mathcal{J}(A_1, A_2) = J_{A_1,A_2}(\xi(A_1, A_2))$.

Lemma 5.7. *Fix a regular connection $A_{ref} \in \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$. Then there are constants $C, \epsilon' > 0$ so that the following holds for all connections $A \in \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$ satisfying (4.6) with respect to any $0 < \epsilon < \epsilon'$. Let $DK_{(A,\zeta)}(W, z)$ denote the linearization at (A, ζ) in the direction (W, z) of the map*

$$(A, \zeta) \longmapsto K_A(\zeta)$$

from Theorem 4.5 (a). Then this satisfies

$$\|W\|_{\mathcal{L}(X),A} \leq C\Big(\|DK_{(A,\zeta)}(W, z)\|_{\mathcal{L}(X),A'} + \|z\|_{L^p_\delta(X)}\Big)$$

for all $W \in T_A\widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$ and all $z \in L^p_\delta(\Omega^+(X))$. The constants C, ϵ can be chosen to depend continuously on A and A_{ref}.

Lemma 5.8. *Fix a regular connection $A_{ref} \in \mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$. Then there are constants $C, \epsilon' > 0$ so that the following holds for all connections $A \in \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$ satisfying (4.6) with respect to any $0 < \epsilon < \epsilon'$. Let $D\zeta_A W$ denote the linearization at A in the direction W of the map*

$$A \longmapsto \zeta(A)$$

from Theorem 4.5 (b). Then this satisfies

$$\|D\zeta_A W\|_{L^p_\delta(X)} \leq C\|d^{*,\delta}_{A_{ref}}(V - V_{ref})\|_{L^p_\delta(X)}\|W\|_{\mathcal{L}(X),A}$$

for all $W \in T_A \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$*. The constants* C, ϵ *can be chosen to depend continuously on* A_{ref}*.*

Now we give the proofs of Lemmas 5.5, 5.7, 5.6, and 5.8, in that order.

Proof of Lemma 5.5. The tangent space $T_{A_k}G_k$ is cut out by linear elliptic equations. In particular, unique continuation holds for the elements of this tangent space, and so the assignment

$$W_k \longmapsto \|W_k\|_{\mathcal{L}(X_k \setminus B_{L\lambda^{1/2}(x_k)});A_k}$$

defines a norm on $T_{A_k}G_k$. Since $T_{A_k}G_k$ is a finite-dimensional vector space, any two norms are equivalent and so there is a constant C_1 so that

$$\|W_k\|_{\mathcal{L}(X_k);A_k} \leq C_1\|W_k\|_{\mathcal{L}(X_k \setminus B_{L\lambda^{1/2}(x_k)});A_k}$$

for all $W_k \in T_{A_k}G_k$. A simple contradiction argument shows that this constant can be taken to be independent of L, λ, provided $L \geq 1$ and λ is sufficiently small.

Now fix tangent vectors $W_k \in T_{A_k}G_k$ for $k = 1, 2$. Since A_k is regular, we can find a $\mathcal{C}^m$-smooth path $A_k(\tau)$ of regular mASD connections with $A_k(0) = A_k$ and $\frac{d}{d\tau}|_{\tau=0}A_k(\tau) = W_k$. Let $W' = \frac{d}{d\tau}|_{\tau=0}A'(A_1(\tau), A_2(\tau))$. Note that the construction of the preglued connection $A'(A_1, A_2)$ implies there is a uniform constant C_2 so that

$$\sum_k \|W_k\|_{\mathcal{L}(X_k \setminus B_{L\lambda^{1/2}(x_k)});A_k} \leq C_2 \left\|W'\right\|_{\mathcal{L}(X);A'(A_1,A_2)}$$

provided $\lambda > 0$ is sufficiently small. Thus we have

$$\sum_k \|W_k\|_{\mathcal{L}(X_k);A_k} \leq C_1 C_2 \left\|W'\right\|_{\mathcal{L}(X);A'}. \tag{5.9}$$

The next claim ties this in with the linearization of the map J at (A_1, A_2, ξ) when $\xi = 0$.

Claim 1: $DJ_{(A_1,A_2,0)}(W_1, W_2, x) = W' + (D\iota)_{\iota^{-1}(A')}Px, \quad \forall x \in L^p_\delta(\Omega^+(X))$.

Here P is the right-inverse from the proof Theorem 3.3. This depends on A_1, A_2, so to emphasize this, we will temporarily write $P_{A_1,A_2} := P$. Consider the map

$$(A, \xi) \longmapsto \iota \circ \exp_{\iota^{-1}(A)}\left(P_{A_1,A_2}\xi\right) \tag{5.10}$$

where A ranges over all connections near $A' = A'(A_1, A_2)$ and ξ ranges over all self-dual 2-forms near 0. The linearization of (5.10) at $(A', 0)$ is the operator

$$(W, x) \longmapsto W + (D\iota)_{\iota^{-1}(A')}(P_{A_1,A_2}x).$$

Recall from the proof of Theorem 3.3 that

$$J_{A_1,A_2}(\xi) = \iota\left(\exp_{\iota^{-1}(A'(A_1,A_2))}\left(P_{A_1,A_2}\xi\right)\right).$$

That is, $(A_1, A_2, \xi) \mapsto J_{A_1,A_2}(\xi)$ is the map (5.10) precomposed with $A'(A_1, A_2)$ in the A-component. Then Claim 1 follows from the chain rule and the fact that we are differentiating at $\xi = 0$, which kills off all terms involving the A_k-derivatives of P_{A_1,A_2}.

In summary, we have

$$\begin{aligned}\sum_k \|W_k\|_{\mathcal{L}(X_k);A_k} &\leq C_1C_2\,\|W'\|_{\mathcal{L}(X);A'(A_1,A_2)}\\ &\leq C_1C_2\Big(\|W' + (D\iota)_{\iota^{-1}(A')}Px\|_{\mathcal{L}(X);A'} + \|(D\iota)_{\iota^{-1}(A')}Px\|_{\mathcal{L}(X);A'}\Big)\\ &= C_1C_2\Big(\|DJ_{(A_1,A_2,0)}(W_1, W_2, x)\|_{\mathcal{L}(X);A'} + \|(D\iota)_{\iota^{-1}(A')}Px\|_{\mathcal{L}(X);A'}\Big).\end{aligned}$$

We will discuss each term on the right individually.

The first term on the right is almost satisfactory, except we linearized at $(A_1, A_2, 0)$ instead of (A_1, A_2, ξ). To account this, note that it follows from our regularity assumptions and Theorem 3.3 that J is $\mathcal{C}^m$-smooth. In particular, Taylor's theorem gives

$$\begin{aligned}&\|(DJ_{(A_1,A_2,\xi)} - DJ_{(A_1,A_2,0)})(W_1, W_2, x)\|_{\mathcal{L}(X);A'}\\ &\qquad\qquad \leq C_3\|\xi\|_{L^p_\delta(X)}\Big(\|x\|_{L^p_\delta(X)} + \sum_k \|W_k\|_{\mathcal{L}(X_k);A_k}\Big)\end{aligned}$$

for some constant C_3 that depends continuously on the A_k and λ.

Claim 2: The constant C_3 can be taken to be independent of λ, provided λ is sufficiently small.

To see this, recall that the proof of Taylor's theorem shows that C_3 can be taken to be a constant multiple of the supremum of the operator norm of the second derivative of J at $(A_1, A_2, 0)$. By the chain rule, it therefore suffices to uniformly estimate the first two derivatives of $\iota, \exp_{\iota^{-1}(A'(A_1,A_2))}$ and $P = P_{A_1,A_2}$. Obtaining such estimates for ι and the exponential map follow readily because the gluing region is in the complement of the cylindrical end (e.g., ι is affine-linear over this gluing region). That the derivatives of P are uniformly bounded is addressed in Remark 3.17 (a), above.

With this claim in hand, we have

$$\begin{aligned}\sum_k \|W_k\|_{\mathcal{L}(X_k);A_k} &\leq C_1C_2\Big(\|DJ_{(A_1,A_2,\xi)}(W_1, W_2, x)\|_{\mathcal{L}(X);A'} + \|(D\iota)_{\iota^{-1}(A')}Px\|_{\mathcal{L}(X);A'}\Big)\\ &\quad + C_1C_2C_3\|\xi\|_{L^p_\delta(X)}\Big(\|x\|_{L^p_\delta(X)} + \sum_k \|W_k\|_{\mathcal{L}(X_k);A_k}\Big)\end{aligned}$$

When $\|\xi\|_{L^p_\delta(X)} < \epsilon := 1/2C_1C_2C_3$, this implies that $\sum_k \|W_k\|_{\mathcal{L}(X_k);A_k}$ is bounded by

$$2C_1C_2\Big(\|DJ_{(A_1,A_2,\xi)}(W_1, W_2, x)\|_{\mathcal{L}(X);A'} + \|(D\iota)_{\iota^{-1}(A')}Px\|_{\mathcal{L}(X);A'}\Big) + \|x\|_{L^p_\delta(X)}$$

The lemma now follows from the next claim.

