

Topological 4-manifolds with right-angled Artin fundamental groups

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We classify closed, spin^+ , topological 4-manifolds with fundamental group π of cohomological dimension ≤ 3 (up to s -cobordism), after stabilization by connected sum with at most $b_3(\pi)$ copies of $S^2 \times S^2$. In general, we must also assume that π satisfies certain K -theory and assembly map conditions. Examples for which these conditions hold include the torsion-free fundamental groups of 3-manifolds and all right-angled Artin groups whose defining graphs have no 4-cliques.

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1. Introduction

Freedman [13] classified simply connected, topological 4-manifolds up to homeomorphism, and established a framework for studying non-simply connected 4-manifolds.

For a non-simply connected 4-manifold M , the basic homotopy invariants are the fundamental group $\pi := \pi_1(M)$, the second homotopy group $\pi_2(M)$, the equivariant intersection form s_M , and the first k -invariant, $k_M \in H^3(\pi; \pi_2(M))$. These invariants give the *quadratic 2-type*

$$Q(M) := [\pi_1(M), \pi_2(M), k_M, s_M]$$

which has been shown to determine the classification up to s -cobordism of TOP 4-manifolds with geometrically 2-dimensional fundamental groups (see [20]). For manifolds with finite fundamental groups, it is likely that additional invariants are needed (see [18]).

Question. Are closed, oriented, spin 4-manifolds with isometric quadratic 2-types and torsion-free fundamental groups always s -cobordant?

In this paper, we study spin 4-manifolds with fundamental groups belonging to the interesting class of *right-angled Artin groups* (or RAAGs). We build on the methods of [20, 29], but new difficulties appear since our fundamental groups have cohomological dimension > 2 in general. Recall that a right-angled Artin group is defined by a presentation associated to a finite graph Γ (see Sec. 2). An i -clique in Γ is a complete subgraph of Γ with i vertices, and we let $b_i(\pi)$ denote the number of i -cliques in Γ .

Theorem A. *Let π be a right-angled Artin group defined by a graph Γ with no 4-cliques. Suppose that M and N are closed, spin^+ , topological 4-manifolds with fundamental group π . Then any isometry between the quadratic 2-types of M and N is stably realized by an s -cobordism between $M \# r(S^2 \times S^2)$ and $N \# r(S^2 \times S^2)$, whenever $r \geq b_3(\pi)$.*

A *spin⁺ manifold* is a spin manifold with an additional evenness property for its dual intersection form h_M on $\pi_2(M)^*$ (see Lemma 6.1(ii) and Definition 6.7). We will recall below the notion of (stable) *isometry* for quadratic 2-types.

Theorem A is an application of our main result, Theorem 11.2, which covers the large class of fundamental groups with cohomological dimension $\text{cd } \pi \leq 3$ satisfying certain assembly map conditions (see the properties (W-AA) given in Definition 11.1). For example, this class contains all torsion-free fundamental groups of 3-manifolds [1]. Theorem 11.2 extends the results of [20] which handled the class of geometrically 2-dimensional groups. If the fundamental group π happens to be a “good” group for topological surgery [14–16], then we show that any isometry can be realized by a homeomorphism.

For a right-angled Artin group π , these assembly conditions (W-AA) also hold [2], and π has cohomological dimension $\text{cd } \pi \leq 3$ if and only if the defining graph has no 4-cliques (or equivalently if $H_4(\pi; \mathbb{Z}) = 0$). Thus, Theorem A applies to an infinite number of right-angled Artin groups, including $\pi = \mathbb{Z}^3$ (see the examples following Proposition 3.6).

Here is a brief summary of the definitions in [18, 20].

Definition 1.1. For an oriented 4-manifold M , the *equivariant intersection form* is the triple $(\pi_1(M, x_0), \pi_2(M, x_0), s_M)$, where $x_0 \in M$ is a base point and

$$s_M : \pi_2(M, x_0) \otimes_{\mathbb{Z}} \pi_2(M, x_0) \rightarrow \Lambda,$$

where $\Lambda := \mathbb{Z}[\pi_1(M, x_0)]$. This pairing is derived from the cup product on $H_c^2(\widetilde{M}; \mathbb{Z})$, where \widetilde{M} is the universal cover of M ; we identify $H_c^2(\widetilde{M}; \mathbb{Z})$ with $\pi_2(M)$ via Poincaré duality and the Hurewicz Theorem, and so s_M is defined by

$$s_M(x, y) = \sum_{g \in \pi} \varepsilon_0(\tilde{x} \cup \tilde{y}g^{-1}) \cdot g \in \mathbb{Z}[\pi],$$

where $\tilde{x}, \tilde{y} \in H_c^2(\tilde{M}; \mathbb{Z})$ are the images of $x, y \in \pi_2(M)$ under the composite isomorphism $\pi_2(M) \rightarrow H_2(\tilde{M}; \mathbb{Z}) \rightarrow H_c^2(\tilde{M}; \mathbb{Z})$ and ε_0 is given by $\varepsilon_0 : H_c^4(\tilde{M}; \mathbb{Z}) \rightarrow H_0(\tilde{M}; \mathbb{Z}) = \mathbb{Z}$.

Unless otherwise mentioned, our modules are *right* Λ -modules. This pairing is Λ -hermitian, in the sense that for all $\lambda \in \Lambda$, we have

$$s_M(x, y \cdot \lambda) = s_M(x, y) \cdot \lambda \quad \text{and} \quad s_M(y, x) = \overline{s_M(x, y)},$$

where $\lambda \mapsto \bar{\lambda}$ is the involution on Λ given by the orientation character of M . This involution is determined by $\bar{g} = g^{-1}$ for $g \in \pi_1(M, x_0)$. For later reference, we note that when M is spin the term $\varepsilon_0(\tilde{x}, \tilde{y}) \equiv 0 \pmod{2}$, so s_M is an *even* hermitian form.

Definition 1.2. An *isometry* between quadratic 2-types $Q(M)$ and $Q(M')$ is a pair (α, β) , where $\alpha : \pi_1(M, x_0) \rightarrow \pi_1(M', x'_0)$ is an isomorphism of fundamental groups and

$$\beta : (\pi_2(M, x_0), s_M) \rightarrow (\pi_2(M', x'_0), s_{M'})$$

is an α -invariant isometry of the equivariant intersection forms, such that $(\alpha^*, \beta_*^{-1})(k_{M'}) = k_M$. In addition, the following diagram:

$$\begin{CD} 0 @>>> H^2(\pi; \Lambda) @>>> H^2(M'; \Lambda) @>e_{M'}>> \text{Hom}_\Lambda(H_2(M'; \Lambda), \Lambda) @>>> H^3(\pi; \Lambda) @>>> 0 \\ @. @| @V \cong \beta^* VV @V \cong \beta^* VV @| \\ 0 @>>> H^2(\pi; \Lambda) @>>> H^2(M; \Lambda) @>e_M>> \text{Hom}_\Lambda(H_2(M; \Lambda), \Lambda) @>>> H^3(\pi; \Lambda) @>>> 0 \end{CD}$$

arising from the universal coefficient spectral sequence *commutes*, with maps $e_M, e_{M'}$ induced by evaluation, and β after identifying $\pi := \pi_1(M, x_0) \cong \pi_1(M', x'_0)$ via α . We will assume throughout that our manifolds are connected, so that a change of base points leads to isometric intersection forms. By a *stable* isometry, we mean an isometry of quadratic 2-types after adding a hyperbolic form $H(\Lambda^r)$ to both sides.

Stabilization of quadratic 2-types corresponds geometrically to connected sums with finitely many copies of $S^2 \times S^2$. Two closed topological (or smooth) 4-manifolds M and N are said to be *stably homeomorphic* (or *stably diffeomorphic*) if

$$M \# r(S^2 \times S^2) \approx N \# s(S^2 \times S^2)$$

are homeomorphic (or diffeomorphic) after stabilization, for some $r, s \geq 0$. Manifolds which are s -cobordant are stably homeomorphic, but it turns out that the k -invariant is not needed for the stable classification.

Our main result implies the following (compare [20, Lemma 4.5; 24, Theorem 1.5]):

Theorem B. *Let π be a finitely presented group with $\text{cd } \pi \leq 3$ satisfying the assembly property (W-AA)(iii). If M and N are closed, oriented, spin^+ , topological (respectively, smooth) 4-manifolds with fundamental group π , then M is stably*

homeomorphic (respectively, diffeomorphic) to N if and only if their equivariant intersection forms s_M and s_N are stably isometric.

Remark 1.3. If π is the fundamental group of a closed, oriented aspherical 3-manifold, then Theorem B was proved by Kasprowski, Land, Powell and Teichner [24, Theorem 1.5], as a special case of their stable classification of closed topological 4-manifolds with these fundamental groups. In this setting, the authors also gave explicit invariants to completely describe the stable classification for every w_2 -type.

Here is a short outline of the paper. In Sec. 2, we recall some background on the cohomology of right-angled Artin groups. In Sec. 3, we discuss the notion of *tame cohomology* for discrete groups, which is motivated by studying 4-manifolds and may have some independent interest (see Definition 3.1). Section 4 establishes the key cohomological properties we need for groups with $\text{cd } \pi \leq 3$, and Sec. 5 shows that the Λ -dual $\pi_2(M)^* := \text{Hom}_\Lambda(\pi_2(M), \Lambda)$ is a stably free Λ -module, for every closed spin 4-manifold M with $\text{cd } \pi_1(M) \leq 3$. These results are applied in Sec. 6 to define the *reduced equivariant intersection form* on $\pi_2(M)^*$, and prove that it is non-singular. Examples are provided in Corollary 6.10 of spin^+ manifolds for every such fundamental group.

Our strategy for proving Theorem A uses Kreck’s *modified surgery* [29] (see [20, Sec. 2] for an overview of this theory). In this approach, one studies the bordism groups $\Omega_4(B)$ for a fibration $B \rightarrow B\text{STOP}$. If two manifolds M and N admit suitable B -bordant reference maps, then M and N will be stably homeomorphic after connected sum with finitely many copies of $S^2 \times S^2$. Moreover, M and N will be s -cobordant provided a certain assembly map obstruction vanishes.

In Sec. 7, we define an appropriate fibration $B := B(M) = P \times B\text{TOPSPIN}$ for our setting, where P is the reduced 2-type of M (see Definition 7.1). We show that two manifolds M and N with isometric quadratic 2-types admit reference maps into B . In Secs. 8–10, we compute the bordism groups $\Omega_4(B) = \Omega_4^{\text{Spin}}(P)$ and give explicit criteria for deciding when two spin^+ manifolds M and N are B -bordant. Finally, in Sec. 11, we put these ingredients together to prove our main result, Theorem 11.2, and its application to Theorems A and B. The “extra” stabilization needed in our setting to obtain an s -cobordism classification arises from the fact that our reference maps are not 2-connected unless $H^3(\pi; \Lambda) = 0$. When this cohomology group is nonzero, we must stabilize by $b_3(\pi)$ copies of $S^2 \times S^2$ to complete the argument.

Some background material on homological algebra and the universal coefficient spectral sequence is provided for the reader’s convenience in Appendix A.

Remark 1.4. We restrict to spin^+ 4-manifolds for simplicity, but expect that analogous results hold by including the full w_2 -type in the data (see [20]). To shorten notation in later sections, we will let $\Omega_*^{\text{Spin}}(P)$ denote either the *smooth* or the *topological* spin bordism group of a space P , depending on the context. Note

that the Kirby–Siebenmann invariant [26] is determined by $\text{sign}(M) \pmod{16}$ for spin manifolds.

2. Cohomology of Right-Angled Artin Groups

A *right-angled Artin group* π is a finitely generated group whose relators consist solely of commutators between generators. Right-angled Artin groups are also called *graph groups* since each generator of π can be represented by a vertex of a graph $\Gamma = \Gamma(\pi)$, and pairs of commuting generators in π are represented by edges in Γ between the corresponding vertices.

If π has a presentation with g generators and r relators, we construct a handlebody with fundamental group π using one 0-handle, g 1-handles, and r 2-handles attached to reflect the relations of π . In the case that π is a right-angled Artin group, this handlebody is homotopy equivalent to the 2-skeleton K of a standard classifying space for π , known as the *Salvetti complex* (see Charney [7, Sec. 3.6]).

The integral homology and cohomology ring of π are calculated in [8, 25]. In [8], it is shown that the i th homology group and i th cohomology group are both isomorphic to the i th group of cellular chains of the Salvetti complex, and thus are free abelian groups. In fact, the rank $b_i(\pi)$ of $H_i(\pi; \mathbb{Z})$ is equal to the number of i -cliques (complete subgraphs on i vertices) in the defining graph Γ for π . Thus, the (co)homological dimension $\text{cd } \pi$ of a right-angled Artin group π equals the maximum number of i -cliques in Γ .

In [23], Jensen and Meier calculate the cohomology of right-angled Artin groups with *group ring coefficients* using a simplicial complex $\hat{\Gamma}$ induced from the defining graph Γ of π . In [10], Davis and Okun give a different formulation of the same theorem. We first give some necessary definitions before the statement of their results.

Let Γ be a simplicial graph. Then the *flag complex* $\hat{\Gamma}$ generated by Γ is the minimal simplicial complex in which every complete subgraph in Γ spans a simplex. In other words, the $(i - 1)$ -simplices of $\hat{\Gamma}$ are the i -cliques of Γ . The *link* $\text{Lk}(\sigma)$ of a simplex σ in $\hat{\Gamma}$ is the collection of simplices $\tau \in \hat{\Gamma}$ disjoint from σ , such that σ and τ are sub-complexes of a higher dimensional simplex in $\hat{\Gamma}$. By definition, the link of the empty simplex is the entire flag complex $\hat{\Gamma}$ and by convention, $\text{dim } \emptyset = -1$.

Definition 2.1. For a simplex $\sigma \in \hat{\Gamma}$, we define the subgroup $\pi_\sigma \leq \pi$ to be the right-angled Artin group generated by the subgraph of Γ spanned by the vertices of σ . By convention, $\pi_\emptyset = 1$.

In general, an induced subgraph of Γ defines a subgroup of π which is also a right-angled Artin group, but in this case, since simplices in $\hat{\Gamma}$ are in bijection with complete subgraphs in Γ , we see that π_σ is a free abelian group of rank equal to $\text{dim } \sigma + 1$.

Theorem 2.2 (Jensen–Meier [23], Davis–Okun [10, Theorem 3.3]). *Let Γ be the defining graph for a right-angled Artin group π , and let \mathcal{S} be the set of*

simplices in the induced flag complex $\hat{\Gamma}$. There is a spectral sequence converging to $H^*(\pi; \Lambda)$ whose associated graded groups are given by

$$\text{Gr } H^*(\pi; \Lambda) = \bigoplus_{\sigma \in \mathcal{S}} \left(\tilde{H}^{*-\dim \sigma-2}(\text{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma] \right)$$

when $\hat{\Gamma}$ is not a single simplex. If $\hat{\Gamma}$ is a single simplex, then $\pi \cong \mathbb{Z}^n$, and $H^n(\pi; \Lambda) = \mathbb{Z}$ is the only non-vanishing cohomology group.

Example 2.3. We describe the associated graded group for $H^2(\pi; \Lambda)$ using Theorem 2.2. In the corresponding filtration, the empty simplex is the bottom of the filtration, so we have the following filtration subgroups (indexed so that the top index matches the cohomology degree in question):

$$\begin{array}{ccccccc} 0 & \hookrightarrow & \mathcal{F}_0 & \hookrightarrow & \mathcal{F}_1 & \hookrightarrow & \mathcal{F}_2 = H^2(\pi; \Lambda) \\ & & \parallel & & \downarrow & & \downarrow \\ & & \mathcal{F}_0 & & \mathcal{F}_1/\mathcal{F}_0 & & \mathcal{F}_2/\mathcal{F}_1 \end{array}$$

Note that $\tilde{H}^{-1}(\emptyset) = \mathbb{Z}$ is the only nontrivial cohomology group of the empty simplex. The filtration quotients are given by

- (1) $\mathcal{F}_0 \cong H^1(\hat{\Gamma}) \otimes \mathbb{Z}[\pi]$,
- (2) $\mathcal{F}_1/\mathcal{F}_0 \cong \bigoplus_{\sigma \in \text{Vert}(\hat{\Gamma})} (\tilde{H}^0(\text{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma])$, and
- (3) $\mathcal{F}_2/\mathcal{F}_1 \cong \bigoplus_{\sigma \in \text{Edge}(\hat{\Gamma})} (\tilde{H}^{-1}(\text{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma])$.

The Λ -action on the right-hand side is given by the identity on the integral cohomology of the links, tensored over \mathbb{Z} with the natural action on the induced modules $\mathbb{Z}[\pi/\pi_\sigma]$.

3. Right-Angled Artin Groups With Tame Cohomology

In this section, we investigate certain group cohomological conditions arising in the study of 4-manifolds. Recall that finitely presented groups have either 1, 2, or infinitely many ends, where the number of ends of a group G is equal to the number of ends of any Cayley graph of G . In [33], Stallings proved that a finitely presented group G has more than one end if and only if it decomposes as a nontrivial amalgamated product or an HNN extension over a finite subgroup.

Since a right-angled Artin group π is infinite and torsion-free, π will have more than one end if and only if it decomposes as a free product (see Dunwoody [12]). Thus, right-angled Artin groups are 1-ended, or equivalently $H^1(\pi; \Lambda) = 0$, if and only if $\pi \neq \mathbb{Z}$ and the defining graphs are connected (see [7, Sec. 3.7]). Note that the group cohomology $H^i(\pi; \Lambda)$ of any group π with group ring coefficients is also a Λ -module since Λ is a Λ - Λ bimodule. We say that A is a *torsion* Λ -module whenever $\text{Hom}_\Lambda(A, \Lambda) = 0$.

Definition 3.1. A finitely presented group π has *tame cohomology* if the following conditions hold:

- (i) $\text{Hom}_\Lambda(H^2(\pi; \Lambda), \Lambda) = 0,$
- (ii) $\text{Hom}_\Lambda(H^3(\pi; \Lambda), \Lambda) = 0,$
- (iii) $\text{Ext}_\Lambda^1(H^3(\pi; \Lambda), \Lambda) = 0.$

In Sec. 6, we show that when the fundamental group $\pi_1(M)$ of a closed 4-manifold M has tame cohomology, then the dual of its equivariant intersection form s_M is non-singular. We expect this property to have a key role in extending our classification results to right-angled Artin groups of higher cohomological dimension.

In the special case, when $\text{cd } \pi_1(M) \leq 3,$ our classification results do not assume tame cohomology: we will establish conditions (ii) and (iii) in Proposition 4.1, and prove the non-singularity of this form in Lemma 6.1, without assuming condition (i).

Remark 3.2. In the first version of this paper, we asked whether all right-angled Artin groups have tame cohomology. In response, Jonathan Hillman kindly provided the following example: $H^2(\pi; \Lambda) \cong \Lambda,$ for $\pi = (\mathbb{Z}^2 * \mathbb{Z}^2)^2.$ However, we do not know the answer for right-angled Artin groups with no 4-cliques in the associated graph.

