A REMARK ABOUT DIHEDRAL GROUP ACTIONS ON SPHERES

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Abstract. We show that a finite dihedral group does not act pseudofreely and locally linearly on an even-dimensional sphere S^{2k}, with $k > 1$. This answers a question of R. S. Kulkarni from 1982.

1. Introduction

In this note we let $D_p = \langle a, b \mid a^p = b^2 = 1, bab = a^{-1} \rangle$ denote the finite dihedral group of order $2p$, for p an odd prime. A famous theorem of Milnor [8] states that a finite dihedral group can not act freely on a topological n-manifold with the mod 2 homology of S^n. More generally, a *pseudofree* action is one which is free outside of a discrete set of points. In [6, Theorem 7.4], R. S. Kulkarni studied orientation-preserving, pseudofree actions of finite groups G on manifolds which are $\mathbb{Z}/2$-homology n-spheres, and found that for $n = 2k$ the group G must be (i) a periodic group which acts freely on S^{2k-1}, (ii) dihedral, or (iii) tetrahedral, octahedral or icoshedral (when $k = 1$). The first case occurs as the suspension of any free action of a periodic group on S^{2k-1}, and the other cases already appear for orthogonal actions on S^2. Kulkarni asked whether the second case could actually occur on S^{2k} if $k > 1$. This turns out to be impossible.

Theorem A. *The dihedral group $G = D_p$, p an odd prime, can not act pseudofreely and locally linearly on S^{2k}, preserving the orientation, for $k > 1$.*

For k even, we show that there does not even exist a finite pseudofree G-CW complex $X \simeq S^{2k}$, with $X^G = \emptyset$. For all *odd* integers $k \geq 1$, such complexes do exist, for example by taking the join of S^2 with the action given by $G \subset SO(3)$ and a finite Swan complex for G (see [9], [3]).

Remark 1.1. My interest in this question was prompted by the recent paper of A. Edmonds [2], where he proves this result for k even. Our methods seem rather different. The discussion by Edmonds in [2, 4.1] combined with Theorem A shows that there are no effective pseudo-free dihedral actions on S^n, for $n > 2$, even if some elements of G are allowed to reverse orientation.

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2. The chain complex

In this section we let $G = D_p$ and suppose that X is a finite G-CW complex such that $X \simeq S^{2k}$, with $k > 0$, and $X^G = \emptyset$. We further assume the G-action is pseudofree, and induces the identity on homology. It follows from [6, Prop. 7.3] that every non-identity element of G fixes exactly two points. We assume that $X^G = \emptyset$ since this is a necessary condition for a locally-linear, pseudo-free action on a sphere (by Milnor's theorem).

Let $\mathbf{C} = \mathbf{C}(X^?))$ denote the chain complex of X over the orbit category $\mathbb{Z}\varGamma := \mathbb{Z}\operatorname{Or}_{\mathcal{F}} G$ with respect to the family $\mathcal{F}$ of all proper subgroups of G (see [1] or [7] for this theory). The notation means that $\mathbf{C}_i(G/U) = C_i(X^U)$, for $U \leq G$, and the action of $N_G(U)/U$ on $\mathbf{C}_i(G/U)$ induced by the G-action on X is expressed algebraically through the functorial properties of $\mathbf{C}$..

Our pseudo-free assumption on the G-CW complex X implies that $\mathbf{C}_i(G/U) = 0$, if $U \neq 1$ is a non-trivial subgroup of G, and $i > 0$. Therefore,

$$H_i(\mathbf{C})(G/U) = 0, \text{ if } i > 0, \text{ for all } U \neq 1. \tag{2.1}$$

From the homology of S^{2k}, we have

$$H_0(\mathbf{C})(G/1) = \mathbb{Z}, \text{ and } H_i(\mathbf{C})(G/1) = 0 \text{ for } i \neq 0, 2k. \tag{2.2}$$