Claim 3: There are constants $C_4, L, \lambda_0 > 0$ so that

$$\|(D\iota)_{\iota^{-1}(A')}Px\|_{\mathcal{L}(X,g_{L,\lambda});A'} \leq C_4\|x\|_{L^p_\delta(X,g_{L,\lambda})}$$

for all x and all $0 < \lambda < \lambda_0$. These constants can be chosen to depend continuously on the $A_k \in \mathcal{A}^{1,p}(\mathcal{T}_{k,\Gamma_k})$.

We briefly sketch the proof, leaving the details to the reader. Use the fact that P is uniformly bounded to control the zeroth order terms appearing in the definition of $\|\cdot\|_{\mathcal{L}(X);A'}$. To control the term involving $d^{*,\delta}_{A'}$, use Remark 3.17 (b). To control the $d^+_{A'}$ term, use the fact that P is a right inverse to an operator that is essentially $d^+_{A'}$ plus lower order terms. □

Proof of Lemma 5.7. Fix A and W as in the lemma. Let $A(\tau)$ be a $\mathcal{C}^m$-path in $\widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$ satisfying $A(0) = A$ and $\frac{d}{d\tau}|_{\tau=0}A(\tau) = W$. Let $\mu_\tau = \mu(A(\tau), A_{ref})$ be the 0-form from Proposition 4.3 associated to $A(\tau), A_{ref}$. Set

$$u_\tau := \exp(\mu_\tau), \quad A' := u_0^* A, \quad W' := \frac{d}{d\tau}\Big|_{\tau=0} u_\tau^* A(\tau).$$

By the product rule, we have

$$W' = \mathrm{Ad}(u_0^{-1})W + d_{A'}\Big(\frac{d}{d\tau}\Big|_{\tau=0}\mu_\tau\Big).$$

Now the gauge invariance (5.4) and the definition of our norms give

$$\begin{aligned}
\|W\|_{\mathcal{L}(X);A} &= \|\mathrm{Ad}(u_0^{-1})W\|_{\mathcal{L}(X);A'} \\
&\leq \|W'\|_{\mathcal{L}(X);A'} + \Big\|d_{A'}\Big(\tfrac{d}{d\tau}\Big|_{\tau=0}\mu_\tau\Big)\Big\|_{\mathcal{L}(X);A'} \\
&= \|W'\|_{\mathcal{L}(X);A'} + \Big\|d_{A'}\Big(\tfrac{d}{d\tau}\Big|_{\tau=0}\mu_\tau\Big)\Big\|_{L^p_\delta(X)} \\
&\quad + \Big\|\Big[F^+_{A'}, \Big(\tfrac{d}{d\tau}\Big|_{\tau=0}\mu_\tau\Big)\Big]\Big\|_{L^p_\delta(X)} + \Big\|d^{*,\delta}_{A'} d_{A'}\Big(\tfrac{d}{d\tau}\Big|_{\tau=0}\mu_\tau\Big)\Big\|_{L^p_\delta(X)}.
\end{aligned}$$

Focusing on the second term on the right, we note that the operator $d^{*,\delta}_{A'}$ is injective on $\mathrm{im}(d_{A'})$, so there is a bound of the form

$$\Big\|d_{A'}\Big(\frac{d}{d\tau}\Big|_{\tau=0}\mu_\tau\Big)\Big\|_{L^p_\delta(X)} \leq C_1 \Big\|d^{*,\delta}_{A'} d_{A'}\Big(\frac{d}{d\tau}\Big|_{\tau=0}\mu_\tau\Big)\Big\|_{L^p_\delta(X)}.$$

As for the third term on the right, the fact that A is mASD implies that F^+_A is uniformly bounded in $\mathcal{C}^0$; the same is therefore true of $F^+_{A'} = \mathrm{Ad}(u_0^{-1})F^+_A$. Combining this with the fact that the operator $d^{*,\delta}_{A'} d_{A'}$ is injective on 0-forms, we obtain

$$\begin{aligned}
\Big\|\Big[F^+_{A'}, \Big(\tfrac{d}{d\tau}\Big|_{\tau=0}\mu_\tau\Big)\Big]\Big\|_{L^p_\delta(X)} &\leq C_2 \Big\|\tfrac{d}{d\tau}\Big|_{\tau=0}\mu_\tau\Big\|_{L^p_\delta(X)} \\
&\leq C_3 \Big\|d^{*,\delta}_{A'} d_{A'}\Big(\tfrac{d}{d\tau}\Big|_{\tau=0}\mu_\tau\Big)\Big\|_{L^p_\delta(X)}.
\end{aligned}$$

In summary, we have

$$\|W\|_{\mathcal{L}(X);A} \leq \|W'\|_{\mathcal{L}(X);A'} + (1 + C_1 + C_3)\Big\|d^{*,\delta}_{A'} d_{A'}\Big(\tfrac{d}{d\tau}\Big|_{\tau=0}\mu_\tau\Big)\Big\|_{L^p_\delta(X)}$$

Our hypotheses imply that A' and A_{ref} differ by a term that is controlled by the $\mathcal{C}^0$-norm of μ. This implies we have an estimate of the form

$$\Big\| d^{*,\delta}_{A'} d_{A'} \Big(\frac{d}{d\tau}\Big|_{\tau=0} \mu_\tau \Big) \Big\|_{L^p_\delta} \leq C_4 \Big\| d^{*,\delta}_{A_{ref}} d_{A'} \Big(\frac{d}{d\tau}\Big|_{\tau=0} \mu_\tau \Big) \Big\|_{L^p_\delta}.$$

To estimate this further, differentiate the defining identity $0 = d^{*,\delta}_{A_{ref}}(u^*_\tau A_\tau - A_{ref})$ at $\tau = 0$ to get

$$d^{*,\delta}_{A_{ref}} d_{A'} \Big(\frac{d}{d\tau}\Big|_{\tau=0} \mu_\tau \Big) = -d^{*,\delta}_{A_{ref}} W.$$

Thus

$$\begin{aligned} \Big\| d^{*,\delta}_{A_{ref}} d_{A'} \Big(\tfrac{d}{d\tau}\Big|_{\tau=0} \mu_\tau \Big) \Big\|_{L^p_\delta} &= \Big\| d^{*,\delta}_{A_{ref}} W \Big\|_{L^p_\delta} \\ &\leq \Big\| d^{*,\delta}_{A} W \Big\|_{L^p_\delta} + C_5 \|W\|_{L^p_\delta} \\ &= \Big\| d^{*,\delta}_{A'} W' \Big\|_{L^p_\delta} + C_5 \|W'\|_{L^p_\delta} \\ &\leq \max(1, C_5) \|W'\|_{\mathcal{L}(X);A'}. \end{aligned}$$

Hence

$$\|W\|_{\mathcal{L}(X);A} \leq C_6 \|W'\|_{\mathcal{L}(X);A'}$$

where $C_6 = 1 + (1 + C_1 + C_3) C_4 \max(1, C_5)$. To finish the proof of the lemma, argue exactly as we did in the proof of Lemma 5.5, starting after the estimate (5.9). □

Proof of Lemma 5.6. Let $A_k(\tau)$ be a $\mathcal{C}^m$-smooth path in G_k satisfying $A_k(0) = A_k$ and $\frac{d}{d\tau}|_{\tau=0} A_k(\tau) = W_k$. Set $\xi_\tau := \xi(A_1(\tau), A_2(\tau))$. Note that the τ-derivative

$$\frac{d}{d\tau}\Big|_{\tau=0} \xi_\tau = D\xi_{(A_1,A_2)}(W_1, W_2)$$

is the term that we are looking to bound.