We now apply the results of Jensen–Meier and Davis–Okun cited in Theorem 2.2 to study the low-dimensional cohomology of right-angled Artin groups. For any Λ -module $A,$ we use the notation $A^* = \text{Hom}_\Lambda(A, \Lambda)$ for the *dual module*. We begin with the following observation. Let $\pi_{n,m}$ denote the free product of n copies of \mathbb{Z}^2 and m copies of $\mathbb{Z}.$

Lemma 3.3. *If π is a right-angled Artin group, then there exists a subgroup $\pi_{n,m} \leq \pi$ such that $H^1(\pi; \Lambda) \cong H^1(\pi_{n,m}; \text{Res } \Lambda)$ under restriction. If π is a finitely generated free group, then $H^1(\pi; \Lambda)^* = 0.$*

Proof. If the defining graph $\Gamma = \Gamma(\pi)$ is connected, then $H^1(\pi; \Lambda) = 0$ and there is nothing to prove (take the empty-free product). Otherwise, we can write Γ as a disjoint union of non-empty connected graphs, and let $\Gamma(n, m) \subset \Gamma$ denote a subgraph consisting of all the m singleton vertex components together with one edge from each of the other n connected components. The right-angled Artin group defined by $\Gamma(n, m)$ is isomorphic to $\pi_{n,m}.$ Since the filtration terms for computing $H^1(\pi; \Lambda)$ consist of $\mathcal{F}_0 \cong \tilde{H}^0(\hat{\Gamma}) \otimes \mathbb{Z}[\pi]$ and

$$\mathcal{F}_1/\mathcal{F}_0 \cong \bigoplus_{\sigma \in \text{Vert}(\hat{\Gamma})} (\tilde{H}^{-1}(\text{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma]),$$

it follows that $H^1(\pi; \Lambda)$ is mapped isomorphically under restriction to $H^1(\pi_{n,m}; \text{Res } \Lambda).$

Suppose now that $n = 0$ so that π is a free group of rank $m.$ Let N be the connected sum of m copies of $S^1 \times S^3.$ An easy argument using the universal coefficient spectral sequence now shows that $H^1(\pi; \Lambda)^* = 0.$ □

The next result gives an inductive criterion for the cohomology groups to be torsion Λ -modules.

Lemma 3.4. *If $H^i(\hat{\Gamma}) = 0$ then $\text{Hom}_\Lambda(H^{i+1}(\pi; \Lambda), \Lambda) = 0$.*

Proof. The case $i = 0$ is trivial, so consider the case when $i = 1$. The filtrations for $H^2(\pi; \Lambda)$ are given in Example 2.3, with $\mathcal{F}_0 = 0$ by the assumption that $H^1(\hat{\Gamma}) = 0$. Consequently, $\mathcal{F}_1 = \mathcal{F}_1/\mathcal{F}_0$. By dualizing the short exact sequence

$$0 \rightarrow \mathcal{F}_1/\mathcal{F}_0 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_2/\mathcal{F}_1 \rightarrow 0,$$

we get the exact sequence

$$0 \rightarrow (\mathcal{F}_2/\mathcal{F}_1)^* \rightarrow (\mathcal{F}_2)^* \rightarrow (\mathcal{F}_1/\mathcal{F}_0)^* \rightarrow \text{Ext}_\Lambda^1(\mathcal{F}_2/\mathcal{F}_1, \Lambda) \rightarrow \dots$$

We claim $(\mathcal{F}_1/\mathcal{F}_0)^* = 0$ and $(\mathcal{F}_2/\mathcal{F}_1)^* = 0$: Hom splits over a finite direct sum, and any nonzero summands in $\mathcal{F}_1/\mathcal{F}_0$ or $\mathcal{F}_2/\mathcal{F}_1$ will have torsion elements in the tensor product, and are killed by the Hom functor. For example,

$$\begin{aligned} (\mathcal{F}_1/\mathcal{F}_0)^* &= \text{Hom}_\Lambda \left(\bigoplus_{\sigma \in \text{Vert}(\hat{\Gamma})} \tilde{H}^0(\text{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma], \Lambda \right) \\ &= \bigoplus_{\sigma \in \text{Vert}(\hat{\Gamma})} \text{Hom}_\Lambda(\tilde{H}^0(\text{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma], \Lambda), \end{aligned}$$

where the direct sum is taken over all vertices σ in $\hat{\Gamma}$. By [4, Corollary 2.8.4] (the Eckmann–Shapiro Lemma),

$$\text{Hom}_{\mathbb{Z}[\pi]}(\mathbb{Z}[\pi/\pi_\sigma], \mathbb{Z}[\pi]) \cong \text{Hom}_{\mathbb{Z}[\pi_\sigma]}(\mathbb{Z}, \text{Res}_{\pi_\sigma}^\pi(\mathbb{Z}[\pi])) \tag{3.1}$$

is zero as long as $\pi_\sigma \neq 1$, or equivalently, σ is not the empty simplex. Thus from the long exact sequence, we see $(\mathcal{F}_2)^* = H^2(\pi; \Lambda)^* = 0$.

For any i , the assumption that $H^i(\hat{\Gamma}) = 0$ implies that the bottom filtration term $\mathcal{F}_0 = 0$. By the argument used in (3.1), the subquotients $\mathcal{F}_k/\mathcal{F}_{k-1}$, $1 \leq k \leq i + 1$, in the filtration for $H^{i+1}(\pi; \Lambda)$ are all torsion modules.

$$\begin{array}{ccccccc} 0 = \mathcal{F}_0 & \hookrightarrow & \mathcal{F}_1 & \hookrightarrow & \dots & \hookrightarrow & \mathcal{F}_{i-1} & \hookrightarrow & \mathcal{F}_i & \hookrightarrow & \mathcal{F}_{i+1} = H^{i+1}(\pi; \Lambda) \\ & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ & & \mathcal{F}_1/\mathcal{F}_0 & & & & \mathcal{F}_{i-1}/\mathcal{F}_{i-2} & & \mathcal{F}_i/\mathcal{F}_{i-1} & & \mathcal{F}_{i+1}/\mathcal{F}_i \end{array}$$

By dualizing the short exact sequences

$$0 \rightarrow \mathcal{F}_{k-1} \rightarrow \mathcal{F}_k \rightarrow \mathcal{F}_k/\mathcal{F}_{k-1} \rightarrow 0,$$

we see that $(\mathcal{F}_{i+1})^* = H^{i+1}(\pi; \Lambda)^* = 0$. □

Lemma 3.5. *Let π be a right-angled Artin group with defining graph Γ . Suppose that the induced flag complex $\hat{\Gamma}$ is 2-connected and that $H^1(\text{Lk}(\sigma))$ is zero for every vertex σ . Then π has tame cohomology.*

Proof. Since $\hat{\Gamma}$ is 2-connected, $H^i(\hat{\Gamma}) = 0$ for $i = 1, 2$. By Lemma 3.4, $H^{i+1}(\pi; \Lambda)^* = 0$ for $i = 2, 3$. Thus, the first two conditions for π to have tame cohomology are satisfied. For the last condition, we claim that $\text{Ext}_{\Lambda}^1(H^3(\pi; \Lambda), \Lambda) = 0$. The filtration quotients are described below (with $\mathcal{F}_3 = H^3(\pi; \Lambda)$):

- (i) $\mathcal{F}_0 = H^2(\hat{\Gamma}) \otimes \mathbb{Z}[\pi]$,
- (ii) $\mathcal{F}_1/\mathcal{F}_0$ is a direct sum of $H^1(\text{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma]$ over vertices,
- (iii) $\mathcal{F}_2/\mathcal{F}_1$ is a direct sum of $\tilde{H}^0(\text{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma]$ over edges, and
- (iv) $\mathcal{F}_3/\mathcal{F}_2$ is a direct sum of $\tilde{H}^{-1}(\text{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma]$ over faces.

Consider the dual of the short exact sequence $0 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow \mathcal{F}_3/\mathcal{F}_2 \rightarrow 0$, recalling that $(\mathcal{F}_2)^* = 0$:

$$0 \rightarrow \text{Ext}_{\Lambda}^1(\mathcal{F}_3/\mathcal{F}_2, \Lambda) \rightarrow \text{Ext}_{\Lambda}^1(\mathcal{F}_3, \Lambda) \rightarrow \text{Ext}_{\Lambda}^1(\mathcal{F}_2, \Lambda) \rightarrow \dots$$

Since $H^1(\text{Lk}(\sigma)) = 0$ for every vertex σ , $\mathcal{F}_1/\mathcal{F}_0 = 0$. This implies that $\mathcal{F}_1 = 0$ and so $\mathcal{F}_2 \cong \mathcal{F}_2/\mathcal{F}_1$. The functor Ext_{Λ}^1 splits over a finite direct sum, so the first and third nonzero terms in the above sequence are of the form

$$\text{Ext}_{\Lambda}^1(\mathcal{F}_{i+1}/\mathcal{F}_i, \Lambda) = \bigoplus \tilde{H}^{1-i}(\text{Lk}(\sigma)) \otimes \text{Ext}_{\Lambda}^1(\mathbb{Z}[\pi/\pi_\sigma], \Lambda),$$

and we claim that the Ext term on the right-hand side is zero, hence $\text{Ext}_{\Lambda}^1(\mathcal{F}_3, \Lambda) = 0$. The simplices in question are either faces (involved in the first term of the sequence) or edges (involved in the third term). By the Eckmann–Shapiro Lemma,

$$\text{Ext}_{\mathbb{Z}[\pi]}^1(\mathbb{Z}[\pi/\pi_\sigma], \mathbb{Z}[\pi]) \cong \text{Ext}_{\mathbb{Z}[\pi_\sigma]}^1(\mathbb{Z}, \text{Res}_{\pi_\sigma}^{\pi} \mathbb{Z}[\pi]).$$

The restriction of $\mathbb{Z}[\pi]$ to π_σ is just an infinite direct sum of $\mathbb{Z}[\pi_\sigma]$'s, and

$$\text{Ext}_{\mathbb{Z}[\pi_\sigma]}^1(\mathbb{Z}, \mathbb{Z}[\pi_\sigma]) \cong H^1(\pi_\sigma; \mathbb{Z}[\pi_\sigma]) = 0$$

when σ is an edge or a face, since π_σ is a 1-ended group ($\pi_\sigma = \mathbb{Z}^2$ and \mathbb{Z}^3). □

The condition that π has tame cohomology is perhaps restrictive (see Remark 3.2) but we have many examples, including an algorithm that can produce infinitely many right-angled Artin groups with tame cohomology.

Proposition 3.6. *Let Γ be the simplicial graph obtained by carrying out finitely many of the following operations, starting with a single vertex:*

- (i) *Attach a new 1-simplex or a new 2-simplex to the previous graph Γ_0 by identifying one of its vertices with any (but only one) vertex of Γ_0 .*
- (ii) *Attach a new 2-simplex to the previous graph Γ_0 by identifying one of its 1-simplices with any (but only one) 1-simplex of Γ_0 .*

Then the right-angled Artin group defined by Γ will have tame cohomology.

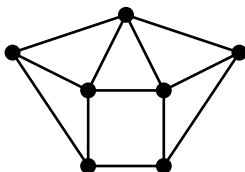
Proof. At each step, we add a new simplex to the previously constructed graph Γ_0 , which can always be contracted to Γ_0 in its flag complex. The algorithm provided

by repeating steps (i) and (ii) produces a simplicial graph Γ with contractible flag complex $\hat{\Gamma}$, which is clearly 2-connected.

Furthermore, this algorithm guarantees that $H^1(\text{Lk}(\sigma)) = 0$ for the link of every vertex σ . We may then apply Lemma 3.5 to conclude that the right-angled Artin group defined by Γ has tame cohomology. \square

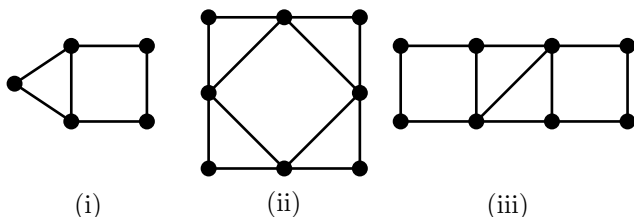
In the setting of Theorem A, we restrict attention to right-angled Artin groups with no 4-cliques in their defining graphs. It is possible that all such right-angled Artin groups have tame cohomology, but we do not yet know if the first condition is always satisfied.

Example 3.7. Does the right-angled Artin group π with the following defining graph have $H^2(\pi; \Lambda)^* = 0$?



The following examples provide graphs with non-simply connected flag complexes whose associated right-angled Artin groups have $H^2(\pi; \Lambda)^* = 0$. This shows at least that Lemma 3.5 is not the best possible result.

Example 3.8. The right-angled Artin groups defined by the following graphs all have $H^2(\pi; \Lambda)^* = 0$.



The argument involves dualizing the short exact sequences derived from the Mayer-Vietoris sequence for π expressed as the amalgamated product of two subgroups. We denote by π_i the right-angled Artin group associated to a graph Γ_i . We first remark that if Γ is the union of two subgraphs Γ_1 and Γ_2 which intersect at Γ_0 , then π is the amalgamated product of π_1 and π_2 over π_0 : this can be seen by comparing the presentations for π_1 and π_2 with the relations of π_0 . Secondly, if Γ_i is the induced subgraph on a collection of vertices from Γ , the groups π_i are subgroups of π .

Proposition 3.9. *If $\Gamma = \Gamma(\pi)$ satisfies all of the following conditions, then $H^2(\pi; \Lambda)^* = 0$.*

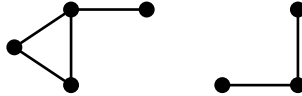
- (i) Γ is the union of two graphs Γ_1 and Γ_2 which intersect in Γ_0 , and all three subgraphs induce subgroups of π .
- (ii) Γ_1 and Γ_2 are connected and have 2-connected flag complexes.
- (iii) Γ_0 is a disjoint union of vertices.

Proof. Since Γ_1 and Γ_2 are connected and Γ_0 is 0-dimensional, we get the short exact sequence

$$0 \rightarrow H^1(\pi_0; \Lambda) \rightarrow H^2(\pi; \Lambda) \rightarrow H^2(\pi_1; \Lambda) \oplus H^2(\pi_2; \Lambda) \rightarrow 0$$

from Mayer–Vietoris. Since $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are 2-connected, $H^2(\pi_1; \Lambda)^*$ and $H^2(\pi_2; \Lambda)^*$ are both zero by Lemma 3.4. Additionally, $H^1(\pi_0; \Lambda)^* = 0$ by Lemma 3.3. Thus, dualizing the short exact sequence above gives the desired result. \square

We can apply this proposition to the graph Γ in Example 3.8(i) with the following two subgraphs below, to be denoted by Γ_1 and Γ_2 , respectively:



Together, their union $\Gamma_1 \cup \Gamma_2$ is Γ and their intersection Γ_0 is two disjoint vertices. Both Γ_1 and Γ_2 are connected and have 2-connected flag complexes. Thus, Proposition 3.9 shows the graph in Example 3.8(i) satisfies the first condition for tame cohomology.

4. Finitely Presented Groups With $\text{cd } \pi \leq 3$

In this section, we will show that any finitely presented group π with $\text{cd } \pi \leq 3$ satisfies the conditions (ii) and (iii) for tame cohomology. In particular, these conditions hold for a right-angled Artin group π with no 4-cliques in its defining graph Γ . The arguments in this section use the homological algebra definition of the Ext-functors via multiple extensions (see MacLane [31, Chap. III, Secs. 5 and 6]).

A particularly useful tool is the hypercohomology spectral sequence defined in Benson [5, Proposition 3.4.3]. If \mathbf{C} and \mathbf{D} are chain complexes over a ring Λ , with \mathbf{C} bounded above and \mathbf{D} bounded below, then there is a spectral sequence

$$\text{Ext}_\Lambda^p(H_q(\mathbf{C}), \mathbf{D}) \Rightarrow \text{Ext}_\Lambda^{p+q}(\mathbf{C}, \mathbf{D})$$

converging to the hypercohomology groups (see Benson and Carlson [4, Sec. 2.7] for the definition of Ext for chain complexes). This is a first quadrant spectral sequence, so the indexing convention above is standard (p runs along the x -axis). The differential d_r has bi-degree $(r, -r + 1)$.

A summary of the information we need about this spectral sequence is provided for the reader’s convenience in Appendix A.

Proposition 4.1. *Let π be a finitely presented group with $\text{cd } \pi \leq 3$. Then $H^3(\pi; \Lambda)^* = 0$ and $\text{Ext}_\Lambda^1(H^3(\pi; \Lambda), \Lambda) = 0$.*

Proof. By Wall [34, Theorem E], a finitely presented group π with $\text{cd } \pi \leq 3$ has geometric dimension ≤ 3 . In other words, there exists a 3-dimensional, finite, aspherical complex L with $\pi_1(L) = \pi$. Let $Y = Y(L)$ be an 8-dimensional thickening of L , and let Z be the double of Y . The long exact sequence of the pair $(Y, \partial Y)$ with Λ -coefficients gives isomorphisms $H_{i-1}(\partial Y) \rightarrow H_{i-1}(Y)$ for $i \leq 4$, since $H_i(Y, \partial Y) \cong H^{8-i}(Y) \cong H^{8-i}(L)$. Furthermore, $H_{i-1}(Y) = 0$ for $i > 1$. Then by the Mayer–Vietoris sequence, $H_i(Z; \Lambda) = 0$ for $1 < i \leq 4$, and $H_1(Z; \Lambda) = 0$ since \tilde{Z} is simply connected.

We use the *universal coefficient spectral sequence* which has E_2 -page

$$E_2^{p,q} = \text{Ext}_\Lambda^p(H_q(Z; \Lambda), \Lambda).$$

This is a special case of the hypercohomology spectral sequence, where $\mathbf{C} = C(Z; \Lambda)$ and $\mathbf{D} = \Lambda$ (a 0-dim chain complex with Λ in degree zero).

Since, $H_q(Z; \Lambda) = 0$ for $q = 1, \dots, 4$, the $E_2^{p,q}$ -terms are all zero for $1 \leq q \leq 4$, and since $H^p(\pi; \Lambda) = 0$ for all $p > 3$, the $E_2^{p,q}$ -terms are zero for $p > 3$. Thus, no differentials affect the $E_2^{0,5}$ - and $E_2^{1,5}$ -terms, which are $E_2^{0,5} \cong H^3(\pi; \Lambda)^*$ and $E_2^{1,5} \cong \text{Ext}_\Lambda^1(H^3(\pi; \Lambda), \Lambda)$ via the isomorphisms $H_5(Z; \Lambda) \cong H^3(Z; \Lambda) \cong H^3(\pi; \Lambda)$. Since $H^5(Z; \Lambda) \cong H_3(Z; \Lambda) = 0$ and $H^6(Z; \Lambda) \cong H_2(Z; \Lambda) = 0$, these two terms must be zero. □

In the universal coefficient spectral sequence computing $H^*(Z; \Lambda)$, since $H^7(Z; \Lambda) = H_1(Z; \Lambda) = 0$, the differential

$$d_2 : E_2^{0,6} = H^2(\pi; \Lambda)^* \xrightarrow{\cong} E_2^{2,5} = \text{Ext}_\Lambda^2(H^3(\pi; \Lambda), \Lambda) \tag{4.1}$$

is an isomorphism. In addition, $\text{Ext}_\Lambda^1(H^2(\pi; \Lambda))$ injects into $\text{Ext}_\Lambda^3(H^3(\pi; \Lambda), \Lambda)$.