In addition, since we assumed that G acts trivially on the homology of S^{2k}, we have

$$H_{2k}(\mathbf{C})(G/1) = \mathbb{Z}, \text{with trivial } G\text{-action}. \tag{2.3}$$

Let $H = \langle a \rangle$ and $K = \langle b \rangle$ denote particular subgroups of G, of order p and 2 respectively. The orbit types give the chain group

$$\mathbf{C}_0 = \mathbb{Z}[G/H^?] \oplus \mathbb{Z}[G/K^?] \oplus \mathbb{Z}[G/K^?],$$

where $\mathbb{Z}[G/V^?]$ denotes the free right module over the orbit category with values

$$\mathbb{Z}[G/V^?](G/U) = \mathbb{Z}\operatorname{Map}_G(G/U, G/V),$$

for all proper subgroups $U \leq G$. In particular, the homology of the fixed sets is given by

$$H_0(\mathbf{C})(G/H) = \mathbb{Z}[G/H^?](G/H) = \mathbb{Z}[N_G(H)/H] = \mathbb{Z}[\mathbb{Z}/2], \tag{2.4}$$

and

$$H_0(\mathbf{C})(G/K) = \big(\mathbb{Z}[G/K^?](G/K)\big)^2 = \big(\mathbb{Z}[N_G(K)/K]\big)^2 = \mathbb{Z} \oplus \mathbb{Z}. \tag{2.5}$$

Definition 2.6. A finite $\mathbb{Z}\varGamma$-chain complex $\mathbf{C}$ of finitely-generated free $\mathbb{Z}\varGamma$-modules, which satisfies the algebraic conditions (2.1)–(2.5), is called a *pseudofree $\mathbb{Z}\varGamma$-chain complex with the $\mathbb{Z}$-homology of S^{2k}.*

One example of such a complex arises from the standard orthogonal action $Y = S(V)$ of the dihedral group on S^2 (for G as a subgroup of $SO(3)$). The $SO(3)$-representation $V = W \oplus \mathbb{R}_-$ is the sum of the standard 2-dimensional real representation W (given by the action on a regular $2p$-gon in the plane), and the non-trivial 1-dimensional real representation $\mathbb{R}_-$. The chain complex $\mathbf{D} = \mathbf{C}(Y^{?})$ over the orbit category has the form

$$\begin{tikzcd}
\left(\mathbb{Z}[G/1^{?}]\right)^2 \arrow[r] \arrow[d, equal] & \left(\mathbb{Z}[G/1^{?}]\right)^3 \arrow[r] \arrow[d, equal] & \mathbb{Z}[G/H^{?}] \oplus \left(\mathbb{Z}[G/K^{?}]\right)^2 \arrow[d, equal] \\
\mathbf{D}_2 \arrow[r] & \mathbf{D}_1 \arrow[r] & \mathbf{D}_0
\end{tikzcd}$$

where $H_2(\mathbf{D}) = \underline{\mathbb{Z}}_0$ is the $\mathbb{Z}\varGamma$-module with value $\underline{\mathbb{Z}}_0(G/1) = \mathbb{Z}$, and zero otherwise. The module $H_0 := H_0(\mathbf{D})$ has value $H_0(G/1) = \mathbb{Z}$, and values at G/H and G/K as listed above. In general, for any pseudofree $\mathbb{Z}\varGamma$-chain complex $\mathbf{C}$ with the $\mathbb{Z}$-homology of S^{2k}, we have $H_{2k}(\mathbf{C}) = \underline{\mathbb{Z}}_0$ and $H_0(\mathbf{C}) = H_0(\mathbf{D})$.