The regularity hypotheses and Theorem 3.3 (c) imply that ξ_τ satisfies $0 = s(J_\tau(\xi_\tau))$ for all τ, where $J_\tau := J_{A_1(\tau),A_2(\tau)}$ is the map from Theorem 5.1 (a). Continuing to use a subscript τ for any term defined in terms of the $A_k(\tau)$ (and hence dependent on τ), we recall the definition of $\widetilde{s}$ from the proof of Theorem 3.3; in particular, this satisfies $s(J_\tau(\cdot)) = \widetilde{s}(P_\tau(\cdot))$. The Taylor expansion of $\widetilde{s}$ therefore gives

$$0 = s(J_\tau(\xi_\tau)) = \widetilde{s}(P_\tau(\xi_\tau)) = \widetilde{s}_\tau(0,0) + \xi_\tau + S_\tau(\xi_\tau).$$

Differentiate the right-hand side at $\tau = 0$ and rearrange to get

$$\frac{d}{d\tau}\Big|_{\tau=0} \xi_\tau = -\frac{d}{d\tau}\Big|_{\tau=0} \widetilde{s}_\tau(0,0) - \frac{d}{d\tau}\Big|_{\tau=0} S_\tau(\xi_0) - (DS_0)_{\xi_0} \Big(\frac{d}{d\tau}\Big|_{\tau=0} \xi_\tau \Big) \tag{5.11}$$

where $(DS_0)_{\xi_0}$ is the linearization at ξ_0 of S_0. We will return to this after we estimate each term on the right individually.

For the first term on the right of (5.11), note that $\widetilde{s}_\tau(0,0) = s(A'_\tau)$ depends on τ only through the preglued connection $A'_\tau := A'(A_1(\tau), A_2(\tau))$. Moreover, the proof of (3.1) shows that $s(A'_\tau)$ is equal to a product of a cutoff function supported in the gluing region, times the connection form for A'_τ in this region. In particular, differentiating this in τ, the same argument used for (3.1) allows us to conclude a uniform bound of the form

$$\begin{aligned}\Big\|\tfrac{d}{d\tau}\Big|_{\tau=0}\widetilde{s}_\tau(0,0)\Big\|_{L^p_\delta(X,g_{L,\lambda})} &\leq C_1 b^{4/p}\Big\|\tfrac{d}{d\tau}\Big|_{\tau=0}A'_\tau\Big\|_{\mathcal{L}(X,g_{L,\lambda}),A'_0}\\ &\leq C_2 b^{4/p}\sum_k\Big\|W_k\Big\|_{\mathcal{L}(X);A_k}\end{aligned}$$

where the second inequality follows by differentiating the defining formula for the preglued connection $A'(A_1,A_2)$. This is the desired bound on the first term.

The second term on the right of (5.11) is similar, albeit a little more involved. The point here is that the quadratic estimates on S_τ give a uniform bound of the form

$$\Big\|\frac{d}{d\tau}\Big|_{\tau=0}S_\tau(\xi_0)\Big\|_{L^p_\delta(X,g_{L,\lambda})} \leq C_3\|\xi_0\|_{L^p_\delta(X,g_{L,\lambda})}\sum_k\Big\|W_k\Big\|_{T_{A_k}\mathcal{A}}.$$

Theorem 3.3 (c) gives $\|\xi_0\|_{L^p_\delta(X,g_{L,\lambda})} \leq C_4 b^{4/p}$, so the desired estimate for this term follows.

Turn now to the last term on the right of (5.11). By the estimate (3.8), the linearization $(DS_0)_{\xi_0}$ satisfies

$$\|(DS_0)_{\xi_0}\xi'\|_{L^p_\delta(X,g_{L,\lambda})} \leq 2\kappa\|\xi_0\|_{L^p_\delta(X,g_{L,\lambda})}\|\xi'\|_{L^p_\delta(X,g_{L,\lambda})}$$

for all ξ'. Since $\|\xi_0\|_{L^p_\delta(X,g_{L,\lambda})} \leq C_4 b^{4/p}$, we may assume that $\|\xi_0\|_{L^p_\delta(X,g_{L,\lambda})} < 1/4\kappa$, which gives

$$\|(DS_0)_{\xi_0}\xi'\|_{L^p_\delta(X,g_{L,\lambda})} \leq \frac{1}{2}\|\xi'\|_{L^p_\delta(X,g_{L,\lambda})}.$$

To see that the above estimates imply the lemma, take the norm of each side of (5.11) and use the estimates just established to obtain

$$\Big\|\frac{d}{d\tau}\Big|_{\tau=0}\xi_\tau\Big\|_{L^p_\delta(X,g_{L,\lambda})} \leq (C_2+C_3C_4)b^{4/p}\sum_k\Big\|W_k\Big\|_{T_{A_k}\mathcal{A}} + \frac{1}{2}\Big\|\frac{d}{d\tau}\Big|_{\tau=0}\xi_\tau\Big\|_{L^p_\delta(X,g_{L,\lambda})}.$$

The corollary follows by subtracting the last term from both sides, and using the identity $D\xi_{(A_1,A_2)}(W_1,W_2) = \frac{d}{d\tau}\Big|_{\tau=0}\xi_\tau$, □

Proof of Lemma 5.8. This follows from the same type of argument given for Lemma 5.6. □

5B. **Proof of Theorem 5.1.** Let $\epsilon > 0$ be small enough so that Theorem 4.5 holds with this value of ϵ. Define $\mathcal{U}$ to be the set of $(A_1,A_2,A)\in\mathcal{E}$ so that

$$\|h-h_0\|_{L^2_2(N)} + \|V-V_0\|_{L^{p^*}_\delta(X)} + \|d^{*,\delta}_{A_0}(V-V_0)\|_{L^p_\delta(X)} < \epsilon/3 \tag{5.12}$$

where $A=\iota(h,V)$ and $A_0=\iota(h_0,V_0) := \mathcal{J}(A_1,A_2)$. Since all elements of the G_k are regular, it follows from Theorem 4.5 that all such A are regular as well.

To show that $\Pi|: \mathcal{U}\to G_1\times G_2$ is locally trivial, fix $(A_{10},A_{20})\in G_1\times G_2$, and set $A_{ref} := A'(A_{10},A_{20})$. By (3.14), by choosing λ sufficiently small, it follows that the coordinates of A_{ref} and $\mathcal{J}(A_{10},A_{20})$ satisfy the estimate (5.12); in fact, this estimate is uniform in λ, in the sense that it holds for all sufficiently small λ. Fix any such λ; we will refine this choice in the next paragraph. Take $\mathcal{V} \subseteq G_1 \times G_2$ to be a neighborhood of (A_{10}, A_{20}) that is small enough so that if $(A_1, A_2) \in \mathcal{V}$, then the components of A_{ref} and $A'(A_1, A_2)$ satisfy (5.12). Though we do not use this observation presently, we note that the set $\mathcal{V}$ can also be chosen to be uniform in λ, provided $\lambda > 0$ is sufficiently small; this is due to the scaling properties the L^{p^*}-norm of 1-forms [5, p.293]. It follows from two applications of the triangle inequality that all $(A_1, A_2, A) \in \Pi^{-1}(\mathcal{V}) \cap \mathcal{U}$ are such that A, A_{ref} satisfy the hypotheses of Theorem 4.5. Thus the map (5.2) is well-defined and indeed provides a local trivialization of $\Pi|$.

To finish the proof, it suffices to show that the map $\Phi = \mathcal{K} \circ \mathcal{J} \colon \mathcal{V} \to \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$ is a local $\mathcal{C}^m$-diffeomorphism onto an open subset. Since all connections present are regular, the dimensions of $\mathcal{V}$ and $\widehat{\mathcal{M}}_{reg}(\mathcal{T}_\Gamma, A_{ref})$ are given by the indices of their defining operators. By Remark 2.12 and the additivity of Fredholm indices on connected sums, these indices agree. It therefore suffices to show that Φ is an immersion. Since $\Phi = \mathcal{K} \circ \mathcal{J}$, it suffices to show that $\mathcal{J}$ and $\mathcal{K}$ are immersions for all $\lambda > 0$ sufficiently small. For $\mathcal{J}$, note that the linearization at (A_1, A_2) is the map

$$(W_1, W_2) \longmapsto DJ_{(A_1, A_2, \xi(A_1, A_2))}(W_1, W_2, D\xi_{(A_1, A_2)}(W_1, W_2)).$$

Suppose this vanishes at some (W_1, W_2). Then by Lemmas 5.5 and 5.6, we would have

$$\sum_k \|W_k\|_{\mathcal{L}(X_k); A_k} \leq C b^{4/p} \sum_k \|W_k\|_{\mathcal{L}(X_k); A_k}$$

By taking $\lambda > 0$ sufficiently small, we may assume $Cb^{4/p} < 1$ and so $W_k = 0$. Thus $\mathcal{J}$ is an immersion. A similar argument, but using Lemmas 5.7 and 5.8, shows that $\mathcal{K}$ is an immersion for small λ.