From the universal coefficient spectral sequence for the 2-skeleton $K \subset L$, we get the exact sequence:

$$0 \rightarrow H^2(\pi; \Lambda) \rightarrow H^2(K; \Lambda) \rightarrow \pi_2(K)^* \rightarrow H^3(\pi; \Lambda) \rightarrow 0, \tag{4.2}$$

where the last nonzero map is a surjection since $H^3(K; \Lambda) = 0$. This sequence can be analyzed by splicing together the short exact sequences:

$$0 \rightarrow H^2(\pi; \Lambda) \rightarrow H^2(K; \Lambda) \rightarrow V \rightarrow 0 \quad \text{and} \quad 0 \rightarrow V \rightarrow \pi_2(K)^* \rightarrow H^3(\pi; \Lambda) \rightarrow 0,$$

where

$$V := \text{coker}(H^2(\pi; \Lambda) \rightarrow H^2(K; \Lambda)) = \ker(\pi_2(K)^* \rightarrow H^3(\pi; \Lambda)).$$

After dualizing, the associated long exact cohomology sequences give two connecting homomorphisms $\delta_1 : H^2(\pi; \Lambda)^* \rightarrow \text{Ext}_\Lambda^1(V, \Lambda)$ and $\delta_2 : \text{Ext}_\Lambda^1(V, \Lambda) \rightarrow \text{Ext}_\Lambda^2(H^3(\pi; \Lambda), \Lambda)$.

Lemma 4.2. *Let π be a finitely presented group with $\text{cd } \pi \leq 3$. The differential (4.1) induces a natural isomorphism $d_2 : H^2(\pi; \Lambda)^* \cong \text{Ext}_\Lambda^2(H^3(\pi; \Lambda), \Lambda)$, where $d_2 = \delta_2 \circ \delta_1$.*

Proof. We refer to Appendix A for the d_2 differential formula (A.1) given by splicing, and apply it to the manifold $Z = Y(L) \cup Y(L)$ defined above. Note that $\mathbf{C} = C_*(Z; \Lambda) \simeq C^{8-*}(Z; \Lambda)$ by Poincaré duality. We claim that the sequence (A.1) for $p = 0$ and $q = 6$, and expressed in terms of the cochain complex of Z , is exactly the sequence (4.2).

Here are some additional details to explain this identification. Let $\mathbf{D} := C^{8-*}(Z; \Lambda)$ denote the cochain complex, and compute the same sequence (A.1) for the complex \mathbf{D} (with $q = 6$). Then $H_6(\mathbf{D}) = H^2(Z; \Lambda)$ and $H_5(\mathbf{D}) = H^3(Z; \Lambda)$. It is not hard to see (by definition of cohomology for Z) that the term $D_6/B_7 \cong H^2(K; \Lambda)$. In fact, the sequence (A.1) in this case is just

$$0 \rightarrow H^2(Z; \Lambda) \rightarrow H^2(K; \Lambda) \rightarrow H^3(Z, K; \Lambda) \rightarrow H^3(Z; \Lambda) \rightarrow 0,$$

which is isomorphic to the sequence (4.2) since $H^3(Z, K; \Lambda) \cong \pi_2(K)^*$, by an application of the universal coefficient spectral sequence to the pair (Z, K) (the only term on the 3 line is at $(0,3)$).

Since $C \simeq D$, the corresponding d_2 in the spectral sequence for computing $H^*(D)$ is given by splicing with (4.2) as claimed. Finally, note that the formula $d_2 = \delta_2 \circ \delta_1$ for $d_2 : H^2(\pi; \Lambda)^* \cong \text{Ext}_\Lambda^2(H^3(\pi; \Lambda), \Lambda)$ is obtained by composing the connecting maps δ_1 and δ_2 obtained from two 3-term exact sequences which splice together to give the 4-term exact sequence (4.2). □

Remark 4.3. These results, which apply to all right-angled Artin groups with $H_4(\pi; \mathbb{Z}) = 0$, are sharper in this special case than our conclusions about tame cohomology from Sec. 3. For example, we no longer require the associated flag complex $\hat{\Gamma}$ to be simply connected.

We are indebted to one of the referees for pointing out that Proposition 4.1 is a special case of a more general cohomological result with an analogous proof. Here is the statement together with an alternate proof.

Proposition 4.4. *Let π be a finitely presented group with $\text{cd } \pi \leq n$. Then $H^n(\pi; \Lambda)^* = 0$ and $\text{Ext}_\Lambda^1(H^n(\pi; \Lambda), \Lambda) = 0$.*

Proof. We begin with the hypercohomology spectral sequence [5, Proposition 3.4.3], which has E_2 -page

$$E_2^{p,q} = \text{Ext}_\Lambda^p(H^{-q}(\pi; \Lambda), \Lambda).$$

This is a fourth quadrant spectral sequence which converges to $H_{p+q}(\pi; \Lambda)$. For all group homology with group ring coefficients, $H_{p+q}(\pi; \Lambda) = 0$ if $p + q \neq 0$ and \mathbb{Z} if $p + q = 0$, and so all entries not lying on the line $p + q = 0$ will die out in the spectral sequence. However, since π has $\text{cd } \pi \leq n$, the terms $E_2^{p,q} = 0$ when $q \leq -(n + 1)$. There are no differentials that affect $E_2^{0,-n} = H^n(\pi; \Lambda)^*$ and $E_2^{1,-n} = \text{Ext}_\Lambda^1(H^n(\pi; \Lambda), \Lambda)$, and so both must be zero. □

Remark 4.5. The hypercohomology spectral sequence above, combined with Lemma 3.3, shows that $H^1(\pi; \Lambda)^* \neq 0$ for a right-angled Artin group π if and only if its defining graph is disconnected and contains at least one edge. The simplest example is $\pi = \mathbb{Z}^2 * \mathbb{Z}$. For the groups $\pi = \pi_{n,m}$, we let Z denote the double of an 8-dimensional thickening of the Salvetti complex. Since $H^8(Z; \Lambda) = \mathbb{Z}$, we obtain an exact sequence

$$0 \rightarrow H^1(\pi; \Lambda)^* \rightarrow \text{Ext}_{\Lambda}^2(H^2(\pi; \Lambda), \Lambda) \rightarrow \mathbb{Z} \rightarrow \text{Ext}_{\Lambda}^1(H^1(\pi; \Lambda), \Lambda) \rightarrow 0.$$

A further calculation with the filtration sequences shows that $\text{Ext}_{\Lambda}^2(H^2(\pi; \Lambda), \Lambda)$ is a direct sum of n copies of $\text{Ext}_{\Lambda}^2(\mathbb{Z}[\pi/\pi_{\sigma}], \Lambda)$, where σ is an edge. If $n > 0$ this is enough to show that $H^1(\pi; \Lambda)^* \neq 0$. If $m = 0$ as well, then the exact sequence above reduces to

$$0 \rightarrow H^1(\pi; \Lambda)^* \rightarrow (\mathbb{Z}[\pi/\pi_{\sigma}])^n \rightarrow \mathbb{Z} \rightarrow 0,$$

and $H^1(\pi; \Lambda)^* \cong \Lambda^{n-1}$. For each of the n edges, $\pi_{\sigma} \cong \mathbb{Z}^2$.

5. Minimal Models and Stabilization

For any finitely presented group π , one can construct a 4-manifold M with fundamental group π by doubling a thickening of a finite 2-complex K with $\pi_1(K) = \pi$. If K has minimal Euler characteristic, then the double of K will have minimal Euler characteristic over all double constructions.

If π is a right-angled Artin group, we can take K to be the 2-skeleton of the Salvetti complex mentioned in Sec. 2. The Salvetti complex has minimal Euler characteristic over all possible $K(\pi, 1)$ since its chain complex gives a minimal resolution for π .

Definition 5.1. We say that a 4-manifold X is *minimal for π* if its Euler characteristic is minimal over all closed, oriented 4-manifolds with fundamental group π .

The Euler characteristic of a minimal 4-manifold for π is the Hausmann–Weinberger invariant [21]. It has been determined for free abelian groups by Kirk and Livingston [28], but is still unknown for most classes of finitely presented groups.

Theorem 5.2 (Hildum [22, Theorem 1.2]). *Let π be a right-angled Artin group with $H_4(\pi; \mathbb{Z}) = 0$. Let K be the 2-skeleton of the Salvetti complex with fundamental group π , and let $Y(K)$ denote a spin 4-dimensional thickening of K . Then the double $M_0 := Y(K) \cup Y(K)$ is a minimal spin 4-manifold for π .*

Corollary 5.3. *Let π be a right-angled Artin group with $H_4(\pi; \mathbb{Z}) = 0$. Then*

$$\text{rk}_{\mathbb{Z}}(\pi_2(M_0) \otimes_{\Lambda} \mathbb{Z}) = b_2(\pi) + b_3(\pi).$$

if M_0 is a minimal spin 4-manifold for π .

Proof. The spectral sequence (of the covering) converging to $H_*(M_0)$ yields the following exact sequence:

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & H_3(\pi) & \xrightarrow{d^3} & \pi_2(M_0) \otimes_{\Lambda} \mathbb{Z} & \longrightarrow & H_2(M_0) & \longrightarrow & H_2(\pi) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
 0 & \longrightarrow & \mathbb{Z}^{b_3(\pi)} & \longrightarrow & \pi_2(M) \otimes_{\Lambda} \mathbb{Z} & \longrightarrow & \mathbb{Z}^{2b_2(\pi)} & \longrightarrow & \mathbb{Z}^{b_2(\pi)} & \longrightarrow & 0
 \end{array}$$

The second term in the exact sequence above is the $(0, 2)$ term in the spectral sequence, $H_0(\pi; \pi_2(M_0)) \cong \pi_2(M_0) \otimes_{\Lambda} \mathbb{Z}$. By exactness, the \mathbb{Z} -rank of $\pi_2(M_0) \otimes_{\Lambda} \mathbb{Z}$ is $b_2(\pi) + b_3(\pi)$. □

The thickened double construction will allow us to determine the structure of $\pi_2(M)$ as a Λ -module, for any 4-manifold with fundamental group π of cohomological dimension $\text{cd } \pi \leq 3$. First, we discuss the homology and cohomology of finite 2-complexes.

Lemma 5.4. *Let π be a finitely presented group with $\text{cd } \pi \leq 3$. If K is any finite 2-complex with fundamental group π , then $\pi_2(K) = H_2(K; \Lambda)$ is stably free as a Λ -module.*

Proof. Let X be an aspherical 3-complex model for $K(\pi, 1)$, and let K be the 2-skeleton of X . In this special case, we will show that $H_2(K; \Lambda)$ is a finitely generated free Λ -module. Consider the chain complex $C_*(X) := C_*(X; \Lambda)$ with Λ -module coefficients:

$$0 \longrightarrow C_3(X) \longrightarrow C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{5.1}$$

This chain complex is exact since \tilde{X} is contractible, and so $H_2(K; \Lambda) \cong Z_2(K) = Z_2(X) = C_3(X)$. Since $C_i(X) \cong \Lambda^{b_i(\pi)}$, we have $H_2(K; \Lambda) \cong \Lambda^{b_3(\pi)}$, where $b_i(\pi)$ denotes the number of i -cells in X . The result now follows, since the module $H_2(K; \Lambda)$ is independent of the choice of finite 2-complex K , after stabilization with free modules. □

In general, the available information about $H^2(K; \Lambda)$ is contained in the 4-term exact sequence (4.2):

$$0 \rightarrow H^2(\pi; \Lambda) \rightarrow H^2(K; \Lambda) \rightarrow \text{Hom}_{\Lambda}(H_2(K; \Lambda), \Lambda) \rightarrow H^3(\pi; \Lambda) \rightarrow 0.$$

In certain cases, we can make specific calculations.

Example 5.5. Consider the case of the right-angled Artin group $\pi = \mathbb{Z}^3$ in which the associated graph Γ is a 3-clique and the flag complex $\hat{\Gamma}$ is a single 2-simplex. Then, $H^2(\pi; \Lambda) = 0$ and $H^3(\pi; \Lambda) = \mathbb{Z}$ by Theorem 2.2. This example indicates

that even though $H_2(K; \Lambda)$ is a free Λ -module, $H^2(K; \Lambda)$ may not be free. In this case, (4.2) becomes

$$0 \rightarrow H^2(K; \Lambda) \rightarrow \Lambda \rightarrow \mathbb{Z} \rightarrow 0,$$

which shows that $H^2(K; \Lambda)$ is the augmentation ideal $I(\pi)$. Thus, $\pi_2(M_0) = I(\pi) \oplus \Lambda$ for $\pi = \mathbb{Z}^3$, and the module $I(\pi)$ is not (stably) free since $H_1(\pi; I(\pi)) \cong H_2(\pi; \mathbb{Z}) \cong \mathbb{Z}^3$.

We now consider the structure of $\pi_2(M)$ and s_M for the thickened doubles of finite 2-complexes (see Kreck and Schafer [30, Sec. II]).

Lemma 5.6. *Let π be a finitely-presented group and let K be any finite 2-complex with fundamental group π . If $Y(K)$ denotes a spin 4-dimensional thickening of K , then*

$$\pi_2(M) = H^2(K; \Lambda) \oplus H_2(K; \Lambda)$$

as a Λ -module, for the thickened double $M = Y(K) \cup Y(K)$. The equivariant intersection form s_M is a metabolic form.

Proof. Let $Y = Y(K)$ and note that $M = \partial(Y \times I)$. We start with the long exact sequence in homology for the pair (M, Y) :

$$\cdots \rightarrow H_3(M, Y; \Lambda) \rightarrow H_2(Y; \Lambda) \rightarrow H_2(M; \Lambda) \rightarrow H_2(M, Y; \Lambda) \rightarrow H_1(Y; \Lambda) \rightarrow \cdots$$

The inclusion of Y into M induces a split injective map $H_i(Y; \Lambda) \rightarrow H_i(M; \Lambda)$ in every dimension. Thus, the maps from $H_i(M, Y; \Lambda)$ to $H_i(Y; \Lambda)$ are all zero maps. In addition, using excision properties as well as Poincaré–Lefschetz duality, we have the isomorphisms $H_2(M, Y; \Lambda) \cong H^2(Y, \partial Y; \Lambda) \cong H_2(Y; \Lambda)$. This gives the split short exact sequence

$$0 \rightarrow H^2(K; \Lambda) \rightarrow H_2(M; \Lambda) \rightarrow H_2(K; \Lambda) \rightarrow 0. \tag{5.2}$$

This, along with the Hurewicz isomorphism $H_2(M; \Lambda) \cong H_2(\widetilde{M}; \mathbb{Z}) \cong \pi_2(M)$ yields the desired result. By [30, Proposition II.4], the equivariant intersection form has the structure:

$$s_M((u, x), (v, y)) = u(y) + \overline{v(x)} + \delta_K(u, v), \tag{5.3}$$

for all $u, v \in H^2(K; \Lambda)$ and $x, y \in H_2(K; \Lambda)$. In this formula, δ_K is an even hermitian symmetric form on $H_2(K; \Lambda)$ determined by the thickening. \square

We can now determine the structure of $\pi_2(M)$ in general, after stabilization by free Λ -modules.

Proposition 5.7. *Let N be a closed, oriented, spin, TOP 4-manifold with fundamental group π . If $\text{cd } \pi \leq 3$, then there exists a simply connected, closed 4-manifold*

X such that

$$N' \# r(S^2 \times S^2) \approx M' \# s(S^2 \times S^2),$$

for some $r, s \geq 0$, where $N' := N \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, $M' := M_0 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# X$, and M_0 is a thickened double.

Proof. We note that the thickened double model M_0 represents the zero bordism element in $\Omega_4^{\text{STOP}}(K(\pi, 1)) = \mathbb{Z}$. Since the signature detects elements in this bordism group, it follows that $N' := N \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is stably homeomorphic to $M' := M_0 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# X$ for some closed, simply connected 4-manifold X with $\text{sign}(N') = \text{sign}(X)$. If M_0 is minimal, we can take X to be a connected sum of copies of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ so that N' and M' have the same signature and Euler characteristic. It follows that $N' \# r(S^2 \times S^2) \approx M' \# r(S^2 \times S^2)$, for some $r \geq 0$ (see [29, Theorem C] and [14, Sec. 9.1]). \square

Corollary 5.8. *Let M and N be a closed, oriented, spin, TOP 4-manifolds with fundamental group π . If $\text{cd } \pi \leq 3$, then $\pi_2(N)$ is stably isomorphic to $\pi_2(M)$ as a Λ -module.*

Proof. By Proposition 5.7, the modules $\pi_2(N)$ and $\pi_2(M)$ are both stably isomorphic to $\pi_2(M_0)$ after stabilization by direct sum with free Λ -modules. \square

Remark 5.9. By using thickenings, we have shown that $\pi_2(N)$ is stably isomorphic to $H_2(K; \Lambda) \oplus H^2(K; \Lambda)$, for any K any finite 2-complex with $\pi_1(K) = \pi_1(N)$. This is a special case of a general result (see [18, Proposition 2.4; 17, Theorem 4.2]), that $\pi_2(N)$ is stably isomorphic to a Λ -module E in an extension:

$$\mathcal{E}_N : 0 \rightarrow H_2(K; \Lambda) \rightarrow E \rightarrow H^2(K; \Lambda) \rightarrow 0.$$

For fundamental groups with $\text{cd } \pi \leq 3$, our direct argument verifies that \mathcal{E}_N is stably a split extension.

Remark 5.10. The k -invariant of the thickened double $M = Y(K) \cup Y(K)$ is the image (induced by the inclusion) of the k -invariant of K . More generally, if $i_M : K \subset M$ is the 2-skeleton of a closed topological 4-manifold M , then $(i_M)_*(k_K) = k_M \in H^3(\pi; \pi_2(M))$. This follows from the push-out diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \pi_2(K) & \longrightarrow & C_2(M) & \longrightarrow & C_1(M) & \longrightarrow & C_0(M) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow (i_M)_* & & \downarrow & & \parallel & & \parallel & & \parallel & & \\ 0 & \longrightarrow & \pi_2(M) & \longrightarrow & C_2(M)/B_2(M) & \longrightarrow & C_1(M) & \longrightarrow & C_0(M) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

in which the upper multiple extension represents k_K and the lower extension represents k_M (see Baues [3, p. 101]).

Here is a useful consequence of this description for the k -invariant.

Proposition 5.11. *Let M and N be closed, spin, topological 4-manifolds with fundamental group π of $\text{cd } \pi \leq 3$. If the equivariant intersection forms s_M and s_N are stably isometric, then their quadratic 2-types $Q(M)$ and $Q(N)$ are stably isometric.*

Proof. It is enough to show that we have an isometry $Q(M) \cong Q(N)$ after stabilization by connected sum with sufficiently many copies of $S^2 \times S^2$. After preliminary stabilization, we may assume that $s_M \cong s_N$ and that $i_M : K \subset M$ and $i_N : K \subset N$ is a 2-skeleton for both manifolds. This uses Whitehead’s stable uniqueness theorem for finite 2-complexes with a fixed fundamental group (up to simple homotopy equivalence) after wedging with sufficiently many copies of S^2 . By Remark 5.10, we have $(i_M)_*(k_K) = k_M$ and $(i_N)_*(k_K) = k_N$. By general position, we may assume that the 1-skeleton of K is embedded in M and the 2-cells are all immersed in M , with at most double point intersections and self-intersections.