Lemma 2.7. *Suppose that $\mathbf{C}$ is a pseudofree $\mathbb{Z}\varGamma$-chain complex with the $\mathbb{Z}$-homology of S^{2k}. Then the complex $\mathbf{C}$ is chain homotopy equivalent to a finite free $2k$-dimensional chain complex $\mathbf{C}'$, with $\mathbf{C}'_i = \mathbf{C}_i$ for $i \geq 4$, whose initial part $\mathbf{C}'_2 \to \mathbf{C}'_1 \to \mathbf{C}'_0$ is chain isomorphic to $\mathbf{D}$.*

Proof. Since $H_0(\mathbf{C}) = H_0(\mathbf{D})$, this follows from the version of Schanuel's Lemma over the orbit category given in the proof of [4, Lemma 8.12]. □

An immediate consequence is the statement of Theorem A for k even.

Corollary 2.8 (Edmonds [2])**.** *Let $G = D_p$. If k is even, there is no effective pseudofree G-action on a finite G-CW complex $X \simeq S^{2k}$, inducing the identity on homology.*

Proof. Let $\mathbf{C} = \mathbf{C}(X^{?})$ denote the chain complex over the orbit category of such an action. From the chain equivalent complex $\mathbf{C}' \simeq \mathbf{C}$ we can extract a periodic resolution

$$0 \to \underline{\mathbb{Z}}_0 \to \mathbf{C}_{2k} \to \mathbf{C}_{2k-1} \to \cdots \to \mathbf{C}_4 \to \mathbf{C}''_3 \to \underline{\mathbb{Z}}_0 \to 0$$

since $H_2(\mathbf{D}) = H_{2k}(\mathbf{C}) = \underline{\mathbb{Z}}_0$, where $\mathbf{C}''_3$ is a direct sum of copies of $\mathbb{Z}[G/1^{?}]$. By evaluating at $G/1$ we obtain a periodic projective resolution from $\mathbb{Z}$ to $\mathbb{Z}$ over $\mathbb{Z}G$ of length $(2k-2)$. Since $G = D_p$ has periodic cohomology of period 4 (and not two), we conclude that k is odd. □

The proof of Theorem A, k odd. Suppose, if possible, that we have a locally linear and orientation-preserving pseudofree topological action of G on S^{2k}, for some odd integer $k \geq 3$. Then there exists a finite G-CW complex $X \simeq S^{2k}$, and a chain homotopy equivalence $\mathbf{C}(X^{?}) \simeq \mathbf{C}'$ provided by Lemma 2.7. We may identify the singular set $\operatorname{Sing}(X)$ of X with the singular set of the given action on S^{2k}. Let $\{x_0, x_1, x_2\} \subset \operatorname{Sing}(X)$ denote representatives of the distinct G-orbits of singular points (with $G_{x_0} = H$, and $G_{x_i} = K$ for $i = 1, 2$). Around each singular point x_i, $0 \leq i \leq 2$, we can choose a linearly embedded 2-disk slice $G \times_{G_{x_i}} D^2 \subset S^{2k}$, since the action (S^{2k}, G) is locally linear. This gives a G-equivariant embedding

$$f_0 \colon \bigcup_{0 \leq i \leq 2} \left(G \times_{G_{x_i}} D^2\right) \subset S^{2k}.$$

Since the pseudofree orbit structure of the standard G-action on $S^2 = S(V)$ is the same for any locally linear action on S^{2k}, we can consider f_0 to be a G-equivariant embedding of a tubular neighbourhood of the singular set of $S(V)$ into S^{2k}. By obstruction theory, and since $k \geq 3$, we can extend this embedding f_0 to a G-equivariant embedding $f\colon S(V) \subset S^{2k}$. Non-equivariantly such an embedding of $S^2 \subset S^{2k}$ is isotopic to a standard embedding. We have thus obtained a dihedral action on S^{2k} of the type considered in my earlier joint work with Erik Pedersen [5], namely one conjugate to "a topological action on a sphere which is free off a standard proper subsphere, and given by a $S(V)$ on the subsphere". However, we proved in [5, Theorem 7.11] that such an action exists if and only if the representation V on the subsphere contains two $\mathbb{R}_-$ factors. Since this is not the case for the standard $SO(3)$-representation V of G, we conclude that a pseudofree G-action on S^{2k} does not exist for $k > 1$. $\square$

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