The irreducibility claims follow from the analogous claims appearing in Theorems 3.3 and 4.5. □

6. Existence results

Let X be an oriented cylindrical end 4-manifold with $b^+(X) = 0$ or 1. In this section, we will show how to use the above framework to prove the existence of families of mASD connections on X; the cases $b^+(X) = 0$ and $b^+(X) = 1$ are treated in Sections **6B** and **6C**, respectively. The ASD existence result Theorem A is proved in Section **6D**.

Part of our existence results state that the connections we construct are topologically non-trivial in a certain sense. In the case of closed 4-manifolds, this non-triviality is captured by the non-vanishing of a characteristic class of the bundle supporting the connections. In the present cylindrical end setting, we will use a certain relative characteristic class to measure this non-triviality. The details of this are carried out in Section **6A**.

6A. Relative characteristic classes and adapted bundles. This section reviews topological quantities associated to 4-manifolds with cylindrical ends. We begin with a review of characteristic classes in the closed (compact with no boundary) setting.

Suppose Z is a closed, oriented 4-manifold and $P \to Z$ is a principal G-bundle. We define
$$\kappa(P) := -\frac{1}{8\pi^2}\int_Z \langle F_A \wedge F_A \rangle$$
where $A \in \mathcal{A}(P)$ is any connection and $\langle F_A \wedge F_A \rangle$ is obtained by combining the wedge and the inner product on $\mathfrak{g}$ defined via the immersion (2.1). Then $\kappa(P)$ is independent of the choice of A by the Bianchi identity. Topologically, $\kappa(P) = c_2(P \times_G \mathbb{C}^r)\,[Z]$ is the second Chern number of the $\mathbb{C}^r$-bundle associated to P via the map (2.1) and the standard action of $\mathrm{SU}(r)$ on $\mathbb{C}^r$. In particular, $\kappa(P) \in \mathbb{Z}$ is an integer representing an obstruction to P being trivializable.

Now consider the bundle $Q \to N$ over the 3-manifold N, and fix a gauge transformation $u \in \mathcal{G}(Q)$. We can form the mapping torus $Q_u = [0,1] \times Q/(0, u(q)) \sim (1, q)$ which is a principal G-bundle over $S^1 \times N$. Then we define the *degree* of u to be the integer
$$\deg(u) := \kappa(Q_u).$$
This depends only on the homotopy type of u and so descends to a group homomorphism
$$\deg\colon \pi_0(\mathcal{G}(Q)) \longrightarrow \mathbb{Z}$$
from the group of components of $\mathcal{G}(Q)$. The degree is an obstruction to extending u to a gauge transformation on X_0 (or equivalently X). We denote by $\mathcal{G}_0(Q)$ the subgroup of degree-zero gauge transformations. When G is simply-connected, the degree $\deg\colon \pi_0(\mathcal{G}(Q)) \to \mathbb{Z}$ is injective, and so $\mathcal{G}_0(Q)$ is exactly the identity component of $\mathcal{G}(Q)$.

Since the cylinder End X deformation retracts to the 3-manifold N, we have a natural isomorphism $\pi_0(\mathcal{G}(\mathrm{End}\ X)) \cong \pi_0(\mathcal{G}(N))$ and so the degree provides a homomorphism $\deg\colon \pi_0(\mathcal{G}(\mathrm{End}\ X)) \to \mathbb{Z}$. We denote by $\mathcal{G}_0(\mathrm{End}\ X)$ the kernel of this homomorphism.

We will be working with principal G-bundles on the cylindrical end 4-manifold X. Bundle isomorphism is too course of an equivalence relation to be useful in the cylindrical-end setting (e.g., when G is simply-connected, all principal G-bundles are trivializable since $H^4(X) = 0$). A more useful relation for our purposes deals with *adapted bundles*, which are pairs (E, A_{End}), where $E \to X$ is a principal G-bundle, and A_{End} is a connection on the cylindrical end End X. Then we say that (E, A_{End}) is *equivalent* to (E', A'_{End}) if there is a bundle isomorphism from E to E' that carries A_{End} to A'_{End}. See Donaldson's book [4, Section 3.2] for more details; note that Donaldson only treats flat connections A_{End}, but our applications require that we extend the discussion.

By the above discussion, it follows that $\mathcal{G}_0(\mathrm{End}\ X)$ consists of gauge transformations on End X that have extensions to $E \to X$. Thus, any adapted bundle (E, A_{End}) depends on A_{End} only through its $\mathcal{G}_0(\mathrm{End}\ X)$-equivalence class. The next example illustrates an interplay between the degree and the equivalence classes of adapted bundles; it will be relevant to our gluing discussion below.

Example 6.1. Fix an adapted bundle (E, A_{End}) and a point $x \in X$. Suppose $E_\ell \to S^4$ is a principal G-bundle with $\kappa(E_\ell) = \ell \in \mathbb{Z}$. Taking the connected sum of X and S^4

at x, we recover the same manifold back $X \cong X\#S^4$, up to diffeomorphism. At the bundle level, we can carry out a similar connected sum procedure to obtain a bundle $E' = E\#E_\ell$ over X. Provided $x \in X_0$ is not on the end, the connection A_{End} can be viewed as a connection on E'. Then the admissible bundles (E, A_{End}) and (E', A_{End}) are equivalent if and only if $\ell = 0$. More generally, there is a gauge transformation u on End X with $\deg(u) = \ell$, and so that the adapted bundle $(E, u^* A_{\mathrm{End}})$ is equivalent to (E', A_{End}).

Assume that the connection A_{End} converges on the end in the sense that

$$\lim_{t\to\infty} A_{\mathrm{End}}|_{\{t\}\times N} = \Gamma$$

for some connection Γ on N, where the limit is in $L^2_1(N)$, say. Let A be any connection on E that restricts on End X to A_{End}. Then the quantity

$$\kappa(E, A_{\mathrm{End}}) := \lim_{T\to\infty} -\frac{1}{8\pi^2}\int_{X_0\cup[0,T]\times N} \langle F_A \wedge F_A\rangle$$

is well-defined and independent of the choice of A. We will call $\kappa(E, A_{\mathrm{End}})$ the *relative characteristic number* of the adapted bundle (E, A_{End}). It depends on A_{End} only through the value of Γ and the topological type of E. Indeed, if $E' = E\#E_\ell$ is as in Example 6.1, then

$$\kappa(E', A_{\mathrm{End}}) = \kappa(E, A_{\mathrm{End}}) + \ell.$$

Moreover, if A_{End} is asymptotic to Γ, then working modulo $\mathbb{Z}$, we recover

$$8\pi^2\kappa(E, A_{\mathrm{End}}) = \mathcal{CS}(\Gamma) \qquad \mathrm{mod}\ \mathbb{Z}$$

the Chern–Simons value of Γ as defined in [17, Section 2.1] (here one should interpret the trace in [17] as the one induced from (2.1)).

6B. **Existence when $b^+(X) = 0$.** Let $E_{\mathrm{triv}} \to X$ be the trivial bundle, A_{triv} the trivial connection on E_{triv}, and Γ_{triv} the trivial connection on the end. Fix thickening data $\mathcal{T}_{\Gamma_{\mathrm{triv}}}$. Here we assume δ and β are chosen as in the beginning of Section 5. We recall from Section **2B** that the thickening data also includes the choice of $\epsilon_0 > 0$ so that any two points in the center manifold have Chern–Simons values differing by $\epsilon_0/2$. For each $0 < \epsilon < \epsilon_0$, we will write $\mathcal{T}(\epsilon)$ for the same set of thickening data as $\mathcal{T}_{\Gamma_{\mathrm{triv}}}$, but with ϵ in place of ϵ_0.

Let $E_\ell \to S^4$ be a principal G-bundle with $\kappa(E_\ell) = \ell \in \mathbb{Z}$, where κ is the characteristic number of Section **6A**. We will write

$$\mathcal{M}_\ell(S^4, G) := \left\{A \in \mathcal{A}(E_\ell) \mid F_A^+ = 0\right\}/\mathcal{G}(E_\ell)$$

for the moduli space of ASD connections on E_ℓ; here we are working relative to the standard metric on S^4. Let $\mathcal{M}^*_\ell(S^4, G) \subseteq \mathcal{M}_\ell(S^4, G)$ denote the subset of irreducible ASD connections. The existence of irreducible ASD connections on S^4 was studied extensively in [1, Section 8]. For example, when $G = \mathrm{SU}(r)$ and the embedding (2.1) is the identity, then the space $\mathcal{M}^*_\ell(S^4, \mathrm{SU}(r))$ is nonempty if and only if $\ell \geq r/2$. The most famous situation is when $G = \mathrm{SU}(2)$ and $\ell = 1$, in which case $\mathcal{M}_1(S^4, \mathrm{SU}(2)) =$

$\mathcal{M}_1^*(S^4, \mathrm{SU}(2))$ and this is diffeomorphic to the open unit ball in $\mathbb{R}^5$. The dimension of $\mathcal{M}_\ell^*(S^4, G)$ for general simple, simply-connected G is given in [1, Table 8.1].