Since π has $\text{cd } \pi \leq 3$, we may assume by Lemma 5.4 that $\pi_2(K)$ is a finitely generated free Λ -module, and a direct summand of both $\pi_2(M)$ and $\pi_2(N)$. Finally, after further stabilization, we may assume that the equivariant intersection forms s_M and s_N and their quadratic refinements q_M and q_N vanish on these summands $\pi_2(K)$. This is achieved as follows: at each 2-cell intersection or self-intersection point where two sheets meet, we take the connected sum with a copy of $S^2 \times S^2$. Then we construct a new embedding of K by attaching disjoint 2-cells in $S^2 \times S^2 \setminus D^4$ to the original separate 2-cell sheets. This process removes one intersection point, and may be repeated to embed the same complex K in a stabilization of M .

After applying the same steps to the image of K in N , we may assume that $\pi_2(K)$ is a *totally isotropic* submodule with respect to both quadratic forms. At this stage, K is no longer the full 2-skeleton of M and N , but still carries their k -invariants.

Now suppose that $\phi : (\pi_2(M), s_M) \rightarrow (\pi_2(N), s_N)$ is an isometry of equivariant intersection forms. We claim that there is a self-isometry $\psi : s_N \rightarrow s_N$, after further stabilization, such that the composite $\psi \circ \phi$ satisfies $\psi(\phi(x)) = x$, for all $x \in \pi_2(K) \subset \pi_2(M)$. It will then follow that $(\psi \circ \phi)_*(k_M) = k_N$ and hence that $Q(M) \cong Q(N)$.

To construct the required self-isometry of a stabilized s_N , we use unitary transvections as in [19, Sec. 1]. Let $\{w_1, \dots, w_k\}$ be a free Λ -base for $\pi_2(K)$, which can be considered either as elements in $\pi_2(M)$ or in $\pi_2(N)$. Since $\pi_2(N)^* \rightarrow \pi_2(K)^*$ is a split surjection, we can pick elements $\{v_1, \dots, v_k\} \subset \pi_2(N)$ such that $\{\text{ad } s_N(v_1), \dots, \text{ad } s_N(v_k)\}$ projects to the dual base $\{w_1^*, \dots, w_k^*\}$ for $\pi_2(K)^*$. In other words, $s_N(w_i, v_j) = \delta_{ij}$, for $1 \leq i, j \leq k$. Since s_N admits a quadratic refinement q_N , and $\pi_2(K)$ is a free totally isotropic Λ -module, we may assume that $q_N(v_i) = 0$ and $s_N(v_i, v_j) = 0$, for $1 \leq i, j \leq k$ (see [35, Lemma 5.3]). It follows that the elements $\{w_1, \dots, w_k; v_1, \dots, v_k\}$ span a hyperbolic direct summand of $(\pi_2(N), s_N, q_N)$ isomorphic to $H(\pi_2(K))$.

We consider the orthogonal direct sum $s_N \oplus H(\Lambda)_1 \oplus \dots \oplus H(\Lambda)_k$ with k new hyperbolic summands, and let $\{e_i, f_i\}$ denote a hyperbolic base for the i th

summand. We compute the following composite of unitary transvections (see the formula [19, 1.2]):

$$\theta_i(w_i) := (\sigma_{f_i,0,w_i} \circ \sigma_{e_i,0,v_i})(w_i) = \sigma_{f_i,0,w_i}(\sigma_{e_i,0,v_i}(w_i)) = \sigma_{f_i,0,w_i}(w_i + e_i) = e_i$$

for $1 \leq i \leq k$. This shows that we can move any basis element w_i to the standard position e_i in a new hyperbolic summand. By construction, $\theta_i(w_j) = w_j$, $\theta_i(e_j) = e_j$ and $\theta_i(f_j) = f_j$, if $j \neq i$. Let $\theta = \theta_k \circ \dots \circ \theta_1$, so that $\theta(w_i) = e_i$, for $1 \leq i \leq k$.

We can apply this process to elements $\{\phi(w_1), \dots, \phi(w_k)\}$ of the totally isotropic submodule $\phi(\pi_2(K)) \subset \pi_2(N)$, and construct a self-isometry θ' so that $\theta'(\phi(w_i)) = e_i$, for $1 \leq i \leq k$. The required stabilized self-isometry ψ of s_N , such that $\psi(\phi(x)) = x$ for all $x \in \pi_2(K)$, is given by $\psi := \theta^{-1} \circ \theta'$. □

6. The Reduced Equivariant Intersection Form

In this section, let π denote a finitely presented group with $\text{cd } \pi \leq 3$. Let M denote a closed, oriented, spin 4-manifold with fundamental group π . As above, we can construct a model 4-manifold M_0 by taking the double of a thickening $Y(K)$, where K denotes the 2-skeleton of a minimal aspherical 3-complex L with $\pi_1(L) = \pi$.

For fundamental groups of geometric dimension ≤ 2 , the quotient

$$\pi_2(M)^\dagger := \pi_2(M)/H^2(\pi; \Lambda) \cong \text{Hom}_\Lambda(H_2(M; \Lambda), \Lambda)$$

is stably free since $H^3(\pi; \Lambda) = 0$ (see [20, Corollary 4.4]). In that context, the “reduced” equivariant intersection form s_M^\dagger was a non-singular form on $\pi_2(M)^\dagger$. In our setting, where π can have geometric dimension 3, the quotient $\pi_2(M)^\dagger$ is Λ -finitely generated but may not be stably free as a Λ -module. Instead, we define

$$L_M := \pi_2(M)^* = \text{Hom}_\Lambda(H_2(M; \Lambda), \Lambda),$$

and obtain a non-singular hermitian form $h_M : L_M \times L_M \rightarrow \Lambda$, which we also call the *reduced equivariant intersection form*. For groups with $\text{cd } \pi \leq 3$, this form has a key role in the proof of Theorem 11.2, through a generalization of [20, Theorem 5.13].

Lemma 6.1. *Let $\pi_1(M)$ be a finitely presented group with $\text{cd } \pi_1(M) \leq 3$. Then*

- (i) $(\text{ad } s_M)^*$ is an isomorphism of finitely generated, stably free Λ -modules, and
- (ii) the inverse of $(\text{ad } s_M)^*$ is the adjoint of a non-singular form

$$h_M : L_M \times L_M \rightarrow \Lambda,$$

on $L_M = \pi_2(M)^*$.

Proof. By Corollary 5.8, we may assume that $M = M_0$ is a model manifold with the given fundamental group, obtained as the double of a thickened finite 2-complex

K . We will first show that L_M is a finitely generated, (stably) free Λ -module after dualizing the summands in the isomorphism

$$\pi_2(M) \cong \pi_2(K) \oplus H^2(K; \Lambda)$$

provided by Lemma 5.6. Consider the 4-term exact sequence (4.2) for the 2-skeleton K of an aspherical 3-complex:

$$0 \rightarrow H^2(\pi; \Lambda) \rightarrow H^2(K; \Lambda) \rightarrow \text{Hom}_\Lambda(H_2(K; \Lambda), \Lambda) \rightarrow H^3(\pi; \Lambda) \rightarrow 0.$$

The third nonzero term is a free Λ -module, since $H_2(K; \Lambda) \cong \Lambda^{b_3(\pi)}$. For the following argument, we define $F := \text{Hom}_\Lambda(H_2(K; \Lambda), \Lambda) \cong \Lambda^{b_3(\pi)}$. Recall from Sec. 4 that the above sequence is obtained by splicing together

$$0 \rightarrow H^2(\pi; \Lambda) \rightarrow H^2(K; \Lambda) \rightarrow V \rightarrow 0$$

and

$$0 \rightarrow V \rightarrow F \rightarrow H^3(\pi; \Lambda) \rightarrow 0,$$

where $V := \text{coker}(H^2(\pi; \Lambda) \rightarrow H^2(K; \Lambda)) = \ker(F \rightarrow H^3(\pi; \Lambda))$. Taking the dual of both short exact sequences yields

$$0 \rightarrow V^* \rightarrow H^2(K; \Lambda)^* \rightarrow H^2(\pi; \Lambda)^* \xrightarrow{\delta_1} \text{Ext}_\Lambda^1(V, \Lambda) \rightarrow \dots$$

and

$$0 \rightarrow H^3(\pi; \Lambda)^* \rightarrow F^* \rightarrow V^* \rightarrow \text{Ext}_\Lambda^1(H^3(\pi; \Lambda), \Lambda) \rightarrow \dots$$

By Proposition 4.1, we have $F^* \cong V^*$, and since F is a free Λ -module the connecting map $\delta_2 : \text{Ext}_\Lambda^1(V, \Lambda) \rightarrow \text{Ext}_\Lambda^2(H^3(\pi; \Lambda), \Lambda)$ is an isomorphism. Now, Lemma 4.2 shows that the connecting map $\delta_1 : H^2(\pi; \Lambda)^* \rightarrow \text{Ext}_\Lambda^1(V, \Lambda)$ is also an isomorphism. It follows that $V^* \cong H^2(K; \Lambda)^*$. Together these facts give an isomorphism $H^2(K; \Lambda)^* \cong F^*$ to a free Λ -module. We have now shown that

$$\pi_2(M)^* \cong F \oplus F^* \cong \Lambda^{2b_3(\pi)}$$

is a finitely generated, free Λ -module.

To show that $(\text{ad } s_M)^*$ is an isomorphism, we can perform the same splicing trick for the 4-term exact sequence

$$0 \rightarrow H^2(\pi; \Lambda) \rightarrow \pi_2(M) \xrightarrow{\text{ad } s_M} \pi_2(M)^* \rightarrow H^3(\pi; \Lambda) \rightarrow 0 \tag{6.1}$$

with V_1 denoting the similar cokernel

$$0 \rightarrow H^2(\pi; \Lambda) \rightarrow H^2(M; \Lambda) \rightarrow V_1 \rightarrow 0.$$

We may compare the 4-term exact sequences for $H^2(M; \Lambda)$ and $H^2(K; \Lambda)$, and note that the restriction map $i^* : H^2(M; \Lambda) \rightarrow H^2(K; \Lambda)$ is a split surjection by Lemma 5.6. The connecting maps $\delta_1 : H^2(\pi; \Lambda)^* \rightarrow \text{Ext}_\Lambda^1(V; \Lambda)$ and $\delta'_1 : H^2(\pi; \Lambda)^* \rightarrow \text{Ext}_\Lambda^1(V_1; \Lambda)$ are related by the formula $\delta'_1 = i^* \circ \delta_1$. Since the

induced map $i^* : \text{Ext}_\Lambda^1(V; \Lambda) \rightarrow \text{Ext}_\Lambda^1(V_1; \Lambda)$ is a (split) injection and δ_1 is an isomorphism (by Lemma 4.2), we conclude that δ'_1 is an injection. From the sequence

$$0 \rightarrow V_1^* \rightarrow H^2(M; \Lambda)^* \rightarrow H^2(\pi; \Lambda)^* \xrightarrow{\delta'_1} \text{Ext}_\Lambda^1(V_1, \Lambda) \rightarrow \dots,$$

we obtain

$$V_1^* \cong H^2(M; \Lambda)^* = \pi_2(M)^* = L_M.$$

This yields the isomorphism

$$(\text{ad } s_M)^* : L_M^* \cong (\pi_2(M)^*)^* \xrightarrow{\cong} V_1^* \xrightarrow{\cong} \pi_2(M)^* = L_M. \tag{6.2}$$

Thus, we can define a non-singular form $h_M : L_M \times L_M \rightarrow \Lambda$ by the formula

$$h_M(x, y) = ((\text{ad } s_M)^*)^{-1}(x)(y) \in \Lambda,$$

whose adjoint is the inverse of $(\text{ad } s_M)^* : (\pi_2(M)^*)^* \rightarrow \pi_2(M)^*$. □

It remains to check that the form h_M is hermitian.

Definition 6.2. Let $h : B \times B \rightarrow \Lambda$ be a sesquilinear form on a right Λ -module B . We say that (B, h) is *hermitian* (symmetric) if $h = Th$, where $Th(b, b') := \overline{h(b', b)}$, for all $b, b' \in B$. We say that h is *strongly even* if $h = \lambda + T\lambda$ for some sesquilinear form (B, λ) .

Remark 6.3. It is a standard fact that an even hermitian form (B, h) is strongly even whenever B is a finitely generated free Λ -module.

Let $\gamma : B \rightarrow (B^*)^*$ denote the map $\gamma(b) = \overline{b^{**}}$ defined by the evaluation $\langle \gamma(b), v \rangle = \langle b^{**}, v \rangle = \overline{v(b)}$, for $v \in B^*$ and note that we have a map of right Λ -modules:

$$\langle \gamma(b\lambda), v \rangle := \gamma(b\lambda)(v) = \overline{v(b\lambda)} = \overline{(v\bar{\lambda})(b)} = \gamma(b)(v\bar{\lambda}) = (\gamma(b) \cdot \lambda)(v),$$

where as usual, the right Λ -module structure on $B^* = \text{Hom}_\Lambda(B, \Lambda)$ is given by the formula $(v\lambda)(b) = v(b\bar{\lambda})$, for all $v \in B^*$ and $\lambda \in \Lambda$.

Lemma 6.4. *The form (B, h) is hermitian if and only if $(\text{ad } h)^* \circ \gamma = \text{ad } h$.*

Proof. It suffices to show that $\text{ad}(Th) = (\text{ad } h)^* \circ \gamma$. This follows from the calculation

$$\langle ((\text{ad } h)^* \circ \gamma)(b), b' \rangle = \langle \gamma(b), \text{ad } h(b') \rangle = \overline{\text{ad } h(b')(b)} = \overline{h(b', b)}, \tag{6.3}$$

where $\langle -, - \rangle : B^* \times B \rightarrow \Lambda$ denotes the evaluation pairing. □

Lemma 6.5. *The form (L_M, h_M) is hermitian.*

Proof. Since s_M is hermitian, we have the formula $\text{ad } s_M = (\text{ad } s_M)^* \circ \gamma$ from Lemma 6.4. After dualizing, we obtain $(\text{ad } s_M)^* = \gamma^* \circ (\text{ad } s_M)^{**}$. Note that $\gamma^* : B^{***} \rightarrow B^*$ is the left inverse of $\gamma_1 : B^* \rightarrow (B^{**})^* = (B^*)^{**}$, given by $v \mapsto v^{**}$, for any right Λ -module B . This follows from the calculation

$$\langle (\gamma^* \circ \gamma_1)(v), b \rangle = \langle \gamma^*(v^{**}), b \rangle = \langle v^{**}, \gamma(b) \rangle = v^{**}(b^{**}) = \overline{b^{**}(v)} = v(b),$$

for all $b \in B$ and all $v \in B^*$. Therefore, $\gamma_1 \circ (\text{ad } s_M)^* = (\text{ad } s_M)^{**}$.

By definition, $\text{ad } h_M \circ (\text{ad } s_M)^* = \text{id}$, so we have $(\text{ad } s_M)^{**} \circ (\text{ad } h_M)^* = \text{id}$ by dualizing. Therefore, $(\text{ad } h_M)^* \circ \gamma_1 = \text{ad } h_M$. From Lemma 6.4 applied to $B = \pi_2(M)^*$ and $h = h_M$, we conclude that h_M is hermitian symmetric. \square

Remark 6.6. If $\pi = \pi_1(M)$ is any finitely presented group with tame cohomology, then the splicing argument above shows directly that the dual of the equivariant intersection form s_M induces a well-defined, non-singular hermitian form $h_M : L_M \times L_M \rightarrow \Lambda$ on $\pi_2(M)^*$. However, if $\text{cd } \pi \geq 4$, the module $\pi_2(M)^*$ might not be stably free (take $\pi = \mathbb{Z}^4$).

Definition 6.7 (spin⁺ manifold). If $\text{cd } \pi_1(M) \leq 3$, we define the *reduced Stiefel–Whitney class*

$$w_M : L_M \rightarrow \mathbb{Z}/2$$

by the formula $w_M(x) = \varepsilon_1(h_M(x, x)) \pmod{2}$, for all $x \in L_M$. Here, $\varepsilon_1 : \Lambda \rightarrow \mathbb{Z}$ is the coefficient of $\lambda \in \Lambda$ at the identity element. The form h_M is even if and only if $w_M \equiv 0$. In this case, we say that h_M has *even reduced w_2 -type*, and that M is a *spin⁺ manifold*.

We are indebted to Peter Teichner for pointing out that h_M is not always an even form.

Remark 6.8. Examples of spin 4-manifolds with even reduced w_2 -type were given by Plotnick [32, Sec. 3] using a “twist-spinning” construction applied to any closed, oriented, aspherical 3-manifold (compare [24, Sec. 6.2]). In these examples, s_M is the hyperbolic form on the augmentation ideal $I(\pi) \subset \mathbb{Z}[\pi]$.

We conclude this section by showing that the thickened double construction provides a spin⁺ manifold for every fundamental group π , whenever $\text{cd } \pi \leq 3$.

Note first that the module $\pi_2(M)^{**}$ supports the non-singular hermitian form $\widehat{s}_M(u, v) = \langle (\text{ad } s_M)^{**}(u), v \rangle$ whose adjoint is $(\text{ad } s_M)^{**}$. The hermitian property follows as in Lemma 6.4.

Lemma 6.9. *If s_M is strongly even, then h_M has even reduced w_2 -type.*

Proof. We first show that the homomorphism $\widehat{w}_M : \pi_2(M)^{**} \rightarrow \mathbb{Z}/2$ associated to the form \widehat{s}_M by the formula $u \mapsto \varepsilon_1(\widehat{s}_M(u, u)) \pmod{2}$ is determined by w_M .

To check this, we will write $u = \text{ad } h_M(x)$, for a unique $x \in \pi_2(M)^*$, and use the formula $\gamma_1 \circ (\text{ad } s_M)^* = (\text{ad } s_M)^{**}$ from the proof of Lemma 6.5. Now, we have

$$\begin{aligned} \langle (\text{ad } s_M)^{**}(u), u \rangle &= \langle (\text{ad } s_M)^{**}(\text{ad } h_M(x)), \text{ad } h_M(x) \rangle \\ &= \langle \gamma_1 \circ (\text{ad } s_M)^*(\text{ad } h_M(x)), \text{ad } h_M(x) \rangle \\ &= \langle \gamma_1(x), \text{ad } h_M(x) \rangle = \text{ad } h_M(x)(x), \end{aligned}$$

for all $u \in \pi_2(M)^{**}$, with $u = \text{ad } h_M(x)$. Therefore, $\widehat{w}_M(u) = w_M(x)$ as required.