The following is the first of our main existence results for mASD connections; Theorem C with $b^+(X) = 0$ from the introduction is an immediate consequence.

Theorem 6.2 (Existence of mASD-connections when $b^+ = 0$)**.** *Assume $b^+(X) = 0$ and A_ℓ is an irreducible* ASD *connection on $E_\ell \to S^4$ for some $\ell \in \mathbb{Z}$. Then for every $0 < \epsilon < \epsilon_0$, there is*

- *(a) a neighborhood $\mathcal{V} \subseteq \mathcal{M}_\ell(S^4, G)$ of $[A_\ell] \in \mathcal{M}_\ell(S^4, G)$;*
- *(b) a trivializable principal G-bundle $E \to X$ that is canonically trivial on the end;*
- *(c) a connection A' on E that is flat in the complement of a compact set, asymptotic to Γ on the end, and satisfies $\kappa(E, A'|_{\mathrm{End}\ X}) = \ell$; and*
- *(d) a $\mathcal{C}^m$-embedding*

$$\Phi : \mathcal{V} \longrightarrow \widehat{\mathcal{M}}_{reg}(\mathcal{T}(\epsilon), A').$$

The image of Φ consists of irreducible connections. In particular, there exist an irreducible, regular mASD *connection A on E with $|\kappa(E, A|_{\mathrm{End}\ X}) - \ell| < \epsilon/2$.*

If X is simply-connected, then the map Φ is a $\mathcal{C}^m$-diffeomorphism onto an open subset of $\widehat{\mathcal{M}}_{reg}(\mathcal{T}(\epsilon), A')$.

For $\epsilon < 1$ and $\ell \neq 0$, the condition $|\kappa(E, A|_{\mathrm{End}\ X}) - \ell| < \epsilon/2$ implies that A is not flat. (The analogous statement in the case where X is closed and $G = \mathrm{SU}(r)$ is that A is supported on a bundle P with $c_2(P)\,[X] = \ell$.)

Proof of Theorem 6.2. View S^4 as a cylindrical end 4-manifold with no ends, and let $\mathcal{T}_\emptyset$ be the empty set of thickening data as in Section **2C**. Form the connected sum of S^4 and X at any point in S^4 and any point in X lying in the interior of the compact part. Note that all ASD connections on S^4 are regular (e.g., use the Weitzenböck formula [5, (7.1.23)]). In particular, since A_ℓ is irreducible and regular, there is a neighborhood $\mathcal{V} \subseteq \mathcal{M}_\ell^*(S^4, G)$ of $[A_\ell]$ that is diffeomorphic to a precompact open set $G_1 \subseteq \widehat{\mathcal{M}}_{reg}(\mathcal{T}_\emptyset, A_\ell)$ containing A_ℓ; that is, $\mathcal{V}$ consists of gauge equivalence classes of regular, irreducible ASD connections on S^4, and G_1 consists of their lifts to the Coulomb slice through A_ℓ. Using this diffeomorphism, we identify $\mathcal{V}$ and G_1.

By Proposition 2.13, the assumption $b^+(X) = 0$ implies that A_{triv} is regular. Then the singleton set $G_2 := \{A_{\mathrm{triv}}\}$ plainly consists of regular connections. First assume X is simply-connected. Then A_{triv} is isolated in its gauge slice and so G_2 happens to also be an open subset of the space $\widehat{\mathcal{M}}_{reg}(\mathcal{T}_{\Gamma_{\mathrm{triv}}}, A_{\mathrm{triv}})$ of regular mASD connections on X. Define $E := E_{\mathrm{triv}}\#E_\ell$ to be the connected sum bundle as in Section **6A**, equipped with thickening data $\mathcal{T}(\epsilon)$. Take A' to be the preglued connection, which is plainly asymptotic to Γ_{triv}. In particular, $\kappa(E; A'|_{\mathrm{End}\ X}) = \ell$ due to the discussion of Section **6A**. By possibly shrinking $\mathcal{V}$, if necessary, we define Φ to be the $\mathcal{C}^m$-diffeomorphism of the same name from Theorem 5.1 (b) (here we are using the identifications $\mathcal{V} \cong G_1 \cong G_1 \times G_2$). If A is any connection in the image of Φ, then $|\kappa(E, A|_{\mathrm{End}\ X}) - \ell| < \epsilon/2$ follows from the definition of ϵ as a parameter in the set of thickening data and the fact that $\kappa(E; A|_{\mathrm{End}\ X})$ recovers the Chern–Simons value of the asymptotic limit of A.

In the case where X is not simply-connected, it follows from Remark 5.3 (a) that the same conclusions hold, except Φ is now only a $\mathcal{C}^m$-embedding (i.e., it need not be surjective). □

6C. Existence when $b^+(X) = 1$. Here we consider the case where $b^+(X) = 1$ and $G = \mathrm{SU}(2)$. We assume (2.1) is the identity (so $r = 2$); then the characteristic number κ from Section **6A** is the second Chern number. Fix thickening data $\mathcal{T}_{\Gamma_{\mathrm{triv}}}$ and assume δ and β are chosen as in the beginning of Section 5. Define $\mathcal{T}(\epsilon)$ as in Section **6B**. The following is the second of our main existence results for mASD connections. Theorem C with $b^+(X) = 1$ is an immediate consequence.

Theorem 6.3 (Existence of mASD-connections when $b^+ = 1$)**.** *Assume $b^+(X) = 1$ and fix an integer $\ell \geq 2$. Then for every $0 < \epsilon < \epsilon_0$, there is*

(a) a trivializable principal $\mathrm{SU}(2)$*-bundle $E \to X$ that is canonically trivial on the end;*

(b) a connection A' on E that is flat in the complement of a compact set, asymptotic to Γ on the end, and satisfies $\kappa(E; A'|_{\mathrm{End}\ X}) = \ell$; and

(c) an irreducible mASD *connection $A \in \widehat{\mathcal{M}}(\mathcal{T}(\epsilon), A')$ on E satisfying*

$$|\kappa(E, A|_{\mathrm{End}\ X}) - \ell| < \epsilon/2.$$

Proof. Our proof follows that of [20, Section 7] and [3, pp. 327—334]. The assumption that $b^+ = 1$ combines with Proposition 2.13 to imply that the cokernel of the operator $d^+ : L^p_{1,\delta}(\Omega^1(X, \mathbb{R})) \to L^p_\delta(\Omega^+(X, \mathbb{R}))$ is one-dimensional. As in Section **2C**, this cokernel can be realized as the space $H^+(X, \mathbb{R})$ of closed self-dual 2-forms in $L^2(\Omega^+(X, \mathbb{R}))$ that restrict on each slice $\{t\} \times N$ to be orthogonal to the space of harmonic forms on N. Fix a non-zero element $\omega_0 \in H^+(X, \mathbb{R})$; this is unique up to scaling. By unique continuation for solutions of elliptic equations, it follows that the set of points in X where ω_0 vanishes is open and dense. In particular, we can find two distinct points $x_1, x_2 \in \mathrm{int}(X_0)$ with $\omega_0(x_1) \neq 0$ and $\omega_0(x_2) \neq 0$. When $\ell > 2$, choose additional points $x_3, \ldots, x_{\ell-2} \in \mathrm{int}(X_0)$; these can be arbitrarily chosen, provided the x_i are all distinct. The gluing Theorem 3.3 has a straightforward extension to handle gluing for multiple connected sums that we briefly describe now.

Fix scaling parameters $\lambda_1, \ldots, \lambda_\ell > 0$, and set

$$\lambda := \max(\lambda_1, \ldots, \lambda_\ell).$$

Here we will consider ℓ copies of S^4; denote these copies by $S^4_1, \ldots, S^4_\ell$, and fix points $x'_i \in S^4_i$. Then as we did in Section **3A**, glue $x_i \in X_0$ to $x'_i \in S^4_i$ over balls with radii controlled by λ_i.