Now, suppose that $s_M = \lambda + T\lambda$ for some even hermitian form $(\pi_2(M), \lambda)$. We have the formula

$$(\text{ad } s_M)^{**} = (\text{ad } \lambda)^{**} + (\text{ad } \lambda)^{***} \circ \gamma^{**}$$

and we claim that

$$((\text{ad } \lambda)^{***} \circ \gamma^{**})(u)(v) = \overline{((\text{ad } \lambda)^{**})(v)(u)}.$$

This follows since

$$\begin{aligned} \langle (\text{ad } \lambda)^{***} \circ \gamma^{**}(u), v \rangle &= \langle (\text{ad } \lambda)^{***}(\gamma^{**}(u)), v \rangle = \langle \gamma^{**}(u), (\text{ad } \lambda)^{**}(v) \rangle \\ &= \langle u, \gamma^*((\text{ad } \lambda)^{**}(v)) \rangle = \langle (\text{ad } \lambda)^{**}(v), \gamma(u) \rangle \\ &= \overline{\langle (\text{ad } \lambda)^{**}(v), u \rangle} \end{aligned}$$

for all $u, v \in \pi_2(M)^{**}$. Therefore, s_M strongly even implies $\widehat{w}_M = 0$, and hence $w_M = 0$. □

Corollary 6.10. *Let π be a finitely presented group with $\text{cd } \pi \leq 3$. Then there exists a smooth, closed, oriented spin^+ manifold M^4 with $\pi_1(M) = \pi$.*

Proof. Let M be a smooth, spin, thickened double 4-manifold as constructed in Lemma 5.6. Then the equivariant intersection form s_M is strongly even. This follows from the formula in (5.3), since the metabolic form s_M is determined by an even hermitian form $(\pi_2(K), \delta_K)$ which depends on the particular thickening. However, $\pi_2(K)$ may be assumed to be a free Λ -module by stabilization, and therefore δ_K is strongly even. □

7. The Reduced 2-Type P

In order to prove Theorem A, we will follow the strategy of [20, Sec. 2] involving the reduced normal 2-type and the “modified” surgery theory developed by Kreck [29]. Since we have restricted our attention to spin topological manifolds, the *reduced normal 2-type* for a given 4-manifold M is a fibration $B \rightarrow B\text{STOP}$, with total space

$$B := B(M) = P \times B\text{TOPSPIN}$$

defined by the second factor projection $P \times B\text{TOPSPIN} \rightarrow B\text{TOPSPIN}$ and the natural map $B\text{TOPSPIN} \rightarrow B\text{STOP}$. Here, we are using the stabilized classifying

spaces for topological \mathbb{R}^n -bundles, with $w_1 = 0$ for *BSTOP* and $w_1, w_2 = 0$ for *BTOPSPIN* (see [26, p. 190]).

It remains to describe the reduced 2-type $P = P(M)$.

Definition 7.1. Let M be a closed, oriented, spin 4-manifold with fundamental group π . We define the *reduced 2-type of M* as follows: let $P = P(M)$ be the two-stage Postnikov system with $\pi_1(P) = \pi_1(M) = \pi$ and $\pi_2(P) = L_M = \pi_2(M)^*$. The total space P is defined by a fibration over $K(\pi, 1)$ with fiber $K(L_M, 2)$ and classified by k_P in $H^3(\pi; L_M)$:

$$\begin{array}{ccc} K(L_M, 2) & \longrightarrow & P \\ & & \downarrow \\ & & K(\pi, 1) \longrightarrow H^3(\pi; L_M) \end{array}$$

We have a reference map $c_M : M \rightarrow P$ which factors through the algebraic 2-type of M (classified by $\pi_1(M) = \pi$, $\pi_2(M)$, and $k_M \in H^3(\pi; \pi_2(M))$). The k -invariant k_P is given by the image of k_M under the map induced by $\pi_2(M) \rightarrow \pi_2(P)$.

The map $\pi_2(M) \rightarrow \pi_2(P)$ is neither an injection nor a surjection in general because P is given by the union of cells of dimension ≥ 2 with M . In particular, the map c_M is 1-connected but not 2-connected.

Definition 7.2. For M a closed, spin, TOP 4-manifold, a map $c : M \rightarrow P$ to a space P is called a *reduced 3-equivalence* if c induces an isomorphism $c_* : \pi_1(M) \cong \pi_1(P)$ and an isomorphism $c^* : \pi_2(P)^* \cong \pi_2(M)^*$. If c is a reduced 3-equivalence, then $c \times \nu_M : M \rightarrow P \times \text{BTOPSPIN}$ is called a *reduced normal 2-smoothing*.

We have a result similar to [20, Proposition 2.11].

Proposition 7.3. *Let M and M' be closed, oriented, spin, TOP 4-manifolds with fundamental group π . Suppose that there is an isometry $Q(M) \cong Q(M')$ given by isomorphisms $\alpha : \pi_1(M) \rightarrow \pi_1(M')$ and $\beta : \pi_2(M) \rightarrow \pi_2(M')$. Then there is a 3-coconnected fibration $P \times \text{BTOPSPIN} \rightarrow \text{BSTOP}$ admitting two reduced normal 2-smoothings $M \rightarrow P \times \text{BTOPSPIN}$ and $M' \rightarrow P \times \text{BTOPSPIN}$ that induce (α, β) .*

Proof. After identifying $\pi_1(M) = \pi$ and $\pi_1(M')$ via the given isomorphism α , the map $\beta : \pi_2(M) \rightarrow \pi_2(M')$ is π -equivariant, and $(\text{ad } s_M) = \beta^* \circ (\text{ad } s_{M'}) \circ \beta$. If we let $P = P(M)$ and $P' = P(M')$ denote the reduced 2-types for M and M' , respectively, then we have a fiber homotopy equivalence $P \rightarrow P'$ over $K(\pi, 1)$ induced by the given isometry (α, β) preserving the k -invariants. □

8. The Cohomology of P

In order to prove Theorem A, we need to calculate the topological spin bordism group $\Omega_4^{\text{Spin}}(P)$, and this requires some information about $H^*(P; \Lambda)$ and $H_*(P; \mathbb{Z})$.

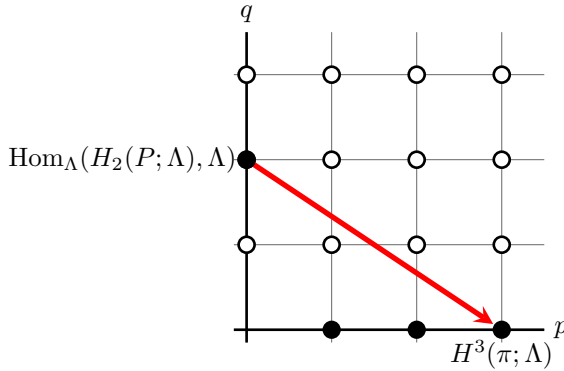


Fig. 1. The E_2 -page of the spectral sequence $E_2^{p,q} = \text{Ext}_\Lambda^p(H_q(P; \Lambda), \Lambda)$ converging to $H^*(P; \Lambda)$. The solid dots represent possible nonzero terms.

If $c : M \rightarrow P$ is a reduced 3-equivalence, we have the same 4-term exact sequence for $P = P(M)$ as we do for K and M . The following diagram commutes:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & H^2(\pi; \Lambda) & \longrightarrow & H^2(P; \Lambda) & \xrightarrow{\text{eval}} & \text{Hom}_\Lambda(\pi_2(P), \Lambda) & \xrightarrow{d_3} & H^3(\pi; \Lambda) & \longrightarrow & 0 \\
 & & \parallel & & \cong \downarrow c^* & & \cong \downarrow c^* & & \parallel & & (8.1) \\
 0 & \longrightarrow & H^2(\pi; \Lambda) & \longrightarrow & H^2(M; \Lambda) & \xrightarrow{\text{eval}} & \text{Hom}_\Lambda(\pi_2(M), \Lambda) & \longrightarrow & H^3(\pi; \Lambda) & \longrightarrow & 0
 \end{array}$$

and we see that $H^2(P; \Lambda) \cong H^2(M; \Lambda)$. Here is a partial calculation of the cohomology of the reduced 2-type $P = P(M)$.

Lemma 8.1. $H^1(P; \Lambda) \cong H^1(\pi; \Lambda)$, $H^2(P; \Lambda)$ is an extension of $\ker d_3 \subseteq \pi_2(P)^*$ by $H^2(\pi; \Lambda)$, and $H^3(P; \Lambda) = 0$.

Proof. We look at the universal coefficient spectral sequence $E_2^{p,q} \cong \text{Ext}_\Lambda^p(H_q(P; \Lambda), \Lambda)$ to compute $H^*(P; \Lambda)$. For $q = 0$, we have $E_2^{p,0} \cong \text{Ext}_\Lambda^p(\mathbb{Z}, \Lambda) = H^p(\pi; \Lambda)$. When q is odd, $H_q(P; \Lambda) \cong H_q(\tilde{P}; \mathbb{Z}) = 0$ since \tilde{P} is a product of copies of $\mathbb{C}P^\infty$. Thus, the $E_2^{p,1}$ - and $E_2^{p,3}$ -terms are zero. Additionally, since $H_2(P; \Lambda) \cong \pi_2(P)$ is stably Λ -free (by Corollary 5.8 and Lemma 6.1), the terms $\text{Ext}_\Lambda^p(H_2(P; \Lambda), \Lambda) = 0$ for $p > 0$. The $E_2^{1,2}$ -term of the spectral sequence will therefore be zero. The only nonzero differential is the d_3 map in Fig. 1, which is surjective by (8.1). Hence the $E_\infty^{3,0}$ -term is zero, and so $H^3(P; \Lambda) = 0$. \square

Lemma 8.2. For a closed, oriented, TOP 4-manifold M with reduced 2-type P ,

$$H^3(P, M; \Lambda) = 0.$$

Proof. We use the long exact sequence in cohomology of the pair (P, M) with Λ -coefficients. We have the isomorphisms $H^3(M; \Lambda) \cong H_1(M; \Lambda) \cong H_1(M; \mathbb{Z}) = 0$,

so the long exact sequence becomes

$$\cdots \rightarrow H^2(P; \Lambda) \rightarrow H^2(M; \Lambda) \rightarrow H^3(P, M; \Lambda) \rightarrow H^3(P; \Lambda) \rightarrow 0.$$

From (8.1), the isomorphism $H^2(P; \Lambda) \cong H^2(M; \Lambda)$ forces the middle map in the above sequence to be zero. This yields the isomorphism $H^3(P, M; \Lambda) \cong H^3(P; \Lambda)$, which is zero by Lemma 8.1. □

9. The Homology of P

In this section, let M be a closed, spin, TOP 4-manifold such that $\pi_1(M) = \pi$ is a finitely presented group with $\text{cd } \pi \leq 3$. To compute the homology of the reduced 2-type $P = P(M)$, we use the Serre spectral sequence for the fibration $\tilde{P} \rightarrow P \rightarrow K(\pi, 1)$. This spectral sequence with integral coefficients has the E^2 -page

$$E_{p,q}^2 = H_p(\pi; H_q(\tilde{P}))$$

and we only need the homology of P up to dimension 5. Note that \tilde{P} is a restricted product of copies of $\mathbb{C}P^\infty$, so $H_q(\tilde{P}) = 0$ for q odd. We have already seen that $H_2(\tilde{P}) = \pi_2(P)$ is a stably free Λ -module. Therefore, by [18, Lemma 2.2], we see that $H_4(\tilde{P}) = \Gamma(\pi_2(P))$ is also stably free.^a The notation $\Gamma(\pi_2(P))$ refers to Whitehead’s quadratic functor, defined in [18, Sec. 5]. In summary:

- (i) $E_{p,0}^2 \cong H_p(\pi; \mathbb{Z})$, which is zero when $p \geq 4$,
- (ii) $E_{0,q}^2 \cong H_0(\pi; H_q(\tilde{P})) \cong H_q(\tilde{P}) \otimes_\Lambda \mathbb{Z}$,
- (iii) $E_{p,q}^2 = H_p(\pi; H_q(\tilde{P})) = 0$ is zero for odd q ,
- (iv) $E_{p,2}^2 = H_p(\pi; H_2(\tilde{P})) = 0$, for $p > 0$,
- (v) $E_{p,4}^2 = H_p(\pi; H_4(\tilde{P})) = H_p(\pi; \Gamma(\pi_2(P))) = 0$, for $p > 0$.

In the E^2 -page, all d^2 maps that affect $H_i(P)$, $i \leq 5$, are zero. In the spectral sequence, the only possibly nonzero differential is $d^3 : H_3(\pi) \rightarrow H_0(\pi; \pi_2(P))$.

Proposition 9.1. $d^3 : H_3(\pi) \rightarrow H_0(\pi; \pi_2(P))$ is injective.

The first step of the proof is to establish this result for the minimal model M_0 .

Lemma 9.2. Let $P = P(M_0)$ be the reduced 2-type of the minimal model. Then the map $d^3 : H_3(\pi) \rightarrow H_0(\pi; \pi_2(P))$ is injective.

Proof. The injectivity argument comes from comparing the same d^3 maps in three spectral sequences, the first of which is for $H_*(K)$. In the spectral sequence converging to $H_*(K)$, since $H_3(K)$ surjects onto the $E_{3,0}^\infty$ term and $H_3(K) = 0$, the differential

$$d^3 : H_3(\pi) \rightarrow H_0(\pi; \pi_2(K))$$

must be injective. Recall that M_0 is the double of a thickening $Y := Y(K)$ of K , and we view M_0 as the boundary of $Y \times I$; this gives a map $M_0 \hookrightarrow Y \times I \simeq K$.

^aThe proof given for this lemma applies without change to infinite fundamental groups.

The reduced 2-type P is constructed by attaching cells of dimension 2 and higher to M_0 . We define

$$P^{(3)} := M_0 \cup \bigcup_{\alpha} e_{\alpha}^2 \cup \bigcup_{\beta} e_{\beta}^3,$$

as the union of M_0 with only the 2-cell and 3-cell from P . Since $H^3(P, M_0; \pi_2(K)) = 0$ by Lemma 8.2, obstruction theory tells us the map $M_0 \rightarrow K$ extends over $P^{(3)}$, and we obtain an induced map $H_0(\pi; \pi_2(P^{(3)})) \rightarrow H_0(\pi; \pi_2(K))$. By commutativity of the diagram below, the d^3 map in the spectral sequence converging to $H_*(P^{(3)})$ must also be injective.

$$\begin{array}{ccc} H_0(\pi; \pi_2(K)) & \longleftarrow & H_0(\pi; \pi_2(P^{(3)})) \\ \uparrow d^3 & & \uparrow d^3 \\ H_3(\pi) & \xlongequal{\quad} & H_3(\pi) \end{array}$$

It remains to compare the d^3 differentials for $H_*(P)$ and $H_*(P^{(3)})$. We claim that $\pi_2(P) \cong \pi_2(P^{(3)})$: the relative homologies $H^i(P, P^{(3)}; \Lambda)$ vanish in dimension $i = 2, 3$, so the isomorphism is given by the long exact sequence of the pair. The injectivity of $d_3 : H_3(\pi) \rightarrow H_0(\pi; \pi_2(P^{(3)}))$ implies that $d^3 : H_3(\pi) \rightarrow H_0(\pi; \pi_2(P))$ is also injective. □

The following lemma is used in the proof of Proposition 9.1.

Lemma 9.3. *Let M be a closed, oriented, TOP 4-manifold with $\pi_1(M) = \pi$, and let X be a closed, simply connected 4-manifold. Then the map $d^3 : H_3(\pi) \rightarrow H_0(\pi; \pi_2(P(M)))$ is injective if and only if the map $d^3 : H_3(\pi) \rightarrow H_0(\pi; \pi_2(P(M\#X)))$ is injective.*

Proof. We begin by comparing M and $M\#X$. By removing the top dimensional cells of M and $M\#X$, we get an inclusion $M^o \hookrightarrow (M\#X)^o$, the latter of which is just M^o wedged with a collection of n 2-spheres arising from X^o . This inclusion induces a split injection $\pi_2(M^o) \rightarrow \pi_2((M\#X)^o)$, and so $\pi_2(M)$ is stably isomorphic to $\pi_2(M\#X)$:

$$\pi_2(M\#X) \cong \pi_2((M\#X)^o) \cong \pi_2(M^o) \oplus \Lambda^n \cong \pi_2(M) \oplus \Lambda^n,$$

where $n = b_2(X)$. If $P(M)$ is the reduced 2-type of M , and $P(M\#X)$ is the reduced 2-type of $M\#X$, then it follows that $\pi_2(P(M))$ is stably isomorphic to $\pi_2(P(M\#X))$, and therefore $H_0(\pi; \pi_2(P(M))) \cong H_0(\pi; \pi_2(P(M\#X)))$. The conclusion about injectivity for the maps d^3 now follows by naturality of the spectral sequences with respect to the map $P(M) \rightarrow P(M\#X)$. □

Proof of Proposition 9.1. By Proposition 5.7, we have

$$M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}\#r(S^2 \times S^2) \approx M'\#_s(S^2 \times S^2).$$

where $M' = M_0\#_p\mathbb{C}P^2\#q\overline{\mathbb{C}P^2}$ is a suitable stabilization of the minimal model M_0 .

This homeomorphism allows us to equate their second homotopy groups, and thus identify their reduced 2-types. By applying Lemma 9.3 several times, the d^3 map for $P(M)$ is injective if and only if the d^3 map for $P(M_0)$ is injective, and Lemma 9.2 completes the proof. \square

Remark 9.4. The same d^3 map in the spectral sequence converging to $H_*(M)$ is injective as well, given by naturality of the spectral sequences under the map $M \rightarrow K$.

Proposition 9.5. *The integral homology $H_i(P)$, for $i \leq 5$, is given as follows:*

- (i) $H_0(P) = \mathbb{Z}$ and $H_1(P) = H_1(\pi)$.
- (ii) $H_2(P)$ is an extension of $H_2(\pi)$ by $\text{coker}\{d^3 : H_3(\pi) \rightarrow H_2(\tilde{P}) \otimes_{\Lambda} \mathbb{Z}\}$.
- (iii) $H_3(P) = H_5(P) = 0$
- (iv) $H_4(P) \cong H_4(\tilde{P}) \otimes_{\Lambda} \mathbb{Z}$.

In addition, $H_3(P; \mathbb{Z}/2) = 0$.

Proof. The table summarizes the calculations above. For right-angled Artin groups, the extension in (ii) is split. The argument showing that $H_3(P; \mathbb{Z}/2) = 0$ follows that same steps as above. The details will be left to the reader. \square

10. The Spin Bordism Groups $\Omega_*^{\text{Spin}}(P)$

The Atiyah–Hirzebruch spectral sequence is used to compute the topological (or smooth) spin bordism groups of the reduced 2-type $P = P(M)$, denoted as above by $\Omega_*^{\text{Spin}}(P)$. The E^2 -page of the spectral sequence is given by $H_p(P; \Omega_q^{\text{Spin}}(\text{pt}))$,

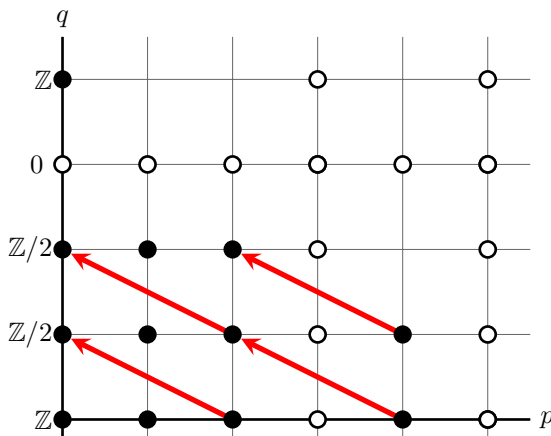


Fig. 2. The E^2 -page of the spectral sequence converging to $\Omega_*^{\text{Spin}}(P)$.

and the relevant spin bordism groups of a point are given below:

$$\Omega_q^{\text{Spin}}(\text{pt}) = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z} \quad \text{for } q = 0, 1, 2, 3, 4$$

The topological case differs from the smooth case only in the divisibility of the signature, so that $\Omega_4^{\text{TopSpin}}(\text{pt}) = 8\mathbb{Z}$ while $\Omega_4^{\text{Spin}}(\text{pt}) = 16\mathbb{Z}$.