At the bundle level, let $E_{\mathrm{triv}} \to X$ be the trivial $\mathrm{SU}(2)$-bundle, and let $E_1 \to S^4$ be the $\mathrm{SU}(2)$-bundle with $\kappa(E_1) = c_2(E_1)\left[S^4\right] = 1$. More concretely, we can take E_1 to be the frame bundle of $\Lambda^+ T^* S^4$ (then $\Lambda^+ T^* S^4$ is the adjoint bundle of E_1). In Section 3, gluing the bundles depended on the choice of fiber isomorphism ρ identifying the fibers of the principal bundles at the gluing points. In the present setting with ℓ gluing points, this corresponds to the choice of a fiber isomorphism

$$\rho_i \in \mathrm{Gl}_i = \mathrm{Hom}_{\mathrm{SU}(2)}((E_{\mathrm{triv}})_{x_i}, (E_1)_{x'_i}),$$

for each $1 \leq i \leq \ell$.

Let A_{triv} be the trivial connection on E_{triv}. Let A' be the preglued connection on E obtained from A_{triv} and the standard "one-instanton" A_{st} on each of the bundles $E_1 \to S^4_i$ for $1 \leq i \leq \ell$. Then $\kappa(E, A'|_{\text{End }X}) = \ell$; note that A' depends on the λ_i and ρ_i. The proof of Theorem 3.3 extends to produce C, L, λ_0, J, π, and $\xi \in L^p_\delta(\Omega^+(X))$, satisfying the conditions of Theorem 3.3 (a)—(c) and Corollary 3.23 whenever $0 < \lambda < \lambda_0$; though we suppress this in the notation, these quantities depend on the connections A_{triv} and A_{st}, as well as the isomorphisms ρ_i. In particular, the connection $A := J(\xi)$ is irreducible and satisfies

$$s(A) = -\sigma\pi\xi, \qquad |\kappa(E, A|_{\text{End }X}) - \ell| < \epsilon/2.$$

It suffices to show that the λ_i and ρ_i can be chosen so that $\sigma\pi\xi = 0$, since this implies that A is mASD. For this, let X' be the complement in X of the $L\lambda_0^{1/2}$-balls around the x_i; we assume λ_0 is small enough so these balls do not intersect and are contained in X_0. Note that the bundles E and E_{triv} are canonically identified over X', and so over X' we can compare 2-forms on E_{triv} with 2-forms on E. The self-dual 2-form $\sigma\pi\xi$ vanishes if and only if the integral

$$\int_{X'} \langle \omega \wedge \sigma\pi\xi \rangle = 0 \tag{6.4}$$

vanishes for all $\omega \in H^+(X, \text{ad}(A_{\text{triv}})) = H^+(X, \mathbb{R}) \otimes \mathfrak{g}$.

Claim:

$$\int_{X'} \langle \omega \wedge \sigma\pi\xi \rangle = q^\ell_\omega(\{(\lambda_i, \rho_i)\}_i) + O(\lambda^3) \tag{6.5}$$

where

$$q^\ell_\omega(\{(\lambda_i, \rho_i)\}_i) := \sum_{i=1}^{\ell} \lambda_i^2 \text{tr}(\rho_i \omega(x_i)).$$

Here $\text{tr}(\rho_i \omega(x_i)) \in \mathbb{R}$ is the pairing of ρ_i and $\omega(x_i)$ as described in [3, Equation (5.3)]. We will prove this claim below, but first we will show how it is used to finish the proof of the theorem. From the discussion leading up to the claim, we are interested in the simultaneous system of equations

$$q^\ell_\omega(\{(\lambda_i, \rho_i)\}_i) = 0, \quad \forall \omega \in H^+(X, \text{ad}(A_{\text{triv}})) \tag{6.6}$$

When $\ell = 2$, the argument of [3, Section V(ii)] carries over verbatim to show that the solutions set of the system (6.6) is non-empty and cut out transversely, whenever $\max(\lambda_1, \lambda_2)$ is sufficiently small. This uses the assumption $\omega_0(x_1), \omega_0(x_2) \neq 0$. (Alternatively, the reader could follow the original argument of Taubes [20, Prop. 7.1], but our notation is more inline with that of [3].) When $\ell > 2$, it was pointed out by Taubes [20, Prop. 6.2] that by taking $\max(\lambda_3, \ldots, \lambda_\ell)$ sufficiently small relative to $\max(\lambda_1, \lambda_2)$, any transverse zero of q^2_ω implies the existence of a transverse zero of q^ℓ_ω. In summary, for each $\ell \geq 2$, there are $\lambda_0' > 0$ and $\mu \in (0,1)$ so that the system (6.6) has a nonempty, transverse solution set, for all $\lambda_1, \ldots, \lambda_\ell > 0$ with

$$\max(\lambda_1, \lambda_2) < \lambda_0', \quad \max(\lambda_3, \ldots, \lambda_\ell) < \mu \max(\lambda_1, \lambda_2).$$

For any such $\lambda_1, \ldots, \lambda_\ell$, since q_ω^ℓ is $O(\lambda^2)$, it then follows from the transversality of $q_\omega^\ell = 0$ and the identity (6.5) that the solution sets to (6.4) and (6.6) are diffeomorphic, provided λ is sufficiently small. In particular, there is a simultaneous zero $\{(\lambda_i, \rho_i)\}_{i=1}^\ell$ of the solution set to (6.4). For this collection of gluing data, the glued connection A is therefore mASD, as desired.

It therefore suffices to verify the above Claim. We will first unpack the notation. Note that the preglued connection A' restricts on X' to equal the trivial flat connection. Let $A'(\lambda_0)$ be the preglued connection defined using λ_0 at every gluing site, and the same ρ_i as was used to define A' (so the only difference between A' and $A'(\lambda_0)$ is that the former uses λ_i at the gluing site x_i, while the latter uses λ_0 at all gluing sites). Define the map i (and hence ι) using $A'(\lambda_0)$ as a reference connection. Then we can write $A' = \iota(\Gamma, V') = i(\Gamma) + V' = A'(\lambda_0) + V'$ for some 1-form V'. It follows that $V'|_{X'} = 0$, and we note also that $A'(\lambda_0)|_{X'} = A_{\mathrm{triv}}$.

Next, recall the map $P : L^p_\delta(\Omega^+) \to T_\Gamma\mathcal{H} \times L^p_{1,\delta}(\Omega^1)$ from Claim 1 in the proof of Theorem 3.3, and write $P\xi = (\eta, V)$. The definition of the map $J = J_{A_{\mathrm{triv}}, A_{\mathrm{st}}}$ gives

$$J(\xi) = i(\exp_\Gamma(\eta)) + V' + V$$

where $\exp_\Gamma : T_\Gamma\mathcal{H} \to \mathcal{H}$ is the exponential. The observations of the previous paragraph combine with the formula (2.10) to give that the restriction of $s(A)$ takes the following form:

$$s(A)|_{X'} = (1 - \beta')F^+_{i(\exp_\Gamma(\eta))} + d^+_{i(\exp_\Gamma(\eta))}V + \frac{1}{2}\left[V \wedge V\right]^+.$$

Returning to the integral (6.4), we can use the defining property of ξ and the above identity for $s(A)$ to get

$$\begin{aligned}
\int_{X'} \langle \omega \wedge \sigma\pi\xi \rangle &= -\int_{X'} \langle \omega \wedge s(A) \rangle \\
&= -\int_{X'} (1-\beta')\langle \omega \wedge F^+_{i(\exp_\Gamma(\eta))} \rangle - \int_{X'} \langle \omega \wedge d^+_{i(\exp_\Gamma(\eta))} V \rangle \\
&\quad -\tfrac{1}{2}\int_{X'} \langle \omega \wedge [V \wedge V]^+ \rangle.
\end{aligned} \tag{6.7}$$

Focus on the last term on the right. Recall from Lemma 2.17 that ω decays in $\mathcal{C}^0$ like $e^{-\delta t}$, and so

$$\int_{X'} \left| \langle \omega \wedge [V \wedge V]^+ \rangle \right| \leq C_1 \|V\|^2_{L^2_\delta(X)} \leq C_1 \|P\xi\|^2_{L^2_\delta(X)}$$

for some constant C_1. By Corollaries 3.23 and 3.27, this term decays like λ^3:

$$-\frac{1}{2}\int_{X'} \langle \omega \wedge [V \wedge V]^+ \rangle = O(\lambda^3).$$