In Fig. 2, we have included the information about $H_*(P)$ from the last section. The two d^2 maps on the left in the E^2 -page are both zero, otherwise $\Omega_*^{\text{Spin}}(\text{pt})$ would not split off in the E^∞ -page of $\Omega_*^{\text{Spin}}(P)$. The other two d^2 maps are the duals of the Sq^2 maps composed with reduction mod 2.

Consider the commutative diagram below that arises from the fibration $\tilde{P} \rightarrow P \rightarrow K(\pi, 1)$. We take homology with $\mathbb{Z}/2$ -coefficients.

$$\begin{array}{ccccccc}
 H_4(\tilde{P}) & \longrightarrow & H_2(\tilde{P}) & & & & \\
 \downarrow & & \downarrow & & & & \\
 H_4(P) & \xrightarrow{DSq^2} & H_2(P) & \longrightarrow & \text{coker}(DSq^2) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \vdots & & \\
 0 = H_4(\pi) & \longrightarrow & H_2(\pi) & \xlongequal{\quad} & H_2(\pi) & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

The above map labeled DSq^2 is the dual of $Sq^2 : H^2(P) \rightarrow H^4(P)$. Since \tilde{P} is a restricted product of copies of $\mathbb{C}P^\infty$ and inverse limits are left exact, $H^2(\tilde{P})$ injects into $H^4(\tilde{P})$, and thus $H_4(\tilde{P})$ surjects onto $H_2(\tilde{P})$. But $H_4(\pi) = 0$ by assumption, so by exactness of the middle vertical sequence, $\text{coker}(DSq^2) \cong H_2(\pi; \mathbb{Z}/2)$. Since $H_5(P; \mathbb{Z}) = 0$, the term $H_2(\pi; \mathbb{Z}/2)$ survives to $\Omega_4^{\text{Spin}}(P)$.

Proposition 10.1. *The topological or smooth spin bordism groups of the reduced normal 2-type $P = P(M)$ for M are detected by an injection*

$$\Omega_4^{\text{Spin}}(P) \subseteq \mathbb{Z} \oplus H_2(\pi; \mathbb{Z}/2) \oplus H_4(P; \mathbb{Z}).$$

The invariants are the signature, an invariant in $H_2(\pi; \mathbb{Z}/2)$, and the fundamental class $c_*[M] \in H_4(P; \mathbb{Z})$.

Proof. This follows from our discussion of the differentials in the spectral sequence. □

The elements of $\Omega_4^{\text{Spin}}(P)$ are represented by pairs (N, f) , where N is a closed, spin 4-manifold and $f : N \rightarrow P$ is a reference map. We will always assume that f induces an isomorphism of fundamental groups.

For spin^+ manifolds (recall Definition 6.7), we will show that the bordism invariant in $H_2(\pi; \mathbb{Z}/2)$ is determined by the other invariants. The method follows [20,

Sec. 5], where the authors define a subset $\Omega_4(P)_M \subset \Omega_4^{\text{Spin}}(P)$, called the *normal structures*. We need to adapt this notion to our setting.

Definition 10.2. Let M be a spin^+ manifold. The set of *normal structures* $\Omega_4(P)_M \subset \Omega_4^{\text{Spin}}(P)$ consists of the spin bordism classes (N, f) , where N has even reduced w_2 -type, such that $f_*[N] = c_*[M]$ and $\text{sign}(N) = \text{sign}(M)$.

We note this set is non-empty, since $(M, c) \in \Omega_4(P)_M$ whenever M is a spin^+ manifold. In Lemma 10.7, we will check that the definition of a normal structure is independent of the choice of representative (N, f) up to spin bordism.

Here is a useful observation.

Lemma 10.3. *Let (N, f) represent an element of $\Omega_4^{\text{Spin}}(P)$. If $f_*[N] = c_*[M]$, then the reduced intersection form $(\pi_2(N)^*, h_N)$ restricted to the image of $f^* : \pi_2(P)^* \rightarrow \pi_2(N)^*$ is isometric to $(\pi_2(M)^*, h_M)$.*

Proof. Since $f_*[N] = c_*[M] \in H_4(P; \mathbb{Z})$, we have the following commutative diagram:

$$\begin{array}{ccccc}
 \pi_2(M) & \xleftarrow[\cap[M]]{\approx} & H^2(M; \Lambda) & \xrightarrow{\text{eval}} & \pi_2(M)^* \\
 \downarrow c_* & & \uparrow c^* & & \uparrow c^* \approx \\
 \pi_2(P) & \xleftarrow[\cap c_*[M]]{} & H^2(P; \Lambda) & \xrightarrow{\text{eval}} & \pi_2(P)^* \\
 \uparrow f_* & & \downarrow f^* & & \downarrow f^* \\
 \pi_2(N) & \xleftarrow[\cap[N]]{\approx} & H^2(N; \Lambda) & \xrightarrow{\text{eval}} & \pi_2(N)^*
 \end{array} \tag{10.1}$$

The composite on the top row defines $\text{ad } s_M$, and the composite on the bottom row defines $\text{ad } s_N$. After dualizing each term in the diagram, each square still commutes, and the upper vertical maps (c_* and c^*) as well as all the horizontal maps become isomorphisms. The composites in the top and bottom rows of the dualized diagram give $(\text{ad } s_M)^*$ and $(\text{ad } s_N)^*$, respectively.

Therefore, the inverses of the dualized top and bottom row composites give $\text{ad } h_M$ and $\text{ad } h_N$, respectively. One can check by a diagram chase that the inverse of the dualized composite in the middle row is isometric via (c^*, c^{**}) to the adjoint of h_M . A similar diagram chase shows that the inverse of this middle row composite is also isometric to the pull-back of the adjoint of h_N via (f^*, f^{**}) . \square

Corollary 10.4. *If $f : N \rightarrow P$ is a reduced 3-equivalence and $f_*[N] = c_*[M]$, then $(\pi_2(M)^*, h_M) \cong (\pi_2(N)^*, h_N)$.*

Proof. We have the isomorphisms:

$$(\pi_2(N)^*, h_N) \cong (\pi_2(P)^*, (h_M)^*) \cong (\pi_2(M)^*, h_M)$$

induced by the maps in the diagram (10.1). \square

We will now define a map $\theta_M : \Omega_4(P)_M \rightarrow L_4(\mathbb{Z}[\pi])$ as in [20, Definition 5.8]. Let (N, f) represent an element of $\Omega_4^{\text{Spin}}(P)$. After preliminary surgeries, we may assume that the map f is 2-connected, so that $(\pi_2(N)^*, h_N)$ is stabilized by direct sum with a hyperbolic form $H(\Lambda^r)$. Note that if N is a spin^+ manifold, then the stabilized manifold also has even reduced w_2 -type. Let

$$V := \ker(f_* : \pi_2(N) \rightarrow \pi_2(P)).$$

Since $\pi_2(P)$ is a stably free Λ -module and f_* is surjective, this sequence splits. After applying $\text{Hom}_\Lambda(-, \Lambda)$, we obtain a (split) short exact sequence

$$0 \rightarrow \pi_2(P)^* \xrightarrow{f^*} \pi_2(N)^* \rightarrow V^* \rightarrow 0. \tag{10.2}$$

Since $\pi_2(N)^*$ is stably free, so is the module V^* .

Definition 10.5. Suppose that (N, f) has 2-connected reference map. Let $(V^*, \lambda_{N,f})$ denote the restriction of the form $(\pi_2(N)^*, h_N)$ to the orthogonal complement of the submodule $\pi_2(P)^*$.

Lemma 10.6. *If M is a spin^+ manifold and $(N, f) \in \Omega_4(P)_M$ is a normal structure with 2-connected reference map, then $(V^*, \lambda_{N,f})$ is a non-singular even hermitian form on a finitely generated, stably free Λ -module.*

Proof. By Lemma 10.3, the form h_N restricted to the image of f^* is isometric to h_M . Since h_M is non-singular, we obtain an isometric splitting

$$(\pi_2(N)^*, h_N) \cong (\pi_2(M)^*, h_M) \oplus (V^*, \lambda_{N,f})$$

and therefore $(V^*, \lambda_{N,f})$ is a non-singular form. Since h_N is even by assumption, the form $(V^*, \lambda_{N,f})$ is also even. □

We now return to detecting the bordism invariant in $H_2(\pi; \mathbb{Z}/2)$ for the elements $(N, f) \in \Omega_4(P)_M$. Let (N, f) be a normal structure representing an element of $\Omega_4(P)_M$, where M is a spin^+ manifold. Lemma 10.6 provides a non-singular even hermitian form $(V^*, \lambda_{N,f})$ on a finitely generated, stably free Λ -module. In particular, the form $\lambda_{N,f}$ admits a unique quadratic refinement. We define $\theta_M : \Omega_4(P)_M \rightarrow \tilde{L}_4(\mathbb{Z}[\pi])$ by setting

$$\theta_M(N, f) = (V^*, \lambda_{N,f}) \in \tilde{L}_4(\mathbb{Z}[\pi]),$$

where $\tilde{L}_4(\mathbb{Z}[\pi])$ denotes the reduced surgery obstruction group (represented by quadratic forms with signature zero), for surgery up to homotopy equivalence.

Lemma 10.7. *Let M be a spin^+ manifold. The map θ_M is well-defined. In addition, $\theta_M(N, f) = 0$ if (N, f) is spin bordant over P to a reduced 3-equivalence.*

Proof. If (N, f) and (N', f') are spin bordant over P with 2-connected reference maps, then N and N' are stably homeomorphic over P by [29, Corollary 3].

However, an arbitrary element $(N, f) \in \Omega_4^{\text{Spin}}(P)$ is spin bordant to one with 2-connected reference map, with the property that s_N is stabilized by a hyperbolic form. Hence the parity (even or odd) of h_N is preserved by this operation.

It follows that any element (N', f') which is spin bordant to $(N, f) \in \Omega_4(P)_M$ is also a normal structure. The stable homeomorphism argument now shows that θ_M is well-defined (compare with [20, Lemma 5.9]).

Now, suppose that $f : N \rightarrow P$ is a reduced 3-equivalence, and that $(N', f') \in \Omega_4(P)_M$ is a 2-connected representative in the same spin bordism class. We have

$$f_*[N] = f'_*[N'] = c_*[M] \in H_4(P).$$

Lemma 10.3 provides the isometric direct sum splittings:

$$(\pi_2(N)^*, h_N) \oplus H(\Lambda^r) = (\pi_2(N')^*, h_{N'}) \cong (\pi_2(M)^*, h_M) \oplus (V^*, \lambda_{N,f})$$

and the isomorphisms:

$$(\pi_2(N)^*, h_N) \cong (\pi_2(P)^*, (h_M)^*) \cong (\pi_2(M)^*, h_M)$$

which give a relation in the L -group, since all these forms are non-singular and even hermitian. It follows that $[(V^*, \lambda_{N,f})] = [H(\Lambda^r)] = 0 \in L_4(\mathbb{Z}[\pi])$, and $\theta_M(N, f) = 0$. □

The next step is to define a map

$$\rho_M : \Omega_4(P)_M \rightarrow H_2(\pi; \mathbb{Z}/2)$$

on an element $[N, f]$ by the projection of the difference $[N, f] - [M, c]$ from

$$\ker(\Omega_4^{\text{Spin}}(P) \rightarrow H_4(P; \mathbb{Z}))$$

to the subquotient $E_{\infty}^{2,2} = H_2(\pi; \mathbb{Z}/2)$ in the Atiyah–Hirzebruch spectral sequence.

Lemma 10.8. $\rho_M : \Omega_4(P)_M \rightarrow H_2(\pi; \mathbb{Z}/2)$ is an injection.

Proof. This follows from Proposition 10.1 and the definition of the subset $\Omega_4(P)_M$. □

Similarly, we define $\Omega_4(M)_M$ and obtain a bijection

$$\hat{\rho}_M : \Omega_4(M)_M \xrightarrow{\cong} H_2(M; \mathbb{Z}/2),$$

since elements of $\Omega_4(M)_M$ are represented by degree one normal maps. The map

$$\hat{\theta}_M : \Omega_4(M)_M \rightarrow \tilde{L}_4(\mathbb{Z}[\pi])$$

is defined by $\hat{\theta}_M(N, g) = \theta_M(N, c \circ g)$, where $g : N \rightarrow M$ represents a bordism element in $\Omega_4(M)_M$. Recall that there is a “universal” assembly map homomorphism

$$\kappa_2 : H_2(\pi; \mathbb{Z}/2) \rightarrow L_4(\mathbb{Z}[\pi])$$

defined for any group (see [27, Sec. 3]). We have a version of [20, Lemma 5.11] in our setting.

Lemma 10.9. $\theta_M = \kappa_2 \circ \rho_M$.

Proof. The following diagram commutes:

$$\begin{array}{ccc}
 & & L_4(\mathbb{Z}[\pi]) \\
 & \hat{\theta}_M \curvearrowright & \\
 \Omega_4(M)_M & \xrightarrow{c_*} & \Omega_4(P)_M \\
 \downarrow \hat{\rho}_M & & \downarrow \rho_M \\
 H_2(M; \mathbb{Z}/2) & \xrightarrow{c_*} & H_2(\pi; \mathbb{Z}/2)
 \end{array}$$

θ_M (arrow from $\Omega_4(P)_M$ to $L_4(\mathbb{Z}[\pi])$)
 κ_2 (arrow from $H_2(\pi; \mathbb{Z}/2)$ to $L_4(\mathbb{Z}[\pi])$)

The outer composition $\hat{\theta}_M = \kappa_2 \circ c_* \circ \hat{\rho}_M$ holds by the same argument given in the proof of [20, Lemma 5.11]. The elements in $\Omega_4(M)_M$ are represented by degree 1 normal maps (N, g) covered by a bundle map $\nu_N \rightarrow \nu_M$, since $g^*(\nu_M) \cong \nu_N$ by [11, 26]. The required formula now follows from Wall’s characteristic class formula for surgery obstructions (see [9]).

The map $c_* : H_2(M; \mathbb{Z}/2) \rightarrow H_2(\pi; \mathbb{Z}/2)$ is surjective, $\hat{\rho}_M$ is a bijection, and ρ_M is an injection. Hence the formula $\theta_M = \kappa_2 \circ \rho_M$ follows from the commutativity of the inner square. □

Corollary 10.10. *Let M be a spin^+ manifold, and let $P = P(M)$. Suppose that $[N, f] \in \Omega_4^{\text{Spin}}(P)$, with f a reduced 3-equivalence and $\text{sign}(N) = \text{sign}(M)$. If $f_*[N] = c_*[M]$ and the map $\kappa_2 : H_2(\pi; \mathbb{Z}/2) \rightarrow L_4(\mathbb{Z}[\pi])$ is injective, then $[N, f] = [M, c] \in \Omega_4^{\text{Spin}}(P)$.*

Proof. This is a version of [20, Corollary 5.12] and the proof is analogous. Since f is a reduced 3-equivalence and $f_*[N] = c_*[M]$, Corollary 10.4 implies that h_N and h_M are isometric (via the given reference maps), and hence (N, f) is a normal structure. The bordism group $\Omega_4^{\text{Spin}}(P)$ is detected by Proposition 10.1, and the difference $[M, c] - [N, f]$ projects to zero in $H_4(P; \mathbb{Z})$ since $f_*[N] = c_*[M]$. By definition, the map $\rho_M(N, f)$ is the projection of the difference $[M, c] - [N, f]$ to the subquotient $H_2(\pi; \mathbb{Z}/2)$. Since f is a reduced 3-equivalence, $\theta_M(N, f) = 0$ by Lemma 10.7, and since κ_2 is injective, $\rho_M(N, f)$ must be zero by Lemma 10.9. Furthermore, since $\text{sign}(N) = \text{sign}(M)$, the elements $[N, f]$ and $[M, c]$ are bordant in $\Omega_4(P)$. □

11. Classification of Spin^+ 4-Manifolds with $\text{cd } \pi_1(M) \leq 3$

In this section, we state and prove our main classification result, Theorem 11.2, for spin^+ topological 4-manifolds with $\text{cd } \pi_1(M) \leq 3$. We conclude by proving Theorems A and B.

For a finitely presented group π with $\text{cd } \pi \leq 3$, let $b_3(\pi)$ denote the minimum number of generators for $H^3(\pi; \Lambda)$ as a Λ -module. Note that $b_3(\pi)$ is bounded above by the Λ -rank of $C_3(L)$, where L is a minimal aspherical 3-complex with $\pi_1(L) = \pi$. We recall a definition from [20].

Definition 11.1. A group π satisfies properties (W-AA) whenever

- (i) The Whitehead group $\text{Wh}(\pi)$ vanishes.
- (ii) The assembly map $A_5 : H_5(\pi; \mathbb{L}_0) \rightarrow L_5(\mathbb{Z}[\pi])$ is surjective.
- (iii) The assembly map $A_4 : H_4(\pi; \mathbb{L}_0) \rightarrow L_4(\mathbb{Z}[\pi])$ is injective.

These properties hold whenever the group π is torsion-free and satisfies the Farrell–Jones isomorphism conjectures in K -theory and L -theory. These conjectures have been verified for many classes of groups, and in particular for all right-angled Artin groups (see [1, 2]).

Theorem 11.2. *Let π be a finitely presented group with $\text{cd } \pi \leq 3$ satisfying the properties (W-AA). If M and N are closed, oriented, spin^+ , TOP 4-manifolds with fundamental group π , then any isometry between the quadratic 2-types of M and N is stably realized by an s -cobordism between $M \# r(S^2 \times S^2)$ and $N \# r(S^2 \times S^2)$, for $r \geq b_3(\pi)$.*

Remark 11.3. Note that the main theorem of [20] applies to groups π with $\text{cd } \pi = 2$, unless π has geometric dimension 3. Any such group would be a counterexample to the famous Eilenberg–Ganea conjecture.

We divide the proof of Theorem 11.2 into the following steps:

- (i) Reduced 3-equivalences $c_M : M \rightarrow P$ and $c_N : N \rightarrow P$ arising from an isometry $(\alpha, \beta) : Q(M) \cong Q(N)$ of quadratic 2-types satisfy

$$(c_M)_*[M] = (c_N)_*[N] \in H_4(P; \mathbb{Z}).$$

This is the corresponding result to [20, Theorem 5.13], and the proof follows a similar outline (see details below). Note however that [20, Lemma 5.16] intended to cite the paper of Whitehead [36, p. 62] for the properties used of the Γ -functor.