We can control the nonlinear parts of the other two terms in (6.7) similarly. Indeed, use $i(\exp_\Gamma(\eta)) = A_{\mathrm{triv}} + (Di)_\Gamma\eta + O(\eta^2)$ and the expansion formulas for the curvature and covariant derivative, to get

$$\int_{X'}\langle\omega\wedge\sigma\pi\xi\rangle \;=\; -\int_{X'}(1-\beta')\langle\omega\wedge d^+(Di)_\Gamma\eta\rangle - \int_{X'}\langle\omega\wedge d^+V\rangle + O(\lambda^3),$$

where $d = d_{A_{\mathrm{triv}}}$. Focus on the first term on the right (there is no analogue of this term in the standard ASD framework). It follows from the definitions of β' and i that $(1-\beta')(d^+(Di)_\Gamma\eta)$ is supported on $[T-1/2, T+1/2]\times N$. Using the formula (2.15), we have

$$-\int_{X'}(1-\beta')\langle\omega\wedge d^+(Di)_\Gamma\eta\rangle = -\int_{X'}(1-\beta')(\partial_t\beta'')\langle\omega\wedge(dt\wedge\eta)^+\rangle = 0,$$

where the last equality uses the facts that (i) $\eta\in H^1_{\Gamma_{\mathrm{triv}}}$ is in the harmonic space on N, and (ii) elements of $H^+(X,\mathrm{ad}(A_{\mathrm{triv}}))$ restrict on each slice to be orthogonal to $H^1_{\Gamma_{\mathrm{triv}}}$. In summary, this gives

$$\begin{aligned}\int_{X'}\langle\omega\wedge\sigma\pi\xi\rangle &= -\int_{X'}\langle\omega\wedge d^+V\rangle + O(\lambda^3)\\ &= \int_{\partial X'}\langle\omega\wedge V\rangle + O(\lambda^3).\end{aligned}$$

What remains is to estimate the integral $\int_{\partial X'}\langle\omega\wedge V\rangle$. This is an integral taking place at the boundary of the disks centered at the gluing sites $x_1,\ldots,x_\ell$. In particular, this integral is identical to the analogous term that arises when gluing in the standard ASD setting (e.g., see the top of [3, p. 328]). Then the argument of [3, pp. 328—331] carries over verbatim to give

$$\int_{\partial X'}\langle\omega\wedge V\rangle = q^\ell_\omega(\{(\lambda_i,\rho_i)\}) + O(\lambda^3).$$

This proves (6.5). □

6D. **An ASD existence result and a proof of Theorem A.** Recall from Section **2A** the definition of the vector field Ξ_Γ on the center manifold. We will be interested in the case where the flat connection Γ satisfies the following hypothesis:

Hypothesis H. *There is a neighborhood $U\subseteq\mathcal{H}_\Gamma$ of Γ so that every $a\in U$ flows under Ξ_Γ to a flat connection in U.*

Example 6.8.

(a) Recall from Section **2A** that U_Γ is a neighborhood of Γ in the Coulomb slice through Γ. Suppose the set of flat connections in U_Γ is smooth in a neighborhood $U'\subseteq U_\Gamma$ of Γ and has the same dimension as $\mathcal{H}_\Gamma$. Then $U := U'\cap\mathcal{H}_\Gamma$ satisfies Hypothesis H.

(b) The assumption of (a) trivially holds when Γ is non-degenerate, since $\mathcal{H}_\Gamma$ consists of a single point. More generally, the assumption of (a) also holds when the Chern–Simons function is Morse–Bott in a neighborhood of Γ.

(c) Suppose $N = T^3$, and let Γ be a flat connection on the trivial SU(2)-bundle. If Γ is not gauge equivalent to the trivial connection, then Γ satisfies the assumption in (a), and hence Hypothesis H; see [10, Lemma 14.2(i)]. However, the trivial connection on T^3 does *not* satisfy Hypothesis H.

The main usefulness of Hypothesis H for us is through the following theorem.

Theorem 6.9. *Consider the situation of Theorem 3.3, and assume A_1 and A_2 are regular. In addition, assume that Γ_1 and Γ_2 each satisfy Hypothesis H. Let $\lambda_0 > 0$ be the constant from Theorem 3.3. Then there is some $0 < \lambda_0' \leq \lambda_0$ so that for all $0 < \lambda < \lambda_0'$, the* mASD *connection $\mathcal{J}(A_1, A_2)$ guaranteed by Theorem 3.3 (and hence by Theorems B and C) is in fact* ASD.

Proof. Fix $0 < \lambda < \lambda_0$ and let $A_\lambda := \mathcal{J}(A_1, A_2)$ be the mASD connection from Theorem 3.3 associated to this value of λ. Recall that $\Gamma = \Gamma_1 \sqcup \Gamma_2$, and so Γ satisfies Hypothesis H since the Γ_k do. It follows from (3.6) that $p_T(A_\lambda) \in \mathcal{H}_{in}$ converges to Γ as λ approaches 0. In particular, by taking λ sufficiently small, Hypothesis H implies that the Ξ_Γ-flow line beginning at $p_T(A_\lambda)$ lies in $\mathcal{H}_{in}$ for all positive time. This implies $i(p_T(A_\lambda))$ is ASD, and so

$$F^+_{A_\lambda} = F^+_{A_\lambda} - \beta' F^+_{i(p_T(A_\lambda))} = s(A_\lambda) = 0.$$

□

Now we can prove our application from the introduction.

Proof of Theorem A. Take Γ to be the trivial flat connection on the trivial SU(2)-bundle. We will show that the two conditions on N stated in Theorem A each imply that Γ satisfies Hypothesis H; it will then be immediate that the mASD connection guaranteed by Theorem C is in fact ASD, as desired.

First assume N is a circle bundle over a surface with positive euler class. Then [17, Corollary 13.2.2] implies that Γ satisfies the condition of Example 6.8 (a), and thus Hypothesis H.

Now assume that $b_1(N) \leq 1$. If $b_1(N) = 0$, then $H^1_\Gamma = H^1(N) \otimes \mathfrak{g} = 0$ and so Γ is nondegenerate. Thus, Γ again satisfies Hypothesis H, but this time by Example 6.8 (b).

Finally, suppose $b_1(N) = 1$. We will show here that Γ satisfies the condition of Example 6.8 (a). Since $b_1(N) = 1$, there is a loop $\gamma : S^1 \to N$ and a harmonic 1-form $\eta \in \Omega^1(N, \mathbb{R})$ so that $\int_\gamma \eta = 1$. For each $\xi \in \mathfrak{g}$, let

$$a_\xi := \Gamma + \xi \otimes \eta.$$

We claim that a_ξ lies in the center manifold $\mathcal{H}_\Gamma$ for all sufficiently small ξ. To see this, first note that a_ξ is flat, since

$$F_{a_\xi} = F_\Gamma + \xi \otimes (d\eta) + \frac{1}{2}[\xi, \xi] \otimes \eta \wedge \eta = 0.$$

This connection also lies in the Coulomb gauge slice for Γ, since

$$d^*_\Gamma(a_\xi - \Gamma) = \xi \otimes d^*\eta = 0.$$

Recall the map Θ and the vector field ∇f_Γ from Section **2A.1**. Since $F_{a_\xi} = 0$, we have

$$\Theta(a_\xi) = 0.$$

Thus $\nabla_{a_\xi} f_\Gamma = 0$. One of the defining features of $\mathcal{H}_\Gamma$ is that it contains all zeros of ∇f_Γ that are sufficiently close to Γ, so this proves the claim.

It thus follows that there is some $\epsilon > 0$ so that the map

$$B_\epsilon(0) \subseteq \mathfrak{g} \longrightarrow \mathcal{H}_\Gamma, \qquad \xi \longmapsto a_\xi$$

is well-defined. It is clearly an immersion, so a dimension count implies that it must be a local diffeomorphism; this uses the fact that $b_1(N) = 1$. This establishes the condition of Example 6.8 (a). □

7. PARTIAL COMPACTIFICATION—THE TAUBES BOUNDARY

Here we give a more global formulation of the result of Theorem 6.2 in the case where $G = \mathrm{SU}(2)$ and $\ell = 1$. Fix a closed set X_0' contained in the interior of the compact part X_0. Let A_{st} be the standard one-instanton on the SU(2)-bundle $E_1 \to S^4$ with $c_2(E_1)\left[S^4\right] = 1$. For $x \in X_0'$, let X_x be the connected sum of X and S^4 obtained by gluing $x \in X$ to the north pole in S^4. Similarly, glue the trivial SU(2)-bundle on X to $E_1 \to S^4$ and let $E_x \to X_x$ be the resulting bundle. Let $A_x' = A'(A_{\mathrm{triv}}, A_{\mathrm{st}})$ be the preglued connection on E_x, where A_{triv} is the trivial connection on X. Note that in the present situation, all auxiliary gluing data can be chosen to be independent of x. For example, the fiber isomorphism ρ of Section **3A** can be taken to be independent of x since we are starting with the trivial bundle on X.