- (ii) The assumption in (W-AA) that the assembly map $A_4 : H_4(\pi; \mathbb{L}_0) \rightarrow L_4(\mathbb{Z}[\pi])$ is injective implies that the map $\kappa_2 : H_2(\pi; \mathbb{Z}/2) \rightarrow L_4(\mathbb{Z}[\pi])$ is injective.
- (iii) Suppose that M and N are closed, oriented, spin^+ , topological (or smooth) 4-manifolds. If M and N have isometric quadratic 2-types, then there are reduced normal 2-smoothings $M \rightarrow P$ and $N \rightarrow P$ which are bordant in $\Omega_4^{\text{Spin}}(P)$. This follows as in [20, Corollary 5.14] adapted to our setting: the details are given in Proposition 7.3 and Corollary 10.10. The assumption that M and N have even reduced w_2 -type is used essentially in this step (see Sec. 10).
- (iv) We show how to apply [29, Theorem 4, p. 735] of Kreck’s modified surgery theory to obtain an s -cobordism between M and N , after a specified stabilization.

For step (i), we recall that the key ingredient in the proof of [20, Theorem 5.13] is the injectivity of the cap product map:

$$\omega_P : H_4(P; \mathbb{Z}) \rightarrow \text{Hom}_\Lambda(H^2(P; \Lambda), H_2(P; \Lambda)).$$

defined by cap product with the class $\text{tr}(\alpha) \in H_4^{lf}(\tilde{P}; \mathbb{Z})^\pi$, for any class $\alpha \in H_4(P; \mathbb{Z})$. The formula

$$b_\alpha(u, v) = \langle u \cup v, \text{tr}(\alpha) \rangle, \quad \text{for } u, v \in H^2(P; \Lambda),$$

defines a symmetric bilinear cup product form

$$H^2(P; \Lambda) \times H^2(P; \Lambda) \rightarrow \mathbb{Z},$$

with $\text{ad } b_\alpha : H^2(P; \Lambda) \rightarrow H^2(P; \Lambda)^*$ given by cap product with $\text{tr}(\alpha)$, followed by the evaluation map $\hat{e} : H_2(P; \Lambda) \rightarrow H^2(P; \Lambda)^*$. Since $H_2(P; \Lambda) \cong \pi_2(P)$ is a stably-free Λ -module, we have the isomorphisms:

$$\text{Hom}_\Lambda(H^2(P; \Lambda), H_2(P; \Lambda)) \xleftarrow{\cong} \text{Hom}_\Lambda(\pi_2(P)^*, \pi_2(P))$$

induced by the dual of the evaluation map $e : H^2(P; \Lambda) \rightarrow H_2(P; \Lambda)^*$ and the calculations in the proof of Lemma 6.1. The two evaluation maps are related by the formula $\hat{e} = e^* \circ \gamma$, where $\gamma : H_2(P; \Lambda) \rightarrow H_2(P; \Lambda)^{**}$ is the ‘‘double-dual’’ isomorphism. We have a natural inclusion

$$\Gamma(\pi_2(P))^\pi \subset \text{Hom}_\Lambda(\pi_2(P)^*, \pi_2(P))$$

by the properties of the Γ -functor (see [36, p. 62; 20, p. 144]).

The same construction applied to the fundamental class $[M] \in H_4(M; \mathbb{Z})$ defines the π -invariant cup product form $b_M(u, v) = \varepsilon_1(s_M(Du, Dv))$, for $u, v \in H^2(M; \Lambda)$, where $\varepsilon_1 : \Lambda \rightarrow \mathbb{Z}$ gives the coefficient at the identity element of π and $D : H^2(M; \Lambda) \rightarrow H_2(M; \Lambda)$ is the Poincaré duality isomorphism. We note that there exists an isometry $s_M \cong s_N$ of equivariant intersection forms if and only if b_M and b_N are equivariantly isometric.

The map ω_P will be applied to the image $(c_M)_*[M]$ of the fundamental class, whose cap product is given by the composite

$$\begin{array}{ccc}
 H^2(M; \Lambda) & \xrightarrow[\cong]{\cap \text{tr}[M]} & H_2(M; \Lambda) \\
 \uparrow c^* & & \downarrow c_* \\
 H^2(P; \Lambda) & \xrightarrow{\cap c_* \text{tr}[M]} & H_2(P; \Lambda),
 \end{array} \tag{11.1}$$

and similarly for ω_P applied to $(c_N)_*[N]$. It follows that there is a commutative diagram (with Λ -coefficients understood):

$$\begin{array}{ccc}
 H_4(M) \oplus H_4(N) & \xrightarrow{\omega_M \oplus \omega_N} & \text{Hom}_\Lambda(H^2(M), H_2(M)) \oplus \text{Hom}_\Lambda(H^2(N), H_2(N)) \\
 \downarrow & & \downarrow \\
 H_4(P; \mathbb{Z}) & \xrightarrow{\omega_P} & \text{Hom}_\Lambda(H^2(P), H_2(P))
 \end{array} \tag{11.2}$$

where the vertical maps are the sum of the natural maps induced by the reference maps c^* and c_* for M and N , respectively. If M and N have isometric quadratic

2-types, then $b_M \cong b_N$ over P and the difference element

$$(c_M)_*[M] - (c_N)_*[N] \in H_4(P; \Lambda)$$

maps to zero under ω_P by the diagram. Therefore, $(c_M)_*[M] = (c_N)_*[N]$ by the injectivity of ω_P . This completes our outline of step (i).

By applying steps (i)–(ii) and Corollary 10.10, we may assume that step (iii) is completed. We note that the argument applied to smooth 4-manifolds produces a smooth spin bordism over P . The final step (iv) only works in the topological category.

The difficulty in step (iv) is that our reduced normal 2-smoothings are not given by 2-connected reference maps $M \rightarrow P$ and $N \rightarrow P$, so the modified surgery result does not apply directly. We now show how a limited amount of stabilization can be used to get 2-connected reference maps, and thus complete the proof of Theorem 11.2.

First, we construct an abstract diagram using an exact sequence

$$0 \rightarrow A \rightarrow B \xrightarrow{g} C \rightarrow D \rightarrow 0$$

and a factorization $B \xrightarrow{j} V \rightarrow C$ of the map $g : B \rightarrow C$.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{j} & V \longrightarrow 0 \\
 & & \downarrow & & \downarrow & \searrow g & \downarrow \\
 0 & \longrightarrow & K & \longrightarrow & B \oplus F & \xrightarrow{f} & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & \nearrow \phi & \downarrow \\
 0 & \longrightarrow & E & \longrightarrow & F & \xrightarrow{\bar{\phi}} & D \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array} \tag{11.3}$$

Let $f(b, x) = g(b) + \phi(x)$, for $b \in B$ and $x \in F$. The map $\phi : F \rightarrow C$ is a lifting of a surjective map $\bar{\phi} : F \rightarrow D$ from a free Λ -module $F \cong \Lambda^r$. Let $E = \ker \bar{\phi}$. We will apply this diagram to the universal coefficient sequence

$$0 \rightarrow H^2(\pi; \Lambda) \rightarrow \pi_2(M) \rightarrow \pi_2(M)^* \rightarrow H^3(\pi; \Lambda) \rightarrow 0$$

from (8.1), so that $A = H^2(\pi; \Lambda)$, $B = \pi_2(M)$, $C = \pi_2(P)$ and $D = H^3(\pi; \Lambda)$. The map $g = \text{ad } s_M$, and in our application $\bar{\phi} : F \rightarrow H^3(\pi; \Lambda)$ will be given by a (minimal) set of r generators for $H^3(\pi; \Lambda)$ as a Λ -module. We denote the lifting ϕ by $\phi_M : F \rightarrow \pi_2(P)$ and the resulting map f by $f_M = \text{ad } s_M + \phi_M$.

Remark 11.4. Let $\beta : \pi_2(M) \rightarrow \pi_2(N)$ be an isometry of the equivariant intersection forms arising from our assumption that $Q(M) \cong Q(N)$. We identify

$\pi_1(M) = \pi_1(N) = \pi$. Then

$$\beta^* \circ \text{ad } s_N \circ \beta = \text{ad } s_M,$$

and we have reference maps $(c_M)_* = \text{ad } s_M$, $(c_N)_* = \beta^* \circ \text{ad } s_N$ as in the proof of Proposition 7.3. It follows that $\beta^* = (c_M)^* \circ ((c_N)^*)^{-1} : \pi_2(N)^* \rightarrow \pi_2(M)^*$ induces an isometry

$$\beta^{**} \circ \text{ad } h_M \circ \beta^* = \text{ad } h_N$$

of the reduced intersection forms. After choosing the map $\phi_M : F \rightarrow \pi_2(P) = \pi_2(M)^*$, we define $\phi_N = (\beta^*)^{-1} \circ \phi_M : F \rightarrow \pi_2(N)^*$. It follows that the pull-back forms $(\phi_M)^*(h_M) = (\phi_N)^*(h_N)$ are equal. The remaining map $f_N : \pi_2(N) \oplus F \rightarrow \pi_2(N)^*$ used in diagram (11.3) is defined by $f_N = \text{ad } s_N + \phi_N$.

Proof of Theorem 11.2. Let (W, Φ) be a spin bordism over P from (M, c_M) to (N, c_N) , with reference map $\Phi : W \rightarrow P$. Since the reference maps c_M and c_N on the boundary of W are not 2-connected (whenever $H^3(\pi; \Lambda) \neq 0$), we modify our problem by applying the algebra above to our geometric setting. We form

$$M' = M \# r(S^2 \times S^2)$$

by performing surgery on null-homotopic circles in M . Let $\theta_M = -(\phi_M)^*(h_M)$ denote the hermitian form on F pulled back from *minus* the reduced intersection form

$$h_M : L_M \times L_M \rightarrow \Lambda$$

on $L_M = \pi_2(M)^*$, so that the map ϕ_M is an isometry with respect to $-h_M$. Since $w_M = 0$ and F is a free Λ -module, we may write $\theta_M = \lambda + \lambda^*$ for some sesquilinear form (F, λ) . Let $\{a_1, \dots, a_r\}$ denote a base for F , and $\{b_1, \dots, b_r\}$ denote the dual base for F^* .

Define a hermitian form $(F \oplus F^*, \psi_M)$ by the formulas $\psi_M(a_i, a_j) = \theta_M(a_i, a_j)$ and $\psi_M(a_i, b_j) = \delta_{ij}$, $\psi_M(b_i, b_j) = 0$. Let $\{e_1, \dots, e_r, f_1, \dots, f_r\}$ denote the standard hyperbolic base for $H(\Lambda^r)$, and define an isometry

$$k : (F \oplus F^*, \psi_M) \rightarrow H(\Lambda^r)$$

by the explicit formulas $k(a_i) = e_i + \sum_j \alpha_{ij} f_j$, where $\lambda = (\alpha_{ij})$ in matrix form, and $k(b_i) = f_i$. Let $k_0 = p_F \circ k^{-1}$, where $p_F : F \oplus F^* \rightarrow F$ is the first factor projection. The reference map $c_M : M \rightarrow P$, with induced map

$$(c_M)_* : \pi_2(M) \xrightarrow{\text{ad } s_M} \pi_2(M)^* = \pi_2(P)$$

can be extended to a 2-connected reference map $c_{M'} : M' \rightarrow P$, with induced map

$$(c_{M'})_* : \pi_2(M') = \pi_2(M) \oplus H(\Lambda^r) \xrightarrow{\text{id} \oplus k_0} \pi_2(M) \oplus F \xrightarrow{f_M} \pi_2(P), \quad (11.4)$$

using the map $f_M : \pi_2(M) \oplus F \rightarrow \pi_2(P)$ from diagram (11.3). Recall that $B = \pi_2(M)$, $C = \pi_2(P)$ and $f_M = \text{ad } s_M + \phi_M$. The trace of the surgeries provides a

spin bordism (Y, φ_M) from (M, c_M) to $(M', c_{M'})$ over P , where $\varphi_M : Y \rightarrow P$ is 2-connected.

By the same construction, the given reference map $c_N : N \rightarrow P$ may be extended to a 2-connected reference map $c_{N'} : N' \rightarrow P$, after forming the connected sum, where $N' = N \# r(S^2 \times S^2)$, with induced map

$$(c_{N'})_* : \pi_2(N') = \pi_2(N) \oplus H(\Lambda^r) \xrightarrow{\text{id} \oplus k_0} \pi_2(N) \oplus F \xrightarrow{f_N} \pi_2(N)^* \xrightarrow{\beta^*} \pi_2(P),$$

using the corresponding f_N from diagram (11.3). Note that since $\phi_N = (\beta^*)^{-1} \circ \phi_M$, we have $\theta_N = -(\phi_N)^* = \theta_M$ and $\psi_N = \psi_M$.

We obtain a spin bordism (Z, φ_N) from $(N', c_{N'})$ to (N, c_N) over P , with reference map $\varphi_N : Z \rightarrow P$. Now, the union of these three bordisms

$$(W', \Phi') = (-Y \cup W \cup Z, \varphi_M \cup \Phi \cup \varphi_N)$$

provides a spin bordism from $(M', c_{M'})$ to $(N', c_{N'})$, with 2-connected reference maps $c_{M'}$ and $c_{N'}$ on the two boundary components.

Definition 11.5. Let P' denote the fibration over $K(\pi, 1)$ defined by $\pi_2(P') = \pi_2(P) \oplus F^*$, with k -invariant pushed forward from P via the inclusion $\pi_2(P) \rightarrow \pi_2(P')$ onto the first summand.

Note that we also have a map $P' \rightarrow P$ induced by the first factor projection

$$p_1 : \pi_2(P') = \pi_2(P) \oplus F^* \rightarrow \pi_2(P).$$

The reference map $c_M : M \rightarrow P$, with induced map

$$(c_M)_* : \pi_2(M) \xrightarrow{\text{ad } s_M} \pi_2(M)^* = \pi_2(P)$$

can be extended to a 2-connected reference map $\hat{c}_{M'} : M' \rightarrow P'$, with induced map

$$\begin{aligned} (\hat{c}_{M'})_* : \pi_2(M') = \pi_2(M) \oplus H(\Lambda^r) &\xrightarrow{\text{id} \oplus k^{-1}} \pi_2(M) \oplus F \oplus F^* \\ &\xrightarrow{f_M \oplus \text{id}_{F^*}} \pi_2(P) \oplus F^* \end{aligned} \tag{11.5}$$

with the property that $(c_{M'})_* = p_1 \circ (\hat{c}_{M'})_*$.

The same construction will be applied to $(N', c_{N'})$ to produce a 2-connected reference map to P' . The next step is to compare $(M', \hat{c}_{M'})$ and $(N', \hat{c}_{N'})$.

Claim 1. *The reference maps $(M', \hat{c}_{M'})$ and $(N', \hat{c}_{N'})$ induce an isometry $s_{M'} \cong s_{N'}$ of equivariant intersection forms over P' .*

Proof. We check directly from the definitions that the isometry

$$\beta \oplus \text{id} : (\pi_2(M), s_M) \oplus H(\Lambda^r) \xrightarrow{\cong} (\pi_2(N), s_N) \oplus H(\Lambda^r)$$

arising from a given isometry $(\text{id}, \beta) : Q(M) \cong Q(N)$ of quadratic 2-types is compatible with the reference maps to P' , as defined above. Since $\psi_M = \psi_N$, this

reduces to checking the equation

$$f_M = \beta^* \circ f_N \circ (\beta \oplus \text{id}) : \pi_2(M) \oplus F \rightarrow \pi_2(P),$$

by verifying that

$$\begin{aligned} (\beta^* \circ f_N \circ (\beta \oplus \text{id}))(b, x) &= (\beta^* \circ f_N)(\beta(b), x) = \beta^*(\text{ad } s_N(\beta(b)) + \beta^*(\phi_N(x))) \\ &= \text{ad } s_M(b) + \phi_M(x) = f_M(b, x) \end{aligned}$$

for all $b \in \pi_2(M)$ and all $x \in F$. □

Claim 2. *The elements $(M', \hat{c}_{M'})$ and $(N', \hat{c}_{N'})$ are spin bordant over P' .*

Proof. We already have a spin bordism (W', Φ') from $(M', c_{M'})$ to $(N', c_{N'})$ over P , so the difference element

$$\alpha := [(N', \hat{c}_{N'})] - [(M', \hat{c}_{M'})]$$

is in the kernel of the natural map $\Omega_4^{\text{Spin}}(P') \rightarrow \Omega_4^{\text{Spin}}(P)$ induced by the map $P' \rightarrow P$ defined above. By the Atiyah–Hirzebruch spectral sequence (as in Sec. 10), we obtain an injection

$$\Omega_4^{\text{Spin}}(P') \rightarrow \Omega_4^{\text{Spin}}(P) \oplus H_4(P'; \mathbb{Z})$$

and the image of $\alpha \in \Omega_4^{\text{Spin}}(P')$ in $H_4(P'; \mathbb{Z})$ is determined by the difference of the fundamental classes $(\hat{c}_{N'})_*[N'] - (\hat{c}_{M'})_*[M'] \in H_4(P'; \mathbb{Z})$.

By Claim 1, we have an isometry $s_{M'} \cong s_{N'}$ over P' , and hence an equivariant isometry $b_{M'} \cong b_{N'}$ over P' . Moreover, we have the analogue of diagram (11.2) for M', N' and P' , and the map $\omega_{P'}$ is again injective. The same argument used in our outline of step (i) now applies to show that $(\hat{c}_{N'})_*[N'] = (\hat{c}_{M'})_*[M'] \in H_4(P'; \mathbb{Z})$, as required. □

We now have a spin bordism $(\widehat{W}', \widehat{\Phi}')$ from $(M', \hat{c}_{M'})$ to $(N', \hat{c}_{N'})$, with 2-connected reference maps $\hat{c}_{M'}$ and $\hat{c}_{N'}$ on the two boundary components. After further surgeries relative to the boundary, we may assume that $\widehat{\Phi}' : \widehat{W}' \rightarrow P'$ is also 2-connected. Diagram (11.3) also determines the structure of

$$KH_2(M') := \ker((\hat{c}_{M'})_* : \pi_2(M') \rightarrow \pi_2(P')).$$

Since the free module $F \cong (\Lambda^r \times \{0\})$ is required to map surjectively onto $H^3(\pi; \Lambda)$, we may take $r = b_3(\pi)$.

Claim 3. *The form $s_{M'}$ restricted to $\ker(\hat{c}_{M'})_* \subset \pi_2(M')$ is identically zero.*

Proof. Since $\ker(\hat{c}_{M'})_* \cong K = \ker f_M$ (from Diagram (11.3)), it is enough to show that the form on $K = \ker f_M$ induced by

$$(\pi_2(M'), s_{M'}) \cong (\pi_2(M), s_M) \oplus (F \oplus F^*, \psi_M)$$

is identically zero. To check this, we compute the hermitian form $s_M \oplus \psi_M \cong s_{M'}$ on elements $z = (b, x, 0)$ and $z' = (b', x', 0)$ of $K \subset \pi_2(M) \oplus F \oplus F^*$ via

$$\begin{aligned} (s_M \oplus \psi_M)(z, z') &= s_M(b, b') + \theta_M(x, x') = s_M(b, b') - h_M(\phi_M(x), \phi_M(x')) \\ &= s_M(b, b') - \text{ad } h_M(\phi_M(x))(\phi_M(x')). \end{aligned}$$

But $\text{ad } s_M(b) = -\phi_M(x)$ and $\text{ad } s_M(b') = -\phi_M(x')$, since our elements lie in $K = \ker f$, so we obtain

$$\begin{aligned} (s_M \oplus \psi_M)(z, z') &= s_M(b, b') - \text{ad } h_M(\text{ad } s_M(b))(\text{ad } s_M(b')) \\ &= s_M(b, b') - \langle \gamma(b), \text{ad } s_M(b') \rangle = 0 \end{aligned}$$

by the formula in (6.3). □

We now have the setting to apply Kreck [29, Theorem 4, p. 735], which measures the obstruction to surgery on $(\widehat{W}', \widehat{\Phi}')$, relative to the boundary, to obtain an s -cobordism between $(M', \hat{c}_{M'})$ and $(N', \hat{c}_{N'})$. Claim 3 above implies that the modified surgery obstruction is zero, by [29, Proposition 8, p. 739]. The role of the first two properties in (W-AA) is explained in [20, Theorem 2.6], showing that we may modify our bordism $(\widehat{W}', \widehat{\Phi}')$ by the action of $L_5(\mathbb{Z}[\pi])$ to obtain an s -cobordism. This completes step (iv) and the proof of Theorem 11.2.

Proof of Theorem A. For π a right-angled Artin group with $\text{cd } \pi \leq 3$, the conditions (W-AA) were established by Bartels and Lück [2]. This shows that Theorem A follows from Theorem 11.2. □

Proof of Theorem B. Suppose that M and N are topological manifolds as in the statement with s_M stably isometric to s_N . Then by Proposition 5.11, we conclude that $Q(M)$ is stably isometric to $Q(N)$. Hence, after stabilization by sufficiently many copies of $S^2 \times S^2$, we may apply all but the final step in the proof of Theorem 11.2 to conclude that M and N are spin bordant over P , and hence stably homeomorphic. For these steps in the argument, we only need the injectivity of κ_2 , which is property (W-AA)(iii). In the smooth case, we apply essentially the same arguments above to show that M and N are smoothly spin bordant over the reduced type $P \times BSpin$, and it follows that M and N are stably diffeomorphic by [29, Theorem C]. □

Appendix A. The Hypercohomology Spectral Sequence

We include some background material and references for homological algebra for the reader's convenience.

(1) The universal coefficient spectral sequence is a special case of the *hypercohomology spectral sequence* defined in Benson [5, Proposition 3.4.3], Another good reference is Benson and Carlson [6, Secs. 2 and 4]. If \mathbf{C} and \mathbf{D} are chain complexes over a ring Λ , with \mathbf{C} bounded above and \mathbf{D} bounded below, then there is a spectral sequence

$$\text{Ext}_\Lambda^p(H_q(\mathbf{C}), \mathbf{D}) \Rightarrow \text{Ext}_\Lambda^{p+q}(\mathbf{C}, \mathbf{D})$$

converging to the hypercohomology groups (see Benson and Carlson [4, Sec. 2.7] for the definition of Ext for chain complexes). This is a first quadrant spectral sequence, so the indexing convention above is standard (p runs along the x -axis). The differential d_r has bi-degree $(r, -r + 1)$.

(1a) For X a connected finite CW-complex with $\pi_1(X) = \pi$ and $\Lambda = \mathbb{Z}[\pi]$, we can take $\mathbf{C} = C(X; \Lambda)$ and $\mathbf{D} = \Lambda$ (a 0-dim chain complex with Λ in degree zero). Then we have the *universal coefficient spectral sequence*

$$\text{Ext}_\Lambda^p(H_q(X; \Lambda), \Lambda) \Rightarrow \text{Ext}_\Lambda^{p+q}(C(X; \Lambda), \Lambda) = H^{p+q}(X; \Lambda).$$

For $q = 0$, the terms along the x -axis are just $\text{Ext}_\Lambda^p(\mathbb{Z}, \Lambda) = H^p(\pi; \Lambda)$. On the y -axis, we have $\text{Hom}_\Lambda(H_q(X; \Lambda), \Lambda)$. To compute $H^2(X; \Lambda)$ for X a closed, oriented 4-manifold, the spectral sequence produces the 4-term exact sequence:

$$0 \rightarrow H^2(\pi; \Lambda) \rightarrow H^2(X; \Lambda) \rightarrow \text{Hom}_\Lambda(H_2(X; \Lambda), \Lambda) \rightarrow H^3(\pi; \Lambda) \rightarrow 0$$

that we use throughout the paper. Note that the map to $H^3(\pi; \Lambda)$ is a d_2 differential, which is surjective since $H^3(X; \Lambda) = H_1(X; \Lambda) = H_1(\tilde{X}; \mathbb{Z}) = 0$.

(1b) There are two usual definitions of the Ext -functors, namely in terms of projective resolutions (the usual homological algebra), or in terms of *multiple extensions* due to MacLane [31, Chap. III, Secs. 5 and 6]. The multiple extensions version is particularly useful in describing connecting maps in long exact sequences (see Benson and Carlson [4, Sec. 2.6]).

(2) *A formula for the d_2 differential:* Let X be a finite CW complex with $\pi := \pi_1(X)$ and integral group ring $\Lambda = \mathbb{Z}\pi$. Let A be a right Λ -module. We use the universal coefficient spectral sequence which has E_2 -page

$$E_2^{p,q} = \text{Ext}_\Lambda^p(H_q(X; \Lambda), A)$$

to calculate the cohomology $H^*(X; A)$. The d_2 differential

$$d_2 : E_2^{p,q} = \text{Ext}_\Lambda^p(H_q(X; \Lambda), A) \rightarrow E_2^{p+2,q-1} = \text{Ext}_\Lambda^{p+2}(H_{q-1}(X; \Lambda), A)$$

is given in terms of the multiple extensions description for Ext by the following “splicing” formula.

(2a) Let $C_* := C(X; \Lambda)$ denote the chain complex of X with Λ -coefficients, and consider the usual sequences of cycles $Z_* := Z_*(C)$ and boundaries $B_* := B_*(C)$

$$\begin{aligned} 0 &\rightarrow B_{q+1} \rightarrow Z_q \rightarrow H_q \rightarrow 0 \\ 0 &\rightarrow Z_q \rightarrow C_q \rightarrow B_q \rightarrow 0 \\ 0 &\rightarrow B_q \rightarrow Z_{q-1} \rightarrow H_{q-1} \rightarrow 0 \end{aligned}$$

for computing $H_* := H_*(X; \Lambda)$. We can put these together to obtain the 4-term exact sequence

$$0 \rightarrow H_q \rightarrow C_q/B_{q+1} \rightarrow Z_{q-1} \rightarrow H_{q-1} \rightarrow 0. \tag{A.1}$$

To define d_2 , we represent an element $\alpha \in \text{Ext}_\Lambda^p(H_q, A)$ by a multiple extension

$$0 \rightarrow A \rightarrow S_{p-1} \rightarrow S_{p-2} \rightarrow \cdots \rightarrow S_0 \rightarrow H_q \rightarrow 0 \tag{A.2}$$

as in the description of Ext given by MacLane [31, Theorem 6.4, Chap. III], and splice the resolutions (A.1) and (A.2) together to get a representative for

$$d_2(\alpha) \in \text{Ext}_\Lambda^{p+2}(H_{q-1}; A).$$

(3) Finally, we relate this description of the d_2 differential as a Yoneda splice to the usual definition. In general (even when C_* is not a projective chain complex), the hypercohomology spectral sequence is defined (see Benson [5, Sec. 3.4]) as a spectral sequence of a double complex

$$E_0^{p,q} = \text{Hom}_\Lambda(P_{p,q}, A)$$

where for each $q \geq 0$, $P_{*,q} \rightarrow C_q$ is a projective resolution of C_q , and the chain maps $P_{*,q} \rightarrow P_{*,q-1}$ are induced by the boundary maps $C_q \rightarrow C_{q-1}$. Applying the horseshoe lemma to the extensions

$$0 \rightarrow B_{q+1} \rightarrow Z_q \rightarrow H_q \rightarrow 0$$

and

$$0 \rightarrow Z_q \rightarrow C_q \rightarrow B_q \rightarrow 0,$$

we can construct a projective resolution for C_q of the form $P_*(B_{q+1}) \oplus P_*(H_q) \oplus P_*(B_q)$, where $P_*(B_{q+1})$, $P_*(H_q)$, and $P_*(B_q)$ are projective resolutions of B_{q+1} , H_q , and B_q , respectively. The boundary map on this projective resolution is of the form

$$\partial_1(a, b, c) = (\partial a + f(b) + g_1(c), \partial b + g_2(c), \partial c),$$

where $f : P_*(H_q) \rightarrow P_*(B_{q+1})$ and $(g_1, g_2) : P_*(B_q) \rightarrow P_*(Z_q) = P_*(B_{q+1}) \oplus P_*(H_q)$ are degree -1 chain maps. Note that $f \circ g_2 = 0$ to check the relation

$\partial_1 \circ \partial_1 = 0$. After applying Hom, we will get the horizontal differential $d_1 : E_0^{p,q} \rightarrow E_0^{p+1,q}$, which can be expressed as the degree 1 map

$$d_1(a, b, c) = (\delta a, f'(a) + \delta(b), g'_1(a) + g'_2(b) + \delta(c)),$$

where f' and g'_1 , and g'_2 are duals of the corresponding maps.

The differential $\partial_0 : P_*(C_q) \rightarrow P_*(C_{q-1})$ is induced by the maps $C_q \rightarrow B_q$ and the composite $B_q \rightarrow Z_{q-1} \rightarrow C_{q-1}$. Hence by the construction of the projective resolution

$$P_*(C_q) = P_*(B_{q+1}) \oplus P_*(H_q) \oplus P_*(B_q),$$

the induced map is $\partial_0(a, b, c) = (c, 0, 0)$ and its dual gives the vertical differential $d_0 : E_0^{*,q-1} \rightarrow E_0^{*,q}$. Then one defines $E_1^{p,q} = H(E_0^{p,q}, d_0)$ and $E_2^{p,q} = H^p(H^q(E_0^{*,*}, d_0), d_1)$.

In our case, we have $E_2^{p,q} = \text{Ext}_\Lambda^p(H_q, A)$. The next differential in the spectral sequence

$$d_2 : \text{Ext}_\Lambda^p(H_q, A) \rightarrow \text{Ext}_\Lambda^{p+2}(H_{q-1}, A)$$

is now defined using the diagram

$$\begin{array}{ccc} \text{Hom}_\Lambda(P_{p+1}(C_{q-1}), A) & \xrightarrow{d_1} & \text{Hom}_\Lambda(P_{p+2}(C_{q-1}), A) \\ & \downarrow d_0 & \\ \text{Hom}_\Lambda(P_p(C_q), A) & \xrightarrow{d_1} & \text{Hom}_\Lambda(P_{p+1}(C_q), A) \end{array}$$

as the map obtained by first applying the horizontal d_1 differential, then lifting via the vertical d_0 differential, and then applying the horizontal d_1 again to the lifted element (see Benson [5, p. 196] for the notation).

Let $u \in \text{Hom}_\Lambda(P_p(H_q), A)$ be a degree p cocycle representing an Ext-class in $\text{Ext}_\Lambda^p(H_q, A)$. On $E_0^{p,q}$, this is represented by the class $(0, u, 0) \in E_0^{p,q} = \text{Hom}_\Lambda(P_p(C_q), A)$. Applying the horizontal differential on it, we obtain $(0, 0, g'_2(u))$ in $E_0^{p+1,q}$, where g'_2 is the dual of the chain map g_2 .

Since the vertical differential d_0 is induced by dualizing ∂_0 , we see that the element $(g'_2(u), 0, 0) \in E_0^{p+1,q-1}$ maps to $(0, 0, g'_2(u))$ under the vertical differential. Now by applying the horizontal differential again, we get $(0, f'(g'_2(u)), g'_1(g'_2(u))) \in E_0^{p+2,q-1}$. The element $f'(g'_2(u)) = (g_2 f)'(u)$ gives a degree $(p + 2)$ -cocycle in $\text{Hom}_\Lambda(P_*(H_{q-1}), A)$. The cohomology class $[(g_2 f)'(u)]$ is the image of $[u]$ under d_2 .

Note that f is the chain map $f : P_*(H_{q-1}) \rightarrow P_*(B_q)$, and g_2 is the second coordinate map of the chain map $(g_1, g_2) : P_*(B_q) \rightarrow P_*(Z_q) = P_*(B_{q+1}) \oplus P_*(H_q)$. The chain map f is induced by the extension

$$0 \rightarrow B_q \rightarrow Z_{q-1} \rightarrow H_{q-1} \rightarrow 0.$$

In addition, we can regard g_2 as the chain map induced by

$$0 \rightarrow H_q \rightarrow C_q/B_{q+1} \rightarrow B_q \rightarrow 0.$$

The multiple extensions description of the Ext-group shows that applying these chain maps corresponds to a Yoneda splice with the corresponding extension classes. This completes the identification of the d_2 differential. \square

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References

1. A. Bartels, F. T. Farrell and W. Lück, The Farrell–Jones conjecture for cocompact lattices in virtually connected Lie groups, *J. Amer. Math. Soc.* **27** (2014) 339–388.
2. A. Bartels and W. Lück, The Borel conjecture for hyperbolic and CAT(0)-groups, *Ann. of Math. (2)* **175** (2012) 631–689.
3. H. J. Baues, *Combinatorial Homotopy and 4-Dimensional Complexes*, de Gruyter Expositions in Mathematics, Vol. 2 (Walter de Gruyter & Co., 1991), With a preface by Ronald Brown.
4. D. J. Benson, *Representations and Cohomology. I*, 2nd edn., Cambridge Studies in Advanced Mathematics, Vol. 30 (Cambridge Univ. Press, 1998), Basic representation theory of finite groups and associative algebras.
5. D. J. Benson, *Representations and Cohomology. II*, 2nd edn., Cambridge Studies in Advanced Mathematics, Vol. 31 (Cambridge Univ. Press, 1998), Cohomology of groups and modules.
6. D. J. Benson and J. F. Carlson, Projective resolutions and Poincaré duality complexes, *Trans. Amer. Math. Soc.* **342** (1994) 447–488.
7. R. Charney, An introduction to right-angled Artin groups, *Geom. Dedicata* **125** (2007) 141–158.
8. R. Charney and M. W. Davis, Finite $K(\pi, 1)$ s for Artin groups, in *Prospects in Topology*, Annals Mathematics Studies, Vol. 138 (Princeton Univ. Press, 1995), pp. 110–124.
9. J. F. Davis, *The Borel/Novikov conjectures and stable diffeomorphisms of 4-manifolds*, in *Geometry and Topology of Manifolds*, Fields Institute Communications, Vol. 47 (Amer. Math. Soc., 2005), pp. 63–76.
10. M. Davis and B. Okun, Cohomology computation for Artin groups, Bestvina–Brady groups, and graph products, *Groups Geom. Dyn.* **6** (2012) 485–531.
11. A. Dold and H. Whitney, Classification of oriented sphere bundles over a 4-complex, *Ann. of Math. (2)* **69** (1959) 667–677.
12. M. J. Dunwoody, The accessibility of finitely presented groups, *Invent. Math.* **81** (1985) 449–457.
13. M. H. Freedman, The topology of four-dimensional manifolds, *J. Differential Geom.* **17** (1982) 357–453.
14. M. H. Freedman and F. Quinn, *Topology of 4-Manifolds*, Princeton Mathematical Series, Vol. 39 (Princeton Univ. Press, 1990).

15. M. H. Freedman and P. Teichner, 4-manifold topology. I. Subexponential groups, *Invent. Math.* **122** (1995) 509–529.
16. M. H. Freedman and P. Teichner, 4-manifold topology. II. Dwyer’s filtration and surgery kernels, *Invent. Math.* **122** (1995) 531–557.
17. I. Hambleton, Intersection forms, fundamental groups and 4-manifolds, in *Proc. Gökova Geometry-Topology Conference 2008*, Gökova Geometry/Topology Conference (GGT) (International Press, Somerville, MA, 2009), pp. 137–150.
18. I. Hambleton and M. Kreck, On the classification of topological 4-manifolds with finite fundamental group, *Proc. London Math. Soc.* (3) **56** (1988) 349–379.
19. I. Hambleton and M. Kreck, Cancellation of hyperbolic forms and topological four-manifolds, *J. Reine Angew. Math.* **443** (1993) 21–47.
20. I. Hambleton, M. Kreck and P. Teichner, Topological 4-manifolds with geometrically 2-dimensional fundamental groups, *J. Topol. Anal.* **1** (2009) 123–151.
21. J.-C. Hausmann and S. Weinberger, Caractéristiques d’Euler et groupes fondamentaux des variétés de dimension 4, *Comment. Math. Helv.* **60** (1985) 139–144.
22. A. Hildum, The minimum b_2 problem for right-angled Artin groups, *Algebr. Geom. Topol.* **15** (2015) 1599–1641.
23. C. Jensen and J. Meier, The cohomology of right-angled Artin groups with group ring coefficients, *Bull. London Math. Soc.* **37** (2005) 711–718.
24. D. Kasprowski, M. Land, M. Powell and P. Teichner, Stable classification of 4-manifolds with 3-manifold fundamental groups, *J. Topol.* **10** (2017) 827–881.
25. K. Kim and F. Roush, Homology of certain algebras defined by graphs, *J. Pure Appl. Algebra* **17** (1980) 179–186.
26. R. C. Kirby and L. C. Siebenmann, *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Annals of Mathematics Studies, Vol. 88 (Princeton Univ. Press, 1977), With notes by John Milnor and Michael Atiyah.
27. R. C. Kirby and L. R. Taylor, A survey of 4-manifolds through the eyes of surgery, in *Surveys on Surgery Theory*, Vol. 2, Annals of Mathematics Studies, Vol. 149 (Princeton Univ. Press, 2001), pp. 387–421.
28. P. Kirk and C. Livingston, The Hausmann–Weinberger 4-manifold invariant of abelian groups, *Proc. Amer. Math. Soc.* **133** (2005) 1537–1546 (electronic).
29. M. Kreck, Surgery and duality, *Ann. of Math.* (2) **149** (1999) 707–754.
30. M. Kreck and J. A. Schafer, Classification and stable classification of manifolds: Some examples, *Comment. Math. Helv.* **59** (1984) 12–38.
31. S. MacLane, Homology, 1st edn., Die Grundlehren der mathematischen Wissenschaften, Vol. 114 (Springer-Verlag, 1967).
32. S. P. Plotnick, Equivariant intersection forms, knots in S^4 , and rotations in 2-spheres, *Trans. Amer. Math. Soc.* **296** (1986) 543–575.
33. J. R. Stallings, On torsion-free groups with infinitely many ends, *Ann. of Math.* (2) **88** (1968) 312–334.
34. C. T. C. Wall, Finiteness conditions for CW-complexes, *Ann. of Math.* (2) **81** (1965) 56–69.
35. C. T. C. Wall, *Surgery on Compact Manifolds*, 2nd edn. (Amer. Math. Soc., 1999), Edited and with a foreword by A. A. Ranicki.
36. J. H. C. Whitehead, A certain exact sequence, *Ann. of Math.* (2) **52** (1950) 51–110.