Fix $\epsilon > 0$, and let $\mathcal{T}(\epsilon)$ be thickening data with this choice of ϵ, as in Section **6B**. By Theorem 3.3, there are $\epsilon_0, \lambda_0 > 0$ so the following holds: For all $0 < \epsilon < \epsilon_0$ and $0 < \lambda \leq \lambda_0$, there is an irreducible, regular mASD connection

$$A(x, \lambda) := \mathcal{J}(A_{\mathrm{triv}}, A_{\mathrm{st}}) \in \mathcal{A}^{1,p}(\mathcal{T}(\epsilon))$$

with the property that $A(x, \lambda) - A_{\mathrm{triv}}|_{X \setminus \mathrm{nbhd}(x)}$ goes to zero in λ in the sense of (3.6). This ϵ_0 depends only on the trivial flat connection on the 3-manifold N; hence ϵ_0 is independent of x. Since X_0' is compact, we can assume this λ_0 is independent of x as well.

We want to allow x to vary, and for this, we form the space

$$\mathcal{E}' := \left\{ (x, \lambda, A) \in X_0' \times (0, \lambda_0] \times \mathcal{A}^{1,p}(\mathcal{T}(\epsilon)) \;\middle|\; A \in \widehat{\mathcal{M}}_{reg}(\mathcal{T}(\epsilon), A(x, \lambda)) \right\}.$$

Let $\Pi' : \mathcal{E}' \to X_0' \times (0, \lambda_0]$ be the projection to the first two factors. Then the assignment $\Psi'(x, \lambda) := (x, \lambda, A(x, \lambda))$ defines a section of Π'. Just as in Theorem 5.1, there is an open neighborhood $\mathcal{U}' \subseteq \mathcal{E}'$ of the image of Ψ' so that the restriction $\Pi'|_{\mathcal{U}'}$ is a locally-trivial $\mathcal{C}^m$-fiber bundle over $X_0' \times (0, \lambda_0]$. By construction, the fiber over (x, λ) is an open subset of $\widehat{\mathcal{M}}_{reg}(\mathcal{T}(\epsilon), A(x, \lambda))$ containing $A(x, \lambda)$.

Remark 7.1. Theorem 6.2 can be viewed as a local version of this fiber bundle construction as follows: Fix a small neighborhood $U_x \subseteq X_0$ around x. The gluing procedure of Section **3A** identifies this with a small neighborhood of the north pole in S^4. The standard description [9, Ch. 6] of the ASD moduli space $\mathcal{M}_1(S^4, \mathrm{SU}(2))$ gives an embedding $S^4 \times (0, \lambda_0] \to \mathcal{M}_1(S^4, \mathrm{SU}(2))$ with the S^4-component specifying the center of mass and $(0, \lambda_0]$ parametrizing the scale of the curvature; here the curvature is concentrating, as λ approaches 0, to a Dirac delta measure supported at the center of mass. Combining these, we have a diffeomorphism

$$f : U_x \times (0, \lambda_0] \longrightarrow \mathcal{V} \subseteq \mathcal{M}_1(S^4, \mathrm{SU}(2))$$

onto an open set $\mathcal{V}$. It follows from this construction that there is a local trivialization of the fiber bundle $\Pi'|_{\mathcal{U}'}$ relative to which Ψ' takes the form $(y, \lambda) \mapsto (y, \lambda, \Phi(f(y, \lambda)))$ where Φ is the map of Theorem 6.2. In fact, by possibly shrinking U_x further, this local trivialization can be taken to be over the full cylinder $U_x \times (0, \lambda_0]$; this due to the fact that the constructions in the proof of Theorem 5.1 can be taken to be uniform in λ. This construction is exploiting a coupling between the parameter λ and the "scale" parameter for the concentration of instantons on $E_1 \to S^4$; see [5, p. 323] for a related discussion.

Now we consider the behavior of this section Ψ' near $\lambda = 0$. For this, suppose $(x_n, \lambda_n) \in X_0' \times (0, \lambda_0]$ is a sequence with $\lambda_n \to 0$; we will call this a *bubbling sequence in* X_0'. By passing to a subsequence, we may assume the x_n converge to some $x_\infty \in X_0'$. It follows from a straight-forward Uhlenbeck-type compactness argument and (3.6) that, after passing to a subsequence, the associated connections $A(x_n, \lambda_n)$ converge *weakly* to the ideal connection (A_{triv}, x_0) in the sense that the curvatures of the $F_{A(x_n,\lambda_n)}$ converge in measure to the delta measure supported at x_0, and

$$\lim_{n\to\infty} \left\| \iota^{-1}(A(x_n, \lambda_n)) - \iota^{-1}(A_{\mathrm{triv}}) \right\|_{L^2_2(N) \times L^p_{1,\delta}(X \setminus B_r(x_0))} = 0$$

for all $r > 0$; see [5, Section 4.4.1] for the analogous ASD case.

Following the lead of [5, Section 4.4.1], the discussion of the previous paragraph can be framed geometrically as follows. Consider the set

$$I(\mathcal{U}') := \mathcal{U}' \cup (X_0' \times \{A_{\mathrm{triv}}\}),$$

which we view as coupling the connections in $\mathcal{U}' \subseteq \mathcal{E}'$ into the same space as the above-mentioned ideal connections. We can extend $\Pi'|_{\mathcal{U}'}$ to a map $I(\Pi') : I(\mathcal{U}') \to X_0' \times [0, \lambda_0]$ by declaring it to send (x, A_{triv}) to $(x, 0)$. Give $I(\mathcal{U}')$ any topology (more below) for which the map $I(\Pi')$ is continuous and so that the notion of weak convergence from the previous paragraph implies convergence in $I(\mathcal{U}')$; we assume also that this topology is first countable. Then the observations of the previous paragraph imply the section Ψ' extends continuously over $X_0' \times \{0\}$ to a section $\overline{\Psi}'$ of $I(\Pi')$. It is due to this that we may view $I(\mathcal{U}')$ as a "partial compactification" for mASD connections: The bubbling sequences in X_0' converge in $I(\mathcal{U}')$.

We end this section with several comments about the construction of the partial compactification $I(\mathcal{U}')$, as well as some of its limitations. This partial compactification is constructed only so that bubbling sequences in X_0' converge—our assumptions on the topology on $I(\mathcal{U}')$ do not necessarily imply subsequential convergence of other types of sequences. The simple reason for this is that we do not yet know how such sequences behave, and what additional limiting objects we would need to include in $I(\Pi')$ to ensure their subsequential convergence. What we are presently lacking is a sufficiently strong version of Uhlenbeck's compactness theorem for mASD connections. In the end, such a theorem would need to (at least) address the following:

(a) *Bubble formation on the end*: To what extent is the mASD condition preserved under Uhlenbeck limits where the curvature concentrates at a point in End X? More fundamentally, is the connections space $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ suitably closed under such limits? This is related to (c) below.

From the gluing perspective, we avoided these questions altogether by only gluing at points in the compact part where mASD connections are ASD; that is, $I(\mathcal{U}')$ only corresponds to the points in the "Taubes boundary" that corresponds to bubbles in $X_0' \subseteq X \backslash$ End X. A more thorough investigation would require not only an understanding of the mASD condition under Uhlenbeck limits, but also an understanding of how to glue at points on End X.

(b) *Energy escaping down the end*: One example of this is bubbling on the end, as discussed in (a). Another example is where a non-trivial amount of energy escapes down the end. This is familiar in the ASD setting, where compactification can be achieved by including spaces of translationally-invariant ASD connections on $\mathbb{R} \times N$ (spaces of "Floer trajectories"); see [8, 4]. In the mASD setting, one would likely need to include spaces of mASD connections on $\mathbb{R} \times N$ to account for energy escaping. The details of this appear to be subtle, since the energy of such connections are not governed by topological quantities, as is the case in the ASD setting. (In the discussion above, where we considered sequences in the image of Ψ', we were able to exclude non-trivial energy on the end by appealing to (3.6).)

(c) *Failure of the slice-wise gauge fixing condition*: In the definition of the space $\mathcal{A}^{1,p}(\mathcal{T}_\Gamma)$ from Section **2B.1**, we restricted attention to connections that restrict on each time slice $\{t\} \times N$, for $t \geq T$, to be gauge equivalent to a connection in the gauge slice U_Γ. This is an open condition in the space of all $L^p_{1,loc}$ connections, and we do not see a reason why this condition should be retained through limits of mASD connections.

It is clear from these observations that $I(\mathcal{U}')$ is by no means the end of the story when it comes to compactification. It is due to this that we have avoided defining a specific topology on $I(\mathcal{U}')$ above, choosing instead to axiomatize a minimal set of desirable properties.